Răzvan Diaconescu

Institution-independent Model Theory

second edition

To the memory of my parents, Elena and Ştefan

Preface

This is a book about doing model theory without a concrete underlying logical system. It teaches us how to live without concrete models, sentences, satisfaction and so on. Our approach is based on the theory of institutions, which has witnessed a vigorous and systematic development over the past few decades and which provides an ideal framework for true abstract model theory. The concept of institution formalises the intuitive notion of a logical system into a mathematical object. Thus our model theory without concrete underlying logical systems and based upon institution theory may be called ‘institution-independent model theory’ because it does not depend on any *concrete* institution.

Institution-independent model theory has several advantages. One is its generality since it can be easily applied to a multitude of logical systems, conventional or less conventional, many of the latter kind getting a proper model theory for the first time through this approach. This is important, especially in the context of the high proliferation of logics in computing science, and especially in the area of formal specification but not only. Then there is the advantage of illuminating the model-theoretic phenomena and its subtle network of causality relationships, thus leading to a deeper understanding which produces new fundamental insights and results even in well-worked traditional areas of model theory.

In this way, we study well-established topics in model theory but also some newly emerged important topics. The former category includes methods (much of model theory can be regarded as a collection of sometimes overlapping methods) such as diagrams, ultraproducts, saturated models and studies about preservation, axiomatizability, interpolation, definability, and possible worlds semantics. The latter category includes also methods of doing model theory ‘by translation’. A part of the book is devoted to extensions of ordinary institution theory oriented towards non-classical model-theoretic phenomena (models ‘with states’ and many-valued truth). The last part of the book digresses from the main topic of the book in that it presents some applications of the institution-independent model theory to specification and programming.

This book is far from being a complete encyclopedia of institution-independent model theory. While several important concepts and results have not been treated here, we believe they can be approached successfully by institutions in the style promoted by our work. Most of all, this book shows *how* to do things rather than provides an exhaustive account of all model theory that can be done institution-independently. It can be used by any working user of model theory but also as a resource for learning model theory.

From a philosophical viewpoint, the institution-independent approach to model theory is based upon a non-substantialist, groundless, perspective on logic and model theory, directly influenced by the doctrine of *śūnyatā* of the Madhyamaka Prāsaṅgika school within Mahāyanā Buddhism. The interested reader may find more about this connection in the essay [69]. This philosophical viewpoint has been developed mainly at Nālandā monastic university many centuries ago by Arya Nāgārjuna and its successors and has been continued to our days by all traditions of Tibetan Buddhism. The relationship between Madhyamaka Prāsaṅgika 's thinking and various branches of modern science are surveyed in [235].

I am grateful to several people who supported the institution-independent model theory project in various ways in general and the writing of this book in particular. I was extremely fortunate to be first the student and later a close friend and collaborator of late Professor Joseph Goguen who together with Rod Burstall introduced institutions. He strongly influenced this work in many ways and at many levels, from philosophical to technical aspects, and was one of the greatest promoters of the non-essentialist approach to science. Andrzej Tarlecki was the true pioneer of model theory in an abstract institutional setting. Till Mossakowski made a lot of useful comments on several preliminary drafts of this book and supported this activity in many other ways too. Grigore Roşu and Marc Aiguier made valuable contributions to this area. Lutz Schröder made several comments and gave some useful suggestions. Achim Blumensath read very carefully a preliminary draft of this book and helped to correct a series of errors. Some of the corrections implemented in the second edition of the book owe to Ionuţ Ţuţu and Andrzej Tarlecki. I am indebted to the late Professor Hans-Jürgen Hoehnke for encouragement and managerial support. Special thanks go to the former students of the Informatics Department of "Şcoala Normală Superioară" of Bucharest, namely Marius Petria, Daniel Găină, Andrei Popescu, Ionuţ Ţuţu, Mihai Codescu, Traian Şerbănuţă and Cristian Cucu. They started as patient students of the institution-independent model theory only to become important contributors to this area. Finally, Jean-Yves Béziau greatly supported the publication and dissemination of this book. I acknowledge the financial support for writing this book from various grants of the Romanian National Council for Scientific Research (CNCS).

Sinaia,

April 2024 *Răzvan Diaconescu*

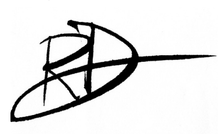


Figure 1: The author with Joseph Goguen at Sinaia, cca. 2000

Contents

Chapter 1

Introduction

Model theory is in essence the mathematical study of semantics, or meaning, of logic systems. As it has a multitude of applications in various areas of classical mathematics, and logic, but also to many areas of informatics and computing science, there are various perspectives on model theory which differ slightly. A rather classical viewpoint is formulated in [42]:

Model theory = logic + universal algebra.

From a formal specification viewpoint, in a similar tone, one may say that

Model theory = logical semantics – specification.

Each such viewpoint implies a specific way of developing the key concepts and the main model theory methods; it also puts different emphasis on results. For example, while forcing is a very important method for the applications of model theory to conventional logic, it plays very little role in computing science. On the other hand, formal specification theory requires a much more abstract view on model theory than the conventional one. The institution theory of Goguen and Burstall [37, 124] arose out of this necessity.

Institutions. The theory of institutions is a categorical abstract model theory which formalises the intuitive notion of a logical system, including syntax, semantics, and the satisfaction relation between them. Institutions constitute a model-oriented meta-theory on logics similarly to how the theory of rings and modules constitutes a meta-theory for classical linear algebra. Another analogy can be made with universal algebra versus particular algebraic structures such as groups, rings, modules, etc., or with mathematical analysis over Banach spaces versus real analysis.

The notion of the institution was introduced by Goguen and Burstall in the late 1970s [37] (with the seminal journal paper [124] being printed rather late) in response to the population explosion of specification logics with the original intention of providing a proper abstract framework for specification of, and reasoning about, software systems.

Since then institutions have become a major tool in development of the theory of specification, mainly because they provide a language-independent framework applicable to a wide variety of particular specification logics. It became standard in the field to have a logic system captured as the institution underlying a particular language or system, such that all language / system constructs and features can be rigorously explained as mathematical entities and to separate all aspects that depend on the details of the particular logic system from those that are general and independent of this logic system by basing the latter on an arbitrary institution. All well-designed specification formalisms follow this path, including for example CASL [14] and CafeOBJ [95].

Recently institutions have also been applied to computing science fields other than formal specification; these include ontologies and cognitive semantics [122], concurrency [192], and quantum computing [39], computational creativity [84].

Institution-independent model theory. This means the development of model theory in the very abstract setting of arbitrary institutions, free of any commitment to a particular logic system, and in a fully axiomatic way. We can safely say that the institution-independent model theory is an axiomatic (approach to) model theory which is based on the formalisation of logical systems as institutions. In this way, we gain another level of abstraction and generality and a deeper understanding of model-theoretic phenomena, not hindered by the largely irrelevant details of a particular logic system, but guided by structurally clean causality. The latter aspect is based on the fact that concepts come naturally as presumed features that "a logic" might exhibit or not and are defined at the most appropriate level of abstraction; hypotheses are kept as general as possible and introduced on a by-need basis, and thus results and proofs are modular and easy to track down regardless of their depth. Access to highly non-trivial results is also considerably facilitated, which is contrary to the impression of some people that such general abstract approaches produce trivial results. As Béziau explains in [23]:

> "This impression is generally because these people have a concrete-oriented mind and that something which is not specified [n.a. concretely] has no meaning for them, and therefore universal logic [n.a. institution-independent model theory in our case] appears as logical abstract nonsense. They are like someone who understands perfectly what is Felix, his cat, but for whom the concept of a cat is a meaningless abstraction. This psychological limitation is a strong defect because, ... [n.a. as this book also shows], what is trivial is generally the specific part, not the universal one [n.a. the institution-independent one] which requires what is the fundamental capacity of human thought: abstraction."

The continuous interplay between the specific and the general in institution-independent model theory brings a large array of new results for particular non-conventional logics, unifies several known results, produces new results in well-studied conventional areas, reveals previously unknown causality relations, and dismantles some which are usually assumed as natural.

The institution-independent model theory also provides a clear and an efficient framework for doing logic and model theory 'by translation (or borrowing)' via a general theory of mappings (homomorphisms) between institutions. For example, a certain property P which holds in an institution I' can be also established in another institution I provided that we can define a mapping $I \to I'$ which 'respects' P.

Institution-independent model theory can be regarded as a form of 'universal model theory', part of the so-called 'universal logic', a recent trend in logic promoted by Béziau and others [24].

Other abstract model theories. Few major abstract approaches to logic have a model-theoretic nature and are therefore comparable to the institution-independent model theory.

The so-called "abstract model theory" developed by Barwise and others [16, 17] however keeps a strong commitment to conventional concrete systems of logic by explicitly extending them and retaining many of their features, hence one may call this framework "half-abstract model theory". In this context even the remarkable Lindström characterization of first-order logic by some of its properties should be rather considered a first-order logic result rather than a true abstract model-theoretic one.

Another framework is given by the so-called "categorical model theory" best represented by the works on sketches [103, 141, 244] or on satisfaction as cone injectivity [8, 9, 10, 171, 168, 166]. The former just develops another language for expressing (possibly infinitary) first-order logic realities. While the latter considers models as objects of abstract categories, it lacks the multi-signature aspect of institutions given by the signature morphism and the model reducts, which leads to severe methodological limitations. Moreover in these categorical model theory frameworks, the satisfaction of sentences by the models is usually defined rather than being axiomatized.

The "general logics" of [12] represents another abstract approach which has a pronounced model-theoretic side. This is strongly motivated by the algebraic logic in the tradition of Traski / Blok / Pigozzi, etc. It also lacks a proper multi-signature aspect and is not fully abstract, especially because of the syntax level.

In contrast to the abstract approaches mentioned above, institutions capture directly the essence of logic systems by axiomatizing fully abstractly the satisfaction relationship between models and sentences without any initial commitment to a particular logic system and by emphasizing properly the multi-signature aspect of logics.

Book content. The book consists of four parts.

1. In the first part, we introduce the basic institution theory including the concept of institution and institution morphisms, and several model-theoretic fundamental concepts such as model amalgamation, diagrams, inclusion systems, and free models. We develop an 'internal logic' for abstract institutions, which includes a semantic treatment for Boolean connectives, quantifiers, atomic sentences, substitutions, and elementary homomorphisms, all of them in an institution-independent setting.

2. The second part is the core of our institution-independent model theoretic study because it develops the main model theory methods and results in an institution-independent setting.

    - The first method considered in this part is that of ultraproducts. Based upon the well-established concept of categorical filtered products, we develop an ultraproduct fundamental theorem in an institution-independent setting and explore some of its immediate consequences, such as ultrapower embeddings and compactness.

    - The chapter on saturated models starts by developing sufficient conditions for directed co-limits of homomorphisms to retain the elementariness. This rather general version of Tarski's elementary chain theorem is a prerequisite for a general result about the existence of saturated models, later used for developing other important results. We also develop the complementary result on the uniqueness of saturated models. Here, the necessary concept of the cardinality of a model is handled categorically with the help of elementary extensions, a concept given by the method of diagrams. We develop an important application for the uniqueness of saturated models, namely a generalized version of the remarkable Keisler-Shelah result in first-order model theory, "two models are elementarily equivalent if and only if they have isomorphic ultrapowers".

    - A good application of the existence result for saturated models is seen in the preservation results, such as "a theory has a set of universal axioms if and only if its class of models is closed under 'sub-models'". We develop a generic preservation-by-saturation theorem. Such preservation results might lead us straight to their axiomatizability versions. One way is to assume the Keisler-Shelah property for the institution and to use a direct consequence of the fundamental ultraproducts theorem which may concisely read as "a class of models is elementary if and only if it is closed under elementary equivalence and ultraproducts".
      Another method to reach an important class of axiomatizability results is by expressing the satisfaction of Horn sentences as categorical injectivity. This leads to general quasi-variety theorems such as "a class of models is closed under products and 'sub-models' if and only if it is axiomatizable by a set of (universal) Horn sentences" and variety theorems such as "a class of models is closed under products and 'sub-models' and 'homomorphic images' if and only if it is axiomatizable by a set of (universal) 'atoms'". All axiomatizability results presented here are collected under the abstract concept of 'Birkhoff institution'.

    - The next topic is interpolation. The institution-independent approach brings several significant upgrades to the conventional formulation. We develop here three main methods for obtaining the interpolation property, the first two having rather complementary application domains. The first one is based on a semantic approach to interpolation and exploits the Birkhoff-style axiomatizability properties of the institution (captured by the above-mentioned concept of Birkhoff institution), while the second, inspired by the conventional methods of first-order logic, is via Robinson consistency. The third one is a borrowing method across institutions.

- We next treat definability, again with rather two complementary methods, via Birkhoff-style axiomatizability and interpolation. While the latter represents a generalization of Beth's theorem of conventional first-order model theory, the former reveals a causality relationship between axiomatizability and definability.

This ends the treatment of somewhat 'classical' topics of model theory in our book. Important institution-independent developments of other 'classical' topics not included in this book are Löwenheim-Skolem theorems, Gödel-Henkin completeness method, forcing, Lindström theorem. We still wanted to keep the material of this book within a reasonable size and it was a matter of personal orientation on what to include and what to omit from the rather vast spectrum of institution-independent model theory works. We hope our choices do satisfy the taste of most of our readers.

3. The third part of the book is devoted to some extensions of the standard institution theory. The chapter on proof theory for institutions introduces the concept of proof in a simple way that suits the model theory, explores proof theoretic versions of compactness, and presents general soundness results for institutions with proofs. The final part of this chapter develops a general sound and complete Birkhoff-style proof system with applications significantly wider than that of the Horn institutions. The other two chapters of this part explore 'models with states' (which applies to various forms of Kripke semantics, but goes much beyond that) and many-valued truth, respectively, in an institution-independent model-theoretic setting. In both cases, it is possible to have a standard institution-theoretic treatment by 'flattening' the non-classical contexts; the high level of abstraction of the concept of institution allows this. In this way, many developments from the previous parts of the book can be applied directly to non-classical contexts. But there are also limitations to that, regarding some finer-grained non-classical aspects. Hence the need for proper non-classical extensions of the concept of institution. The 'stratified institutions' deal with 'models with states', while the '$\mathcal{L}$-institutions' generalise the ordinary institutions by allowing a many-valued satisfaction relation. In both frameworks, we recover some of the themes from the previous parts of the book, including the semantics of logical connectives, ultraproducts, preservation, compactness, interpolation, definability, translation structures, etc. However, some of these do not enjoy yet the same level of development as their ordinary institution-theoretic counterparts, and this is due to inherent increased technical difficulties.
4. The last part of the book presents a few from the multitude of applications of institution-independent model theory to computing science, especially in the areas of formal specification and logic programming. This includes heterogeneous multi-logic frameworks through a Grothendieck construction on institutions, a systematic study of lifting of important properties such as theory co-limits, model amalgamation, and interpolation, from the level of the 'local' institutions to the 'global' Grothendieck institution, structured specifications over arbitrary institutions, the lifting of a complete calculus from the base institution to structured specifications, Herbrand theorems and modularization for logic programming, and the semantics of logic programming with pre-defined types. The fact that the relationship between model theory and computing science is

a two-way street, is illustrated by an unlikely application of Grothendieck institutions to interpolation in the Horn fragment of first-order logic, a surprising interpolation property is obtained through Grothendieck institutions.

The concepts introduced and the results obtained are systematically illustrated in the main text by their applications to the model theory of conventional logic (which includes first-order logic but also fragments and extensions of it). There are only two reasons for doing this. The first is to build a bridge between our approach and the conventional model theory culture. Logicians have great familiarity with first-order logic and model theory, so examples from first-order model theory may support an easier understanding of abstract developments. The second reason has to do with keeping the material within reasonable size. Otherwise, while the conventional (first-order) model theory has been historically the framework for the development of the main concepts and methods of model theory, one of the main messages of this book is that these do not depend on that framework. Any other concrete logic or model theory could be used as a benchmark example in this book, and we do this systematically in the exercise sections with several less conventional logics.

How to use this book. The material of this book can be used in various ways by various audiences both from logic and computing science. Students and researchers of logic can use the material from the first two parts as an institution-independent introduction to model theory. Working logicians and model theorists will find in this monograph a novel view and a new methodological approach to model theory. Computer scientists may use the material of the first part as an introduction to institution theory, and material from the third and the fourth parts for an advanced approach to topics from the semantics of formal specification and logic programming. Also, the institution-independent model theory constitutes a powerful tool for workers in formal specification to perform a systematic model-theoretic analysis of the logic underlying the particular system they employ.

Each section comes with some exercises. While some of them are meant to help the reader accommodate the concepts introduced, others contain quite important results and applications. To keep the book within a reasonable size, much of the knowledge had to be exiled to the exercise sections.

Regarding dependencies between various parts of the material in this book, in general this is more or less quite linear, especially in the first parts of the book. When relevant, we will state explicitly what is specifically needed for studying a certain chapter or section.

Remarks on the style. In the second edition, we have introduced some novelties in the style of presentation. Two of the most important are as follows. The first one concerns the use of parentheses. Here we adopt a minimalist style, using them strictly to avoid ambiguities even if this violates the common notational habits. For instance, we may denote the application of a function f to an element x by fx rather than $f(x)$. However, this is hardly a novelty, for instance, people in type theory or in category theory often do this. The aim of this is to reduce the complexity of formulas in terms of number of symbols used. The second novelty, which we consider more important than the first one, concerns the style of presenting proofs of results. The aim of this is to enhance the readability of the proofs, as we are very well aware that in general model theory is a mathematically difficult area, and in particular the institution theoretic approach adds another layer of sophistication to that. To support the understanding of proofs we adopt a structured style of presenting them such that milestones and main ideas are clearly outlined. Also, we abandon the common ‘always forward’ reasoning in favour of a more realistic style in which we first formulate a statement to be proved and then present the arguments. However, this is balanced and integrated with the common style of presenting proofs.

Part I

Basics

Chapter 2

Categories

Institution-independent model theory as a categorical model theory relies heavily on category theory. This preliminary chapter gives a brief overview of the categorical concepts and results used in this book also allows us to fix some notations and terminology. The reader without enough familiarity with category theory is advised to use one of the textbooks on category theory available in the literature. The references [165] and [31] are among the standard references for category theory. A reference for indexed categories discussing many examples from the model theory of algebraic specification is [232], while [152] contains a rather compact presentation of fibred category theory.

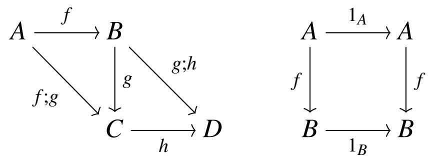
2.1 Basic concepts

Categories

A *category* $\mathbb{C}$ consists of

- a class $|\mathbb{C}|$ of *objects*,
- a class of *arrows* (sometimes also called 'morphisms' or 'homomorphisms'), denoted just as $\mathbb{C}$,
- two maps $dom, cod : \mathbb{C} \to |\mathbb{C}|$ giving the *domain* and *codomain* of each arrow such that for each pair of objects A and B, $\mathbb{C}(A,B) = \{f \in \mathbb{C} \mid dom(f) = A, cod(f) = B\}$ is a *set*,
- for all objects A,B,C, a *composition* map $\_;\_ : \mathbb{C}(A,B) \times \mathbb{C}(B,C) \to \mathbb{C}(A,C)$,
- an *identity* arrow map $1 : |\mathbb{C}| \to \mathbb{C}$ such that $1_A \in \mathbb{C}(A,A)$ for each $A \in |\mathbb{C}|$,

such that the (arrow) composition $\_;\_$ is associative and with identity arrows as left and right identities.



Notice that we prefer to use the diagrammatic notation $f;g$ for composition of arrows in categories, rather than the alternative set theoretic one $g \circ f$ used in many category theory works.

Categories arise everywhere in mathematics. A most typical example is that of sets (as objects) and functions (as arrows) with the usual (functional) composition. We denote this category by $\mathbb{S}et$. Notice that $|\mathbb{S}et|$, the collection of all sets, is *not* a set, it is a proper *class*.

The arrows of a category in general reflect the structure of objects in the sense of preserving that structure. However, this should not always be the case. One can go further by saying that, in reality, a particular category is determined only by its arrows, the objects being a derived rather than a primary concept.

A category $\mathbb{C}$ is *small* when its class of objects $|\mathbb{C}|$ is a set. Note that this implies that $\mathbb{C}$, the class of arrows, is also a set.

$\mathbb{C}$ is *connected* when it is empty or there exists at most one equivalence class for the equivalence generated by the relation on objects given by "there exists an arrow $A \to B$".

Isomorphisms. An arrow $f: A \to B$ is an *isomorphism* when there exists an arrow $g: B \to A$ such that $f;g = 1_A$ and $g;f = 1_B$. The *inverse* g is denoted as f^{-1}. Two objects A and B are *isomorphic*, and we denote this by $A \cong B$, when there exists an isomorphism $f: A \to B$. Isomorphisms in $\mathbb{S}et$ are precisely the bijective (injective and surjective) functions. However, this is not true in general; structure-preserving mappings that are bijective are not necessarily isomorphisms. A simple counterexample is given by the category of partial orders (objects) with order-preserving functions as arrows.

Monoids are exactly the categories with only one object. Then *groups* are exactly the monoids for which all elements (arrows) are isomorphisms.

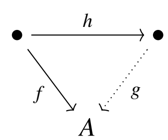
Being isomorphic is an equivalence relation on objects; the equivalence classes of $\cong$ are called *isomorphism classes*.

Epis and monos. A family of arrows $(f_i : A \to B)_{i \in I}$ is *epimorphic* when for each pair of parallel arrows $g_1, g_2 : B \to C$, $f_i;g_1 = f_i;g_2$ for all $i \in I$ implies $g_1 = g_2$, and it is *monomorphic* when for each pair of parallel arrows $g_1, g_2 : C \to A$, $g_1;f_i = g_2;f_i$ for all $i \in I$ implies $g_1 = g_2$. An arrow $f: A \to B$ is *epi* / *mono* when it is epimorphic / monomorphic as a (singleton) family, i.e., (f) is epimorphic / monomorphic.

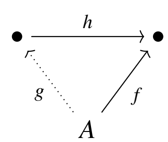
In $\mathbb{S}et$ epis are exactly the surjective functions and the monos are exactly the injective ones. Note that while, in general, whenever arrows appear as functions with additional structure, the injectivity (respectively surjectivity) of the underlying function is a sufficient condition for a function to be mono (respectively epi), the converse is not true. For example, the inclusion $\mathbb{Z} \to \mathbb{Q}$ of integers into the rationals is epi in the category of rings but it is not surjective. This is also an example of an arrow which is both epi and mono but is not an isomorphism.

An arrow $f: A \to B$ is a *retract* to $g: B \to A$ when $g;f = 1_B$. Notice that each retract is an epi. The converse, which is not true in general, is one of the categorical formulations of the Axiom of Choice. Note that $\mathbb{S}et$ has the Axiom of Choice in this sense.

An object A is *injective* with respect to an arrow h when for each arrow $f: dom(h) \to A$ there exists an arrow g such that $h;g = f$. A is simply *injective* when it is injective with respect to all mono arrows.



Dually, an object A is *projective* with respect to an arrow h when for each arrow $f: A \to cod(h)$ there exists an arrow g such that $g;h = f$. A is simply *projective* when it is projective with respect to all epi arrows.



Note that in $\mathbb{S}et$ all objects (sets) are both injective and projective except for the empty set that is only projective.

Functors

A *functor* $\mathcal{U}: \mathbb{C} \to \mathbb{C}'$ between categories $\mathbb{C}$ and $\mathbb{C}'$ maps

- objects to objects, $|\mathcal{U}|: |\mathbb{C}| \to |\mathbb{C}'|$, and
- arrows to arrows, $\mathcal{U}_{A,B}: \mathbb{C}(A,B) \to \mathbb{C}'(\mathcal{U}A, \mathcal{U}B)$ for all objects $A, B \in |\mathbb{C}|$

such that

– $\mathcal{U}1_A = 1_{\mathcal{U}A}$ for each object $A \in |\mathbb{C}|$, and

– $\mathcal{U}(f;g) = (\mathcal{U}f);(\mathcal{U}g)$ for all composable arrows $f, g \in \mathbb{C}$.

Most of the time we will denote $|\mathcal{U}|$ and $\mathcal{U}_{A,B}$ simply by $\mathcal{U}$.

The couple of equations above reveal also our style of using parentheses in formulas. Whenever there is no danger of ambiguity we will refrain from using them, instead relying on juxtaposition. For instance instead of $\mathcal{U}(A)$ we wrote $\mathcal{U}A$. So we will use parentheses only to avoid parsing ambiguities. In this way, we will enhance readability by avoiding heavily parenthesised expressions. Sometimes it is even convenient to use subscripts or superscripts for the application of functors to objects or arrows.

A simple example of a functor is the power-set functor $\mathcal{P}:\ \mathbb{S}et \to \mathbb{S}et$ which maps each set S to the set of its subsets $\{X \mid X \subseteq S\}$ and maps each function $f:\ S \to S'$ to the function $\mathcal{P}f:\ \mathcal{P}S \to \mathcal{P}S'$ such that $(\mathcal{P}f)X = fX = \{fx \mid x \in X\}$.

Another example of a functor is given by 'cartesian product with A'. For any fixed set A, let $A \times - :\ \mathbb{S}et \to \mathbb{S}et$ be the functor mapping each set B to $A \times B = \{(a,b) \mid a \in A, b \in B\}$ and each function $f:\ B \to C$ to $(A \times f):\ A \times B \to A \times C$ defined by $(A \times f)(a,b) = (a, fb)$.

$$
\begin{array}{ccc}
A \times B & \xrightarrow{(a,b)\mapsto b} & B \\
{\scriptstyle A\times f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
A \times C & \xrightarrow[(a,c)\mapsto c]{} & C
\end{array}
$$

A third example is that of 'hom-functors'. For any category $\mathbb{C}$ and any object $A \in |\mathbb{C}|$, the hom-functor $\mathbb{C}(A,-):\ \mathbb{C} \to \mathbb{S}et$ maps any object $B \in |\mathbb{C}|$ to the set of arrows $\mathbb{C}(A,B)$ and each arrow $f:\ B \to B'$ to the function $\mathbb{C}(A,f):\ \mathbb{C}(A,B) \to \mathbb{C}(A,B')$ defined by $\mathbb{C}(A,f)g = g;f$.

When regarding preorders as categories (that have at most one arrow between any two given objects), each preorder-preserving (or monotonic) function between two preorders $(P,\leq) \to (Q,\leq)$ yields another example of a functor. Functors between preorders are precisely the monotonic functions.

A functor $\mathcal{U}:\ \mathbb{C} \to \mathbb{C}'$ is *full* when for each objects A and B, the mapping on arrows $\mathcal{U}_{A,B}:\ \mathbb{C}(A,B) \to \mathbb{C}'(\mathcal{U}A, \mathcal{U}B)$ is surjective and is *faithful* when $\mathcal{U}_{A,B}$ is injective. Note that both functors of the first and the second examples are faithful but not full.

Functors can be composed in an obvious way and each category has an identity functor with respect to functor composition. By discarding the foundational issues (for the interested reader we recommend [147] or [165]), we let $\mathbb{C}at$ be the 'hyper-category' of categories (as objects) and functors (as arrows).

$\mathbb{C} \subseteq \mathbb{C}'$ is a *subcategory* (of $\mathbb{C}'$) when $|\mathbb{C}| \subseteq |\mathbb{C}'|$, $\mathbb{C}(A,B) \subseteq \mathbb{C}'(A,B)$ for all $A,B \in |\mathbb{C}|$, the identities in $\mathbb{C}$ are identities in $\mathbb{C}'$ too, and the composition in $\mathbb{C}$ is a restriction of the composition in $\mathbb{C}'$. A subcategory $\mathbb{C} \subseteq \mathbb{C}'$ is *broad* when $|\mathbb{C}| = |\mathbb{C}'|$.

Concrete categories. A *concrete category* $(\mathbb{A}, \mathcal{U})$ consists of a category $\mathbb{A}$ and a faithful functor $\mathcal{U}:\ \mathbb{A} \to \mathbb{S}et$. This is the most commonly accepted definition for concrete categories, although in [1] this is called 'concrete over $\mathbb{S}et$' or 'construct'.

A functor of concrete categories $\mathcal{F}:\ (\mathbb{A}, \mathcal{U}) \to (\mathbb{B}, \mathcal{V})$ is just a functor $\mathcal{F}:\ \mathbb{A} \to \mathbb{B}$ such that $\mathcal{U} = \mathcal{F};\mathcal{V}$. Let $\mathbb{CC}at$ denote the category that has the concrete categories as objects and functors of concrete categories as arrows.

When it is clear from the context we may omit $\mathcal{U}$ and simply refer to $(\mathbb{A}, \mathcal{U})$ as $\mathbb{A}$. This implies also that for $A \in |\mathbb{A}|$ we may write $a \in A$ instead of $a \in \mathcal{U}A$.

Natural transformations

Fixing categories $\mathbb{A}$ and $\mathbb{B}$, $\mathbb{C}at(\mathbb{A},\mathbb{B})$ can be regarded as a category with functors as objects and *natural transformations* as arrows. A natural transformation $\tau : \ \mathcal{S} \Rightarrow \mathcal{T}$ between functors $\mathcal{S},\mathcal{T} : \ \mathbb{A} \to \mathbb{B}$ is a map $|\mathbb{A}| \to \mathbb{B}$ such that $\tau_A \in \mathbb{B}(\mathcal{S}A, \mathcal{T}A)$ for each $A \in |\mathbb{A}|$ and the following diagram commutes (in $\mathbb{B}$)

$$\begin{array}{ccc} \mathcal{S}A & \xrightarrow{\tau_A} & \mathcal{T}A \\ {\scriptstyle \mathcal{S}f}\downarrow & & \downarrow{\scriptstyle \mathcal{T}f} \\ \mathcal{S}B & \xrightarrow[\tau_B]{} & \mathcal{T}B \end{array}$$

for each arrow $f \in \mathbb{A}(A,B)$. Although the classical notation for the component $\tau(A)$ is τ_A, in the literature the diagrammatic notation $A\tau$ is also frequently used. We will also employ this kind of notation when convenient.

A simple example of a natural transformation is given by considering a function $A \xrightarrow{f} A'$ which determines a natural transformation $nt(f) : \ (A \times -) \Rightarrow (A' \times -)$ given by $nt(f)_B = f \times 1_B$ for each set B, where $(f \times 1_B)(a,b) = (fa,b)$ for each $(a,b) \in A \times B$.

An additional example is given by the natural transformation $\mathbb{C}(f,-) : \ \mathbb{C}(A,-) \Rightarrow \mathbb{C}(B,-)$ for each arrow $B \xrightarrow{f} A$ in a category $\mathbb{C}$. For each $D \in |\mathbb{C}|$, $\mathbb{C}(f,-)_D = \mathbb{C}(f,D) : \mathbb{C}(A,D) \to \mathbb{C}(B,D)$ where $\mathbb{C}(f,D)g = f;g$.

The composition of natural transformations is defined component-wise, i.e., $A(\sigma;\tau) = A\sigma;A\tau$ where $\sigma : \ \mathcal{R} \Rightarrow \mathcal{S} : \ \mathbb{A} \to \mathbb{B}$ and $\tau : \ \mathcal{S} \Rightarrow \mathcal{T} : \ \mathbb{A} \to \mathbb{B}$. This is called the *'vertical' composition* of natural transformations.

Given the natural transformations $\tau : \ \mathcal{S} \Rightarrow \mathcal{T} : \ \mathbb{A} \to \mathbb{B}$ and $\tau' : \ \mathcal{S}' \Rightarrow \mathcal{T}' : \ \mathbb{B} \to \mathbb{C}$

$$\mathbb{A} \underset{\mathcal{T}}{\overset{\mathcal{S}}{\rightrightarrows}} \!\!{\scriptstyle \Downarrow \tau}\; \mathbb{B} \underset{\mathcal{T}'}{\overset{\mathcal{S}'}{\rightrightarrows}} \!\!{\scriptstyle \Downarrow \tau'}\; \mathbb{C}$$

we may define their 'horizontal' composition $\tau\tau' : \ \mathcal{S};\mathcal{S}' \Rightarrow \mathcal{T};\mathcal{T}'$ by

$$A(\tau\tau') = (A\mathcal{S})\tau';(A\tau)\mathcal{T}' = (A\tau)\mathcal{S}';(A\mathcal{T})\tau'.$$

When τ, respectively τ', is an identity natural transformation we may replace it in notation by $\mathcal{S}$, respectively $\mathcal{S}'$. Note also that in the formula above instead of writing $\mathcal{T}(A\tau)$ and $\mathcal{T}A$ we rather wrote $(A\tau)\mathcal{T}'$ and $A\mathcal{T}$, respectively. We did it like that to align the notations to the diagrammatic style which is more convenient in some situations, especially when 'compositions' between natural transformation and functors are involved.

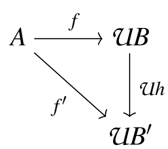
Given two categories $\mathbb{A}$ and $\mathbb{B}$, the *functor category* $\mathbb{C}at(\mathbb{A},\mathbb{B})$ (also denoted $\mathbb{B}^{\mathbb{A}}$) has the functors $\mathbb{A} \to \mathbb{B}$ are objects and the natural transformation between those as arrows.

Other basic categorical constructions

The *opposite* (or *dual*) $\mathbb{C}^{\mathrm{op}}$ of a category $\mathbb{C}$ is obtained just by reversing the arrows and the arrow composition. This means $|\mathbb{C}^{\mathrm{op}}| = |\mathbb{C}|$, $\mathbb{C}^{\mathrm{op}}(A,B) = \mathbb{C}(B,A)$. Identities in $|\mathbb{C}^{\mathrm{op}}|$

are the same as in $\mathbb{C}$.

Given a functor $\mathcal{U}\colon \mathbb{C}' \to \mathbb{C}$, for any object $A \in |\mathbb{C}|$, the *comma category* $A/\mathcal{U}$ has as objects the pairs (f,B) with $f\colon A \to \mathcal{U}B$ and as arrows $(f,B) \to (f',B')$ the $h \in \mathbb{C}'(B,B')$ with $f;\mathcal{U}h = f'$.



When $\mathbb{C} = \mathbb{C}'$ and $\mathcal{U}$ is the identity functor, the category $A/\mathcal{U}$ is denoted by $A/\mathbb{C}$. $\mathbb{C}/A$ is just $(A/\mathbb{C}^{\mathrm{op}})^{\mathrm{op}}$.

Given a class $\mathcal{D} \subseteq \mathbb{C}$ of arrows of a category $\mathbb{C}$ we say that $\mathbb{C}$ is *$\mathcal{D}$-well-powered* when for each object $A \in |\mathbb{C}|$ the isomorphism classes of $\{(B,f) \in |\mathbb{C}/A| \mid f \in \mathcal{D}\}$ form a set (rather than a proper class). Dually, $\mathbb{C}$ is *$\mathcal{D}$-co-well-powered* when for each $A \in |\mathbb{C}|$ the isomorphism classes of $\{(f,B) \in |A/\mathbb{C}| \mid f \in \mathcal{D}\}$ form a set.

2.2 Limits and co-limits

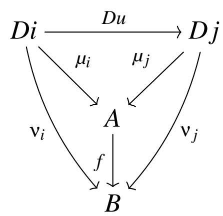
An object 0 is *initial* in a category $\mathbb{C}$ when for each object $A \in |\mathbb{C}|$ there exists a unique arrow in $\mathbb{C}(0,A)$. Dually, an object 1 is *final* in $\mathbb{C}$ when it is initial in $\mathbb{C}^{\mathrm{op}}$, which means that for each object $A \in |\mathbb{C}|$ there exists a unique arrow in $\mathbb{C}(A,1)$.

In $\mathbb{S}et$, the empty set $\emptyset$ is initial and each singleton set $\{*\}$ is final. In $\mathbb{G}rp$, the category of groups, the trivial groups (with only one element) are both initial and final.

Given a functor $\mathcal{U}\colon \mathbb{A} \to \mathbb{X}$, for each $X \in |\mathbb{X}|$, a *universal arrow from X to $\mathcal{U}$* is just an initial object in the comma category $X/\mathcal{U}$. Notice that universal arrows are unique up to isomorphism.

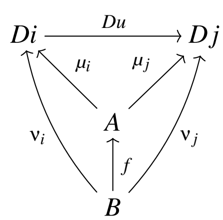
For any categories J and $\mathbb{C}$, the *diagonal* functor $\Delta\colon \mathbb{C} \to \mathbb{C}at(J,\mathbb{C})$ maps any $A \in |\mathbb{C}|$ to the functor $A\Delta\colon J \to \mathbb{C}$ such that $(A\Delta)j = A$ for each object $j \in |J|$ and $(A\Delta)u = 1_A$ for each arrow $u \in J$, and maps any $f \in \mathbb{C}(A,B)$ to the natural transformation $f\Delta\colon A\Delta \Rightarrow B\Delta$ with $(f\Delta)_j = f$ for each $j \in |J|$.

Co-limits. For any functor $D\colon J \to \mathbb{C}$, a *co-cone* to D is just an object of the comma category D/Δ, while a *co-limit* of D is a universal arrow from D to the diagonal functor Δ. As universal arrows, co-limits of functors are unique up to isomorphism. A co-limit $\mu\colon D \Rightarrow A\Delta$ of D may be therefore denoted as $\mu\colon D \Rightarrow A$ (by omitting the diagonal functor from the notation). More explicitly, a co-limit of D consists of an object A and a family of arrows $(\mu_i)_{i\in|J|}$ to A such that $\mu_i = (Du);\mu_j$ for each $u \in J(i,j)$ which behaves like a lowest upper bound for D, i.e., for any family $(\nu_i)_{i\in|J|}$ such that $\nu_i = (Du);\nu_j$ for each $u \in J(i,j)$, there exists a unique arrow f such that $\mu_i; f = \nu_i$ for each $i \in |J|$.



We may denote the vertex A by $Colim(D)$.

Limits. Limits are dual to co-limits. For any functor $D\colon\ J \to \mathbb{C}$, a *limit* $\mu\colon\ A \Rightarrow D$ of D is the 'greatest lower bound' of the *cones* over D, i.e. $\mu = (\mu_i)_{i\in|J|}$ such that $\mu_i;(Du) = \mu_j$ for each $u \in J(i,j)$ and for any family $(\nu_i)_{i\in|J|}$ with the same property, there exists a unique arrow f such that $f;\mu_i = \nu_i$ for each $i \in |J|$.



We may denote the vertex A by $Lim(D)$.

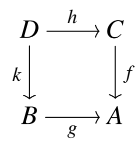
Diagrams as functors. The functors $D\colon\ J \to \mathbb{C}$ for which we have considered limits and co-limits are often called *categorical diagrams* (in $\mathbb{C}$), or just *diagrams* for short. Such a diagram D may be denoted $(Di \xrightarrow{Du} Dj)_{(i \xrightarrow{u} j)\in J}$. Note that the meaning of the functoriality of D, that $D(u;u') = (Du);(Du')$, is the commutativity of D regarded as a diagram in $\mathbb{C}$.

Products and co-products. When J is discrete (has no arrows except the identities), J-limits are called *products* and J-co-limits are called *co-products*; when J is a finite set then the corresponding products or co-products are referred to as finite. Notice that when $J = \emptyset$, then the products are the final objects and the co-products are initial objects.

In $\mathbb{S}et$ the cartesian products are categorical products, while the disjoint unions $A \uplus B$ (defined as $\{(a,1) \mid a \in A\} \cup \{(b,2) \mid b \in B\}$) are co-products.

Pullbacks. When J is the category $\bullet \longrightarrow \bullet \longleftarrow \bullet$ with three objects and two non-identity arrows, J-limits are called *pullbacks*.

In $\mathbb{S}et$, the pullback square



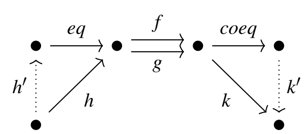
of $C \xrightarrow{f} A \xleftarrow{g} B$ can be defined by $D = \{(b,c) \in B \times C \mid gb = fc\}$, $k(b,c) = b$, and $h(b,c) = c$.

For any arrow f, the pullback of a span $\bullet \xrightarrow{f} \bullet \xleftarrow{f} \bullet$ is called a *kernel of* f. For a function $f : A \to B$, $\{(a,a') \in A \times A \mid fa = fa'\}$ is a kernel.

Pushouts. When J is the category $\bullet \longleftarrow \bullet \longrightarrow \bullet$ with three objects and two non-identity arrows, J-co-limits are called *pushouts*.

In $\mathbb{S}et$, pushouts of any *span* of functions $B \xleftarrow{f} A \xrightarrow{g} C$ always exist. Such pushouts may be given by the quotient of the disjoint union $B \uplus C$ which identifies all the elements fa and ga for each $a \in A$.

Equalizers and co-equalizers. When J is the category with two objects and a pair of parallel arrows between these objects, then J-limits are called *equalizers* and J-co-limits are called *co-equalizers*.



In $\mathbb{S}et$, an equalizer of any pair of parallel arrows $f, g : A \to B$ is just the subset inclusion $\{a \mid fa = ga\} \subseteq A$. A co-equalizer k is the quotient of B by the equivalence generated by $\{(fa, ga) \mid a \in A\}$.

Directed co-limits. When J is a directed partially ordered set (i.e., for each $i, i' \in |J|$ there exists $j \in |J|$ such that $i \leq j$ and $i' \leq j$), then J-co-limits are called *directed* co-limits. For the special case when J is a total order, the J-co-limits are called *inductive* co-limits.

In $\mathbb{S}et$, directed co-limits can be thought of as a generalized kind of union. For any directed diagram of sets $(A_i \xrightarrow{f_{i,j}} A_j)_{(i \leq j) \in (J, \leq)}$ a co-limit is given by the quotient of the disjoint union $\uplus\{A_i \mid i \in |J|\}$ which identifies the elements a_i and $f_{i,j} a_i$.

A category that has all J-(co-)limits is called *J-(co-)complete*. Also, by *small* (co-)-limits we mean all J-(co-)limits for all J that are small categories.

Theorem 2.1. *In any category the following conditions are equivalent:*

1. *the category has finite (co-)limits,*
2. *the category has finite (co-)products and (co-)equalizers, and*
3. *the category has a final (initial) object and pullbacks (pushouts).*

Limits and co-limits in functor categories. Limits and co-limits can be lifted 'point-wise' from the base categories to the functor categories.

Proposition 2.2. *If the category $\mathbb{B}$ has J-(co-)limits, then for any category $\mathbb{A}$, the category $\mathbb{C}at(\mathbb{A},\mathbb{B})$ of functors $\mathbb{A} \to \mathbb{B}$ has small J-(co-)limits (which can be calculated separately in $\mathbb{B}$ for each object $A \in |\mathbb{A}|$).*

Lifting, creation, preservation, reflection of (co-)limits

Let $D: J \to \mathbb{C}$ and $\mathcal{U}: \mathbb{C} \to \mathbb{C}'$ be functors. Then

- $\mathcal{U}$ *preserves* a (co-)limit μ of D when $\mu\mathcal{U}$ is a (co-)limit of $D;\mathcal{U}$. For instance in $\mathbb{S}et$ the 'product with A', $A \times -$, preserves all co-limits.
- $\mathcal{U}$ *lifts (uniquely)* a (co-)limit μ' of $D;\mathcal{U}$, if there exists a (unique) (co-)limit μ of D such that $\mu\mathcal{U} = \mu'$. Notice that if $\mathcal{U}$ lifts J-(co-)limits and $\mathbb{C}'$ has J-(co-)limits, then $\mathbb{C}$ has J-(co-)limits which are preserved by $\mathcal{U}$.
- Slightly stronger than lifting is the following notion. The functor $\mathcal{U}$ *creates* a (co-)limit μ' of $D;\mathcal{U}$, when there exists a unique (co-)cone μ of D such that that $\mu\mathcal{U} = \mu'$ and, furthermore, μ is a (co-)limit. For instance the forgetful functor $\mathbb{G}rp \to \mathbb{S}et$ creates all limits.
- $\mathcal{U}$ *reflects* (co-)limits of D when μ is a (co-)limit of D whenever $\mu\mathcal{U}$ is a (co-)limit of $D;\mathcal{U}$.

Proposition 2.3. *If the functor $\mathcal{U}: \mathbb{C}' \to \mathbb{C}$ preserves J-limits, then for each object $A \in |\mathbb{C}|$, the forgetful functor $A/\mathcal{U} \to \mathbb{C}'$ creates J-limits.*

The dual of Prop. 2.3, for co-limits and with $\mathcal{U}/A \to \mathbb{C}'$, also holds.

Co-limits of final functors

A functor $L: J' \to J$ is called *final* if for each object $j \in |J|$ the comma category j/L is non-empty and connected. Consequently, a subcategory $J' \subseteq J$ is final when the corresponding inclusion functor is final.

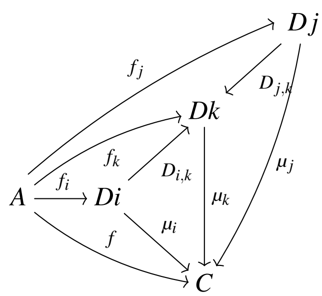
For example, for each natural number n, $(n \to n+1 \to n+2 \to \dots)$ is a final subcategory of $\omega = (0 \to 1 \to 2 \to \dots)$. More generally, for each directed poset $(P,\leq)$ and each $p \in P$, $\{p' \in P \mid p \leq p'\}$ is final in $(P,\leq)$.

Theorem 2.4. *For each final functor $L: J' \to J$ and each functor $D: J \to \mathbb{C}$ when a co-limit $\mu': L;D \Rightarrow Colim(L;D)$ exists, there exists a co-limit $\mu: D \Rightarrow Colim(D)$ and the canonical arrow $h: Colim(L;D) \to Colim(D)$ (given by the universal property of the co-limit of $L;D$) is an isomorphism.*

Finitely presented objects

An object A in a category $\mathbb{C}$ is *finitely presented* if and only if the hom-functor $\mathbb{C}(A,-)$: $\mathbb{C} \to \mathbb{S}et$ preserves directed co-limits. This is equivalent to the following condition:

- for any arrow $f: A \to C$ to the vertex of a co-limiting co-cone $\mu: D \Rightarrow C$ of a directed diagram $D: (J,\leq) \to \mathbb{C}$, there exists $i \in J$ and an arrow $f_i: A \to Di$ such that $f = f_i;\mu_i$, and
- for any two arrows f_i and f_j as above, there exists $k \geq i,j$ such that $f_i;D_{i,k} = f_j;D_{j,k}$.

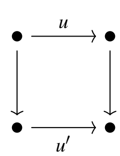


In $\mathbb{S}et$ the finitely presented objects are precisely the finite sets. In the category of groups $\mathbb{G}rp$, the finitely presented groups are exactly the quotients of finitely generated groups by finitely generated congruences.

A category is *locally presentable* when each object is a directed co-limit of finitely presented objects. $\mathbb{S}et$ is locally presentable because each set is the (directed) co-limit of its finite subsets. The concept of a finitely presented object can be considered relative to a particular class $\mathcal{F}$ of directed posets; we call these $\mathcal{F}$*-finitely presented* objects.

Stability under pushouts / pullbacks

A class of arrows $\mathcal{S} \subseteq \mathbb{C}$ in a category $\mathbb{C}$ is *stable under pushouts* if for any pushout square in $\mathbb{C}$



$u' \in \mathcal{S}$ whenever $u \in \mathcal{S}$. *Stability under pullbacks* in $\mathbb{C}$ is stability under pushouts in $\mathbb{C}^{\mathrm{op}}$. When in the definition we replace 'for any' with 'there exists a' we call that *weak stability*.

In general, the epis are stable under pushouts and the monos under pullbacks. In $\mathbb{S}et$, the monos (injective functions) are stable under pushouts too. Injective functions $f: A \to B$ such that $B \setminus fA$ is finite are also stable under pushouts. On the other hand, the set inclusions are only weakly stable under pushouts and pullbacks.

We also say that $\mathcal{S}$ is *(weakly) stable under isomorphisms* when it is (weakly) stable under those pushouts of a span consisting of an arrow from $\mathcal{S}$ and an isomorphism.

Weak limits and co-limits

These are weaker variants of the concepts of limits and co-limits, respectively, obtained by dropping the uniqueness requirement from the universal property of the limits and co-limits, respectively. For example, in $\mathbb{S}et$ for any two nonempty sets A and B, any super-set C of their disjoint union, i.e., $A \uplus B \subseteq C$, is a weak co-product for A and B. Weak limits and co-limits, respectively, are no longer unique up to isomorphism.

2.3 Adjunctions

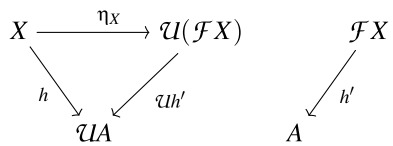
Adjoint functors are a core concept of category theory. Mathematical practice abounds with examples of adjoint functors.

Proposition 2.5. *For any functor $\mathcal{U}\colon\ \mathbb{A} \to \mathbb{X}$ the following conditions are equivalent:*

1. *For each object $X \in \mathbb{X}$ there exists a universal arrow from X to $\mathcal{U}$.*
2. *There exists a functor $\mathcal{F}\colon\ \mathbb{X} \to \mathbb{A}$ and a bijection $\varphi_{X,A}\colon\ \mathbb{A}(\mathcal{F}X, A) \to \mathbb{X}(X, \mathcal{U}A)$ indexed by $|\mathbb{X}| \times |\mathbb{A}|$ and natural in X and A.*
3. *There exists a functor $\mathcal{F}\colon\ \mathbb{X} \to \mathbb{A}$ and natural transformations $\eta\colon\ 1_{\mathbb{X}} \Rightarrow \mathcal{F};\mathcal{U}$ (called the* unit*) and $\varepsilon\colon\ \mathcal{U};\mathcal{F} \Rightarrow 1_{\mathbb{A}}$ (called the* co-unit*) such that the following* triangular equations *hold: $\eta\mathcal{F};\mathcal{F}\varepsilon = 1_{\mathcal{F}}$ and $\mathcal{U}\eta;\varepsilon\mathcal{U} = 1_{\mathcal{U}}$.*

If the conditions above hold, then $\mathcal{U}$ is called a *right adjoint*, and the functor $\mathcal{F}$ is called a *left adjoint* to $\mathcal{U}$. The tuple $(\mathcal{U}, \mathcal{F}, \eta, \varepsilon)$ is called an *adjunction* from (the category) $\mathbb{X}$ to (the category) $\mathbb{A}$.

Very often the notion of adjunction is used in the following "freeness" form. Given an adjunction $(\mathcal{U}, \mathcal{F}, \eta, \varepsilon)$, for any object $X \in |\mathbb{X}|$ there exists an object $\mathcal{F}X$, called *$\mathcal{U}$-free over A* and an arrow $\eta_X\colon\ X \to \mathcal{U}(\mathcal{F}X)$ such that for each object $A \in |\mathbb{A}|$ and arrow $h\colon\ X \to \mathcal{U}A$, there exists a unique arrow $h'\colon\ \mathcal{F}X \to A$ such that $h = \eta_X;\mathcal{U}h'$.



When a category $\mathbb{C}$ has J-(co-)limits, then these are adjoints to the diagonal functor $\Delta\colon\ \mathbb{C} \to \mathbb{C}at(J, \mathbb{C})$. More precisely, *Lim* is a right adjoint to Δ, while *Colim* is a left adjoint to Δ.

The forgetful functor $\mathbb{G}rp \to \mathbb{S}et$ is right adjoint, its left adjoint constructing the groups freely generated by sets.

Galois connections. Let $(P,\leq)$ and $(Q,\leq)$ be preorders. Two preorder preserving functions $L\colon\ (P,\leq)\to(Q,\leq)^{\mathrm{op}}$ and $R\colon\ (Q,\leq)^{\mathrm{op}}\to(P,\leq)$ constitute an adjunction when $Lp\geq q$ if and only if $p\leq Rq$ for all $p\in P$ and $q\in Q$. Notice that triangular equations mean $Lp\geq L(R(Lp))\geq Lp$ and $Rq\leq R(L(Rq))\leq Rq$. The pair (L,R) is called a *Galois connection* between $(P,\leq)$ and $(Q,\leq)$.

Persistent adjunctions. Given an adjunction $(\mathcal{U},\mathcal{F},\eta,\varepsilon)$, the object $\mathcal{F}X$ is called *persistently* $\mathcal{U}$-free when the unit component η_X is an isomorphism, and is called *strongly persistently* $\mathcal{U}$-free when η_X is identity. We can easily see that an object of $\mathbb{A}$ is persistently free if and only if it is strongly persistently free. An adjunction such that for each object X of $\mathbb{X}$, $\mathcal{F}X$ is [strongly] persistently $\mathcal{U}$-free, is called a *[strongly] persistent adjunction* .

Composition of adjunctions. Given two adjunctions $(\mathcal{U},\mathcal{F},\eta,\varepsilon)$ from $\mathbb{X}$ to $\mathbb{A}$, and $(\mathcal{U}',\mathcal{F}',\eta',\varepsilon')$ from $\mathbb{A}$ to $\mathbb{A}'$, note that $(\mathcal{U}';\mathcal{U},\ \mathcal{F};\mathcal{F}',\ \eta;\mathcal{F}\eta'\mathcal{U},\ \mathcal{U}'\varepsilon\mathcal{F}';\varepsilon')$ is an adjunction from $\mathbb{X}$ to $\mathbb{A}'$. This is called the *composition* of the two adjunctions. Adjunctions thus form a 'hyper-category' $\mathbb{A}dj$ with categories as objects and adjunctions as arrows.

The following is one of the most useful properties of adjoint functors.

Proposition 2.6. *Right adjoints preserve all limits and, dually, left adjoints preserve all co-limits.*

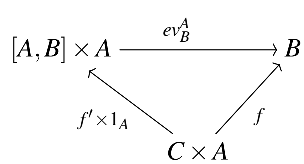
Special adjunctions

Categorical equivalences. The following equivalent conditions define a functor $\mathcal{U}\colon\ \mathbb{X}\to\mathbb{X}'$ as an *equivalence of categories*:

Proposition 2.7. *For any functor $\mathcal{U}\colon\ \mathbb{X}\to\mathbb{X}'$ the following conditions are equivalent:*

- *$\mathcal{U}$ belongs to an adjunction with unit and co-unit being natural isomorphisms, and*
- *$\mathcal{U}$ is full and faithful and each object $A'\in|\mathbb{X}'|$ is isomorphic to $\mathcal{U}A$ for some object $A\in|\mathbb{X}|$.*

Cartesian closed categories. A category $\mathbb{C}$ is *cartesian closed* when it has all finite products, by designation denoted $\_\times\_$, and for each object A the product functor $\_\times A\colon\ \mathbb{C}\to\mathbb{C}$ has a right adjoint $[A,\_]$. If we denote the co-unit of this adjunction by ev^A, it means that for each pair of objects A and B, and for each arrow $f\colon\ C\times A\to B$, there exists a unique arrow $f'\colon\ C\to[A,B]$ such that $f=(f'\times 1_A);ev^A_B$,

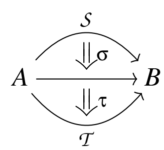


In the examples the co-unit components ev_B^A play the role of 'evaluation maps'. We have that $\mathbb{S}et$ is cartesian closed with $[A,B]$ being the set of all functions $A \to B$, and $ev_B^A(f,a) = fa$. $\mathbb{C}at$ is also cartesian closed with $[A,B]$ being the category $\mathbb{C}at(A,B)$ of the functors $A \to B$ and with the natural transformations between them as arrows.

2.4 2-categories

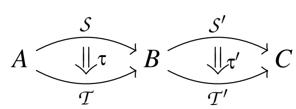
A *2-category* $\mathbb{C}$ is an ordinary category whose objects are called *0-cells*, whose arrows are called *1-cells*, and in addition to ordinary objects and arrows, for each pair of 1-cells $\mathcal{S}, \mathcal{T}$ there is a set $\mathbb{C}(\mathcal{S}, \mathcal{T})$ of *2-cells* (denoted by $\mathcal{S} \Rightarrow \mathcal{T}$) together with two compositions for the 2-cells that are associative and have identities:

- a 'vertical' one $\sigma;\tau : \mathcal{S} \Rightarrow \mathcal{T}$

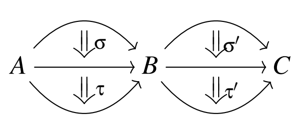


  and

- a 'horizontal' one (denoted by simple juxtaposition) $\tau\tau' : \mathcal{S};\mathcal{S}' \Rightarrow \mathcal{T};\mathcal{T}'$



such that every identity arrow for the first composition is also an identity for the second composition, $1_{\mathcal{S};\mathcal{T}} = 1_{\mathcal{S}}1_{\mathcal{T}}$ for all composable 1-cells $\mathcal{S}$ and $\mathcal{T}$, and such that the following *Interchange Law* holds: given three 0-cells and four 2-cells



the 'vertical' compositions and the 'horizontal' compositions are related by

$$(\sigma;\tau)(\sigma';\tau') = (\sigma\sigma');(\tau\tau').$$

Any category is trivially a 2-category without proper 2-cells. The typical non-trivial example of a 2-category is $\mathbb{C}at$ with categories as 0-cells, functors as 1-cells, and natural transformations as 2-cells.

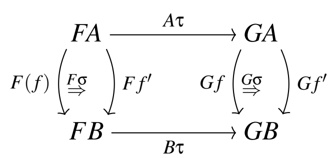
Adjunctions, natural transformations, (co-)limits

The concept of adjunction can be defined abstractly in any 2-category: $(\mathcal{U},\mathcal{F},\eta,\varepsilon)$ is an *adjunction* if $\mathcal{U}\colon A\to X$ and $\mathcal{F}\colon X\to A$ are 1-cells, $\eta\colon 1_X\Rightarrow\mathcal{F};\mathcal{U}$ and $\varepsilon\colon \mathcal{U};\mathcal{F}\Rightarrow 1_{\mathbb{A}}$ are 2-cells such that the *triangular equations* are satisfied:

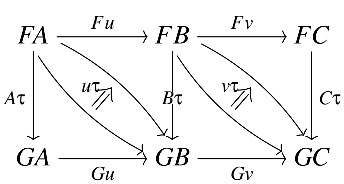
$$\eta\mathcal{F};\mathcal{F}\varepsilon = 1_{\mathcal{F}} \text{ and } \mathcal{U}\eta;\varepsilon\mathcal{U} = 1_{\mathcal{U}}.$$

The proper mappings between 2-categories are 2-functors. A *2-functor* $F\colon \mathbb{C}\to\mathbb{C}'$ between 2-categories $\mathbb{C}$ and $\mathbb{C}'$ maps 0-cells to 0-cells, 1-cells to 1-cells, and 2-cells to 2-cells, such that $F\mathcal{S}\colon FA\to FB$ for any 1-cell $\mathcal{S}\colon A\to B$, and $F\sigma\colon F\mathcal{S}\Rightarrow F\mathcal{T}$ for any 2-cell $\sigma\colon \mathcal{S}\Rightarrow\mathcal{T}$, and such that it preserves both the 'vertical' and the 'horizontal' compositions as well as the identity cells.

A *2-natural transformation* $\tau\colon F\Rightarrow G$ between 2-functors $F,G\colon \mathbb{A}\to\mathbb{B}$ maps any object A of $|\mathbb{A}|$ to a 1-cell $A\tau\colon FA\to GA$ such that $(A\tau)(G\sigma)=(F\sigma)(B\tau)$ for each 2-cell $\sigma\colon f\Rightarrow f'\colon A\to B$.



Lax natural transformations relax the commutativity of the natural transformation square above to the existence of 2-cells. Therefore a lax natural transformation τ between 2-functors F and G maps any object $A\in|\mathbb{A}|$ to $A\tau\colon FA\to GA$ and any 1-cell $u\colon A\to B$ to $u\tau\colon A\tau;(Gu)\Rightarrow(Fu);B\tau$ such that $((F\sigma)(B\tau));f'\tau = f\tau;((A\tau)(G\sigma))$ for each 2-cell $\sigma\colon f\Rightarrow f'\colon A\to B$ and



$(u;v)\tau = (u\tau)(Gv);(Fu)(v\tau)$ for each $u\colon A\to B$ and $v\colon B\to C$.

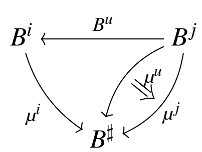
2-categorical limits and co-limits can be defined similarly to the conventional limits and co-limits as universal arrows from/to a diagonal functor. However, in the 2-categorical framework, different concepts of natural transformations determine different concepts of (co-)limits. Therefore, when we employ 2-natural transformations we get the concepts of *2-(co-)limit* as a final (initial) *2-(co-)cone*, and when we employ lax natural transformations we get the concepts of *lax (co-)limit* as a final/initial *lax cone / co-cone*.

2.5 Indexed categories and fibrations

An *indexed category* is a functor $B\colon I^{\mathrm{op}}\to\mathbb{C}at$; sometimes we denote $B(i)$ as B_i (or B^i) for an index $i\in|I|$ and $B(u)$ as B^u for an index morphism $u\in I$. Given an indexed category $B: I^{\mathrm{op}} \to \mathbb{C}at$, let the category d $B^\sharp$ – called the *Grothendieck construction* – have $\langle i, \Sigma\rangle$, with $i \in |I|$ and $\Sigma \in |B^i|$, as objects and $\langle u, \varphi\rangle: \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$, with $u \in I(i,i')$ and $\varphi: \Sigma \to B^u\Sigma'$, as arrows. The composition of arrows in $B^\sharp$ is defined by $\langle u, \varphi\rangle; \langle u', \varphi'\rangle = \langle u;u', \varphi;(B^u\varphi')\rangle$.

Proposition 2.8. *The Grothendieck construction $B^\sharp$ of an indexed category $B: I^{\mathrm{op}} \to \mathbb{C}at$ is the vertex of the lax co-limit $\mu: B \rightsquigarrow B^\sharp$ of B in $\mathbb{C}at$, where*

- *for each index $i \in |I|$, $\mu^i: B^i \to B^\sharp$ is the canonical inclusion of categories, and*
- *for each index morphism $u \in I(i,j)$, $\mu^u: B^u;\mu^i \Rightarrow \mu^j$ is defined by $\mu^u_b = \langle u, 1_{B^u b}\rangle$ for each object $b \in |B^j|$.*



Grothendieck constructions in 2-categories. Prop. 2.8 allows us to internalize the concept of Grothendieck construction to any 2-category. Given a (1-)functor $B: I^{\mathrm{op}} \to V$, where V is an arbitrary 2-category, a *Grothendieck construction* for B is a lax co-limit $\mu: B \rightsquigarrow B^\sharp$. Then the vertex $B^\sharp$ is called the *Grothendieck object* associated to B. We say that a 2-category V *admits Grothendieck constructions* when each (1-)functor $B: I^{\mathrm{op}} \to V$ has a lax co-limit.

Notice also that any 2-functor $B: I^* \to \mathbb{C}at$, where I^* is the *2-dimensional opposite* changing the direction of 2-cells both horizontally and vertically, induces a canonical 2-category structure on the Grothendieck construction $B^\sharp$ of the (1)-functor $B: I^{\mathrm{op}} \to \mathbb{C}at$.

Fibrations

Given a functor $p: \mathbb{B} \to I$, an object / arrow $f \in \mathbb{B}$ is said to be *above* an object/arrow $u \in I$ when $pf = u$. An arrow above an identity is called *vertical*. Every object $i \in |I|$ determines a *fibre* category $\mathbb{B}_i$ consisting of objects above i and vertical morphisms above 1_i. An arrow $f \in \mathbb{B}(A,C)$ is called *cartesian* over an arrow $u \in I$ when f is above u and every $f' \in \mathbb{B}(A',C)$ with $pf' = v;u$ uniquely determines a $g \in \mathbb{B}(A',A)$ above v such that $f' = g;f$. p is called a *fibred category* or *fibration* when for every $A \in |\mathbb{B}|$ and $u \in I(i,pA)$ there is a cartesian arrow (called *cartesian lifting* or *critical lifts* in [1]) with codomain A above u.

Each indexed category $B: I^{\mathrm{op}} \to \mathbb{C}at$ naturally determines a fibration $p: B^\sharp \to I$ as the index projection, i.e., $p\langle i, \Sigma\rangle = i$, such that for each index i, the fibre $B^\sharp_i$ is B^i and $\langle u, \varphi\rangle \in B^\sharp$ is cartesian over u when φ is isomorphism. Notice that for each index morphism $u: i \to i'$ and $\langle i', \Sigma'\rangle \in B^\sharp$, $\langle u, 1_{B^u\Sigma'}\rangle: \langle i, B^u\Sigma'\rangle \to \langle i', \Sigma'\rangle$ is a cartesian lifting of u with codomain $\langle i', \Sigma'\rangle$.

Conversely, if $p: \mathbb{B} \to I$ is a fibration, for each $u \in I(i,i')$ and $A \in \mathbb{B}_{i'}$, we chose a cartesian lifting $\overline{u}: u^*A \to A$ (called the *distinguished cartesian morphism* corresponding to u and A). Such choice determines a functor $u^* : \mathbb{B}_{i'} \to \mathbb{B}_i$ called an *inverse image* functor. Notice that two inverse image functors corresponding to the same u are naturally isomorphic, $(u;v)^* \cong v^*;u^*$ for each $u,v \in I$, and $(1_i)^* \cong 1_{\mathbb{B}_i}$ for each $i \in |I|$. When these natural isomorphisms are identities we say that the fibration is *split*.

Proposition 2.9. *The fibred category given by the forgetful functor from a Grothendieck construction to its category of indices is split. Conversely, each split fibration is a Grothendieck construction and each fibration is equivalent to a Grothendieck construction.*

Cartesian functors are 'morphisms of fibrations'. Given fibrations $p : \mathbb{B} \to I$ and $p' : \mathbb{B}' \to I$, a cartesian functor $U : \mathbb{B} \to \mathbb{B}'$ commutes with the fibrations, i.e., $U;p' = p$, and preserves the cartesian arrows, i.e., maps any cartesian arrow for p to a cartesian arrow for p'.

Limits / co-limits in Grothendieck constructions / fibred categories can be obtained from (co-)limits in the 'local' categories or fibres.

Theorem 2.10. *Given an indexed category $B : I^{\mathrm{op}} \to \mathbb{C}at$, then for each category J the Grothendieck construction $B^\sharp$ has*

- *J-limits when I has J-limits, B^i has J-limits for each index i, and B^u preserves J-limits for each index morphism u, and*
- *J-co-limits when I has J-co-limits, B^i has J-co-limits for each index i, and B^u has a left adjoint for each index morphism u.*

Chapter 3

Institutions

In this chapter, we first give a model-theoretic presentation of classical first-order logic with equality considered in an extended form and develop some of its structural properties that make it an institution. We then introduce the abstract concept of institution and illustrate it by a list of other examples from logic and computing science. The next section introduces morphisms and comorphisms of institutions, which are mappings preserving the structure of an institution with rather complementary meaning in actual situations. The final section of this chapter, which is intended for the more category-theoretic-minded readers, provides a more advanced categorical definition for the concept of institution, which eases considerably our access to the structural properties of categories of institutions. As an application, we prove the existence of limits of institutions.

3.1 From concrete logic to institutions

Perhaps the most representative concrete logic system is the first-order logic. This is the area in which many of the broader ideas of model theory were first worked out. Here we present it in its many-sorted variant and in a particularly structured way which will serve our goal of capturing it as an institution.

Many-sorted first order logic with equality ($\mathcal{FOL}$)

Signatures. In many logic or model theory texts these are called 'languages'. In this book, we use the algebraic specification terminology. For any set S let S^* denote the set of the strings formed with elements of S. The empty string is denoted by $[]$. A (many-sorted) *signature* in $\mathcal{FOL}$ is a tuple (S,F,P) where

- S is the set of *sort* symbols,
- $F = (F_{w\to s})_{w\in S^*,s\in S}$ is a family of sets of (S-sorted) *operation* symbols such that $F_{w\to s}$ denotes the set of operations with *arity* w and *sort* s (in particular, when the arity w is empty, $F_{[]\to s}$ (also denoted $F_{\to s}$) represents the set of *constants* of sort s), and

- $P = (P_w)_{w \in S^*}$ is a family of sets of (S-sorted) relation symbols where P_w denotes the set of relations with *arity* w.

We may sometimes omit the word 'symbol' and simply refer to sort symbols as sorts, to operation symbols as operations, and to relation symbols as relations. That a symbol of operation σ belongs to some $F_{w \to s}$ for some arity w and some sort s may be imprecisely but compactly denoted by '$\sigma \in F$'. The same may be of course applied to the relation symbols. When P is empty, then we may write (S,F) rather than $(S,F,\emptyset)$ and we call this an *algebraic signature*.

The fact that the sets $F_{w \to s}$ (or P_w) are not required to be disjoint reflect the possibility of the so-called *overloading* of symbols. A simple example is given by the following choice for a signature (S,F,P) for specifying natural and integer numbers:

- $S = \{\mathsf{N}, \mathsf{Z}\}$ (with N denoting the natural numbers and Z the integers),
- $F_{\mathsf{NN} \to \mathsf{N}} = \{+\}$, $F_{\mathsf{ZZ} \to \mathsf{Z}} = \{+,-\}$, $F_{\mathsf{Z} \to \mathsf{Z}} = F_{\mathsf{NN} \to \mathsf{Z}} = \{-\}$, and $F_{w \to s} = \emptyset$ otherwise,
- $P_{\mathsf{NN}} = P_{\mathsf{ZZ}} = \{\leq\}$ and $P_w = \emptyset$ otherwise.

This example owes much to the algebraic specification tradition and style. A notorious example from mainstream mathematics is that of the two-sorted signature of the vector spaces, with one sort for the scalars and another one for the vectors. This involves also overloading of operation symbols, for instance $\_+\_$ may be used both for the scalars and for the vectors. But unlike in the previous example, these two additions do not overlap.

Models. Given a $\mathcal{FOL}$ signature (S,F,P), a *model* M interprets:

- each sort symbol s as a set M_s, called the *carrier set of sort* s,
- each operation symbol $\sigma \in F_{w \to s}$ as a function $M_{\sigma : w \to s} : M_w \to M_s$, where M_w stands for $M_{s_1} \times \cdots \times M_{s_n}$ for $w = s_1 \dots s_n$ with $s_1, \dots, s_n \in S$, and
- each relation symbol $\pi \in P_w$ as a subset $M_{\pi : w} \subseteq M_w$.

The models of algebraic signatures are called *algebras*.

To simplify notation we will often write M_σ instead of $M_{\sigma : w \to s}$ and M_π instead of $M_{\pi : w}$.

An (S,F,P)-*model homomorphism* $h : M \to M'$ is an indexed family of functions $(h_s : M_s \to M'_s)_{s \in S}$ such that

- h is an (S,F)-algebra homomorphism $M \to M'$, i.e., $h_s(M_\sigma m) = M'_\sigma(h_w m)$ for each $\sigma \in F_{w \to s}$ and each $m \in M_w$,[1]

$$\begin{array}{ccc} M_w & \xrightarrow{M_\sigma} & M_s \\ {\scriptstyle h_w}\downarrow & & \downarrow{\scriptstyle h_s} \\ M'_w & \xrightarrow[M'_\sigma]{} & M'_s \end{array}$$

[1] $h_w : M_w \to M'_w$ is the canonical component-wise extension of h, i.e., $h_w(m_1, \dots, m_n) = (h_{s_1} m_1, \dots, h_{s_n} m_n)$ where $w = s_1 \dots s_n$ and $m_i \in M_{s_i}$.

and

- $h_w m \in M'_\pi$ if $m \in M_\pi$ (i.e. $h_w M_\pi \subseteq M'_\pi$) for each relation $\pi \in P_w$ and each $m \in M_w$.

Fact 3.1. *For any signature (S,F,P), the (S,F,P)-model homomorphisms form a category under the obvious composition (component-wise as many-sorted functions). The category of (S,F,P)-models is denoted by $Mod(S,F,P)$.*

Sentences. An (S,F)-term t of sort s is a syntactic structure $\sigma(t_1 \dots t_n)$ where $\sigma \in F_{s_1 \dots s_n \to s}$ is an operation symbol and $t_1, \dots, t_n$ are (S,F)-terms of sorts $s_1, \dots, s_n$. By $T_{(S,F)}$ let us denote the set of (S,F)-terms. This is of course a definition by induction the base case being when the σ's are constants.

Given a signature (S,F,P), the set of (S,F,P)-*sentences* is the least set containing the (quantifier-free) atoms and which is closed under Boolean / propositional connectives and quantification as follows:

- An *equation* is an equality $t = t'$ between (S,F)-terms t and t' of the same sort. A *relational atom* is an expression $\pi(t_1, \dots, t_n)$ where $\pi \in P$ and $(t_1, \dots, t_n) \in (T_{(S,F)})_w$ is any list of (S,F)-terms for the arity w of π (i.e., $w = s_1 \dots s_n$ where s_k is the sort of t_k for $1 \leq k \leq n$). An atom is either an equation or a relational atom.
- For ρ_1 and ρ_2 any (S,F,P)-sentences, let $\rho_1 \wedge \rho_2$ be their conjunction which is also an (S,F,P)-sentence. Other Boolean / propositional connectives are the disjunction $(\rho_1 \vee \rho_2)$, implication $(\rho_1 \Rightarrow \rho_2)$, negation $(\neg\rho)$, and equivalence $(\rho_1 \Leftrightarrow \rho_2)$.
- Any finite block X of *variables* for a signature (S,F,P) can be added to the signature as new constants. The formal definition of a *variable for* (S,F,P) is that of a triple $(x,s,(S,F,P))$ where x is the *name of the variable*, which is a natural number, and $s \in S$ is the *sort of the variable*. 'Block' here means that X is a set of variables such that any two different variables in X have different names. Let us denote by $(S,F+X,P)$ the extension of (S,F,P) with X as new constants, where a variable of sort s is a new constant of sort s. Then $(\forall X)\rho$ and $(\exists X)\rho$ are (S,F,P)-sentences for each $(S,F+X,P)$-sentence ρ.

The $\mathcal{FOL}$ concept of variable may seem unnecessarily complex when compared with how variables are considered in traditional logic. This is a cost of having first-order logic specified as a mathematical object (i.e., institution). Let us note the following aspects:

– The qualification of the variables by their signature context guarantees automatically, by a simple set-theoretic argument, that when added as new constants to the signature they indeed do not clash with the already existing constants.

– As the set of the names of the variables (i.e., ω) is countable, for each signature Σ, $Sen\Sigma$ is always a set (and never a proper class). Our choice of ω is rather arbitrary, any infinite set instead of ω would do the job. In order to have a precise definition of $\mathcal{FOL}$, a specific choice for the set of names of the variables is necessary and perhaps ω is the most notorious infinite set.

– The complexity of the $\mathcal{FOL}$ variables does not prevent us to write sentences in the usual simple manner, such as $(\forall x)x + (-x) = 0$, because x *denotes* a $\mathcal{FOL}$ variable which is different from saying that x *is* a variable. Let us apply this convention to any concrete institution that has quantifications.

Signature morphisms. A *signature morphism* $\varphi = (\varphi^{\mathrm{st}}, \varphi^{\mathrm{op}}, \varphi^{\mathrm{rl}}) : (S, F, P) \to (S', F', P')$ consists of

- a function between the sets of sorts $\varphi^{\mathrm{st}} : S \to S'$,
- a family of functions between the sets of operation symbols $\varphi^{\mathrm{op}} = (\varphi^{\mathrm{op}}_{w\to s} : F_{w\to s} \to F'_{\varphi^{\mathrm{st}}w\to\varphi^{\mathrm{st}}s})_{w\in S^*, s\in S}$,[2] and
- a family of functions between the sets of relation symbols $\varphi^{\mathrm{rl}} = (\varphi^{\mathrm{rl}}_w : P_w \to P'_{\varphi^{\mathrm{st}}w})_{w\in S^*}$.

Almost always the authors of model theory textbooks prefer to introduce first-order logic in a single sorted variant. However a more substantial difference between our definition of first order logic – which owes to the algebraic specification tradition – and the conventional ones occurs at the concept of signature morphism. In conventional model theory, this is a rather salient concept which corresponds to φ being an inclusion (more precisely all components of φ, i.e., φ^{st}, $\varphi^{\mathrm{op}}_{w\to s}$, φ^{rl}_w, are inclusions). But the practice of algebraic specification requires our extended concept of signature morphism and in this book we will also see how this conceptual extension brings in many new and interesting perspectives to first-order logic.

Given signature morphisms $\varphi : (S, F, P) \to (S', F', P')$ and $\theta : (S', F', P') \to (S'', F'', P'')$ their *composition* $\varphi ; \theta : (S, F, P) \to (S'', F'', P'')$ is defined component-wise as follows:

- $(\varphi ; \theta)^{\mathrm{st}} = \varphi^{\mathrm{st}} ; \theta^{\mathrm{st}}$;
- for each $w \in S^*,\ s \in S$, the definition of $(\varphi ; \theta)^{\mathrm{op}}_{w\to s}$ is shown in the following diagram

$$F_{w\to s} \xrightarrow{\varphi^{\mathrm{op}}_{w\to s}} F'_{\varphi^{\mathrm{st}}w\to\varphi^{\mathrm{st}}s} \xrightarrow{\theta^{\mathrm{op}}_{\varphi^{\mathrm{st}}w\to\varphi^{\mathrm{st}}s}} F''_{\theta^{\mathrm{st}}(\varphi^{\mathrm{st}}w)\to\theta^{\mathrm{st}}(\varphi^{\mathrm{st}}s)}, \qquad F_{w\to s} \xrightarrow{(\varphi;\theta)^{\mathrm{op}}_{w\to s}} F''_{\theta^{\mathrm{st}}(\varphi^{\mathrm{st}}w)\to\theta^{\mathrm{st}}(\varphi^{\mathrm{st}}s)}$$

- and likewise for the relation symbols.

Fact 3.2. *The signature morphisms considered with the composition defined as above form a category denoted Sig.*

Notations such as $\varphi^{\mathrm{st}}s$, $\varphi^{\mathrm{op}}_{w\to s}\sigma$ or $\varphi^{\mathrm{rl}}_w\pi$ are quite heavy but when there is no danger of ambiguity we may simplify them to φs, $\varphi\sigma$ or $\varphi\pi$, respectively.

[2] Here $(\_)^{\mathrm{op}}$ should not be confused with the similar notation for the opposite of a category; also for any string of sorts $w = s_1 \ldots s_n$, by $\varphi^{\mathrm{st}}w$ we mean the string of sorts $\varphi^{\mathrm{st}}s_1 \ldots \varphi^{\mathrm{st}}s_n$.

Model reducts. Given a signature morphism $\varphi\colon (S,F,P) \to (S',F',P')$, the φ-*reduct* $M'\restriction_\varphi$ of a (S',F',P')-model M' is the (S,F,P)-model defined as follows:

- $(M'\restriction_\varphi)_s = M'_{\varphi^{\mathrm{st}} s}$ for each sort $s \in S$,
- $(M'\restriction_\varphi)_\sigma = M'_{\varphi^{\mathrm{op}}_{w\to s}\sigma}$ for each operation symbol $\sigma \in F_{w\to s}$, and
- $(M'\restriction_\varphi)_\pi = M'_{\varphi^{\mathrm{rl}}_w \pi}$ for each relation symbol $\pi \in P_w$.

Conversely, if M is the φ-reduct of M' then M' is called a φ-*expansion of* M. Moreover, when φ is an inclusion we may also write $M'\restriction_{(S,F,P)}$ instead of $M'\restriction_\varphi$.

The *reduct* $h'\restriction_\varphi$ *of a model homomorphism* is also defined by $(h'\restriction_\varphi)_s = h'_{\varphi s}$ for each sort $s \in S$.

Fact 3.3. *For each signature morphism* $\varphi\colon (S,F,P) \to (S',F',P')$*, the model reduct* $\_\restriction_\varphi$ *is a functor* $Mod(S',F',P') \to Mod(S,F,P)$*. Moreover, Mod becomes a functor* $Sig^{\mathrm{op}} \to \mathbb{C}at$ *when for each signature morphism* φ *we denote* $M\restriction_\varphi$ *by* $(Mod\ \varphi)M$*.*

Sentence translations. Given a signature morphism φ, the *sentence translation* $Sen\ \varphi\colon\ Sen(S,F,P) \to Sen(S',F',P')$ along φ is defined inductively on the structure of the sentences by replacing the symbols from (S,F,P) with symbols from (S',F',P') as defined by φ. At the level of terms, this defines a function $T_{(S,F)} \to T_{(S',F')}$ which we may denote by φ^{tm}, or simply by φ. This can be formally defined by

$$\varphi^{\mathrm{tm}}\ \sigma(t_1,\dots,t_n) = (\varphi^{\mathrm{op}}\ \sigma)(\varphi^{\mathrm{tm}}\ t_1,\dots,\varphi^{\mathrm{tm}}\ t_n).$$

Then

- $(Sen\ \varphi)(t = t') = (\varphi^{\mathrm{tm}}\ t = \varphi^{\mathrm{tm}}\ t')$ for equations,
- $(Sen\ \varphi)\pi(t_1,\dots,t_n) = (\varphi^{\mathrm{rl}}\ \pi)(\varphi^{\mathrm{tm}}\ t_1,\dots,\varphi^{\mathrm{tm}}\ t_n)$ for relational atoms,
- $(Sen\ \varphi)(\rho_1 \wedge \rho_2) = (Sen\ \varphi)\rho_1 \wedge (Sen\ \varphi)\rho_2$ and similarly for all other Boolean connectives, and
- $(Sen\ \varphi)\ (\forall X)\rho = (\forall X^\varphi)(Sen\ \varphi')\rho$ for each finite block of variables X, each $(S,F+X,P)$-sentence ρ, and where

  $$X^\varphi = \{(x,\varphi^{\mathrm{st}}s,(S',F',P')) \mid (x,s,(S,F,P)) \in X\},$$

  and where $\varphi'\colon\ (S,F+X,P) \to (S',F'+X^\varphi,P')$ extends φ canonically by mapping each variable $(x,s,(S,F,P)) \in X$ to $(x,\varphi^{\mathrm{st}}s,(S',F',P'))$.

Fact 3.4. *Sen is a functor* $Sig \to \mathbb{S}et$*.*

The proof of this Fact 3.4 consists of a straightforward check by induction on the structure of sentences, with the only interesting case being the induction step corresponding to quantifiers. So we may skip this proof.

Satisfaction. The satisfaction relation between models and sentences represents the core concept of model theory. In philosophy, it is known as the 'semantic concept of truth'. It is denoted by the symbol '$\models$' which represents a true logo of model theory in the sense that its occurrence indicates clearly that we are in the presence of a model-theoretic argument. It is perhaps rather difficult to find another mathematical area that can be identified so clearly by a single mathematical symbol. Let us recall the satisfaction relation of first-order logic.

First let us note that each term $t = \sigma(t_1,\dots,t_n)$ of sort s gets interpreted by any (S,F,P) model M as an element $M_t \in M_s$ defined by

$$M_t = M_\sigma(M_{t_1},\dots,M_{t_n}).$$

The *satisfaction* between models and sentences is the Tarskian satisfaction defined inductively on the structure of sentences. Given a fixed arbitrary signature (S,F,P),

- for equations: $M \models t = t'$ if $M_t = M_{t'}$,
- for relational atoms $M \models \pi(t_1,\dots,t_n)$ if $(M_{t_1},\dots,M_{t_n}) \in M_\pi$,
- $M \models \rho_1 \wedge \rho_2$ if and only if $M \models \rho_1$ *and* $M \models \rho_2$,
- $M \models \neg\rho$ if and only if $M \not\models \rho$,
- $M \models \rho_1 \vee \rho_2$ if and only if $M \models \rho_1$ *or* $M \models \rho_2$,
- $M \models \rho_1 \Rightarrow \rho_2$ if and only if $M \models \rho_2$ whenever $M \models \rho_1$,
- $M \models (\forall X)\rho$ if $M' \models \rho$ for each expansion M' of M along the signature inclusion $(S,F,P) \hookrightarrow (S,F+X,P)$, and
- $M \models (\exists X)\rho$ if and only if $M \models \neg(\forall X)\neg\rho$.

The result below shows that, in first-order logic, satisfaction is an invariant with respect to changes of signatures.

Proposition 3.5. *For any signature morphism* $\varphi\colon (S,F,P) \to (S',F',P')$*, any* (S',F',P')*-model* M'*, and any* (S,F,P)*-sentence* ρ*,*

$$M'\!\restriction_\varphi \models \rho \;\; \textit{if and only if} \;\; M' \models (Sen\,\varphi)\rho.$$

Proof. We prove this by induction on the structure of the sentences. First notice that, by induction on the structure of terms, we get that for any (S,F)-term t, $M'_{\varphi^{\mathrm{tm}}\,t} = (M'\!\restriction_\varphi)_t$. The Satisfaction Condition for atoms follows immediately, while the preservation of the Satisfaction Condition by Boolean connectives can also be checked very easily.

Now we show that the Satisfaction Condition is preserved by quantification too, and this is the only interesting part of this proof. Universal quantification would be enough since the semantics of existential quantification has been defined in terms of the semantics of negation and of universal quantification. Consider an (S,F,P)-sentence $(\forall X)\rho$ and an (S',F',P')-model M'.

$$\begin{array}{ccc} (S,F,P) & \xrightarrow{\varphi} & (S',F',P') \\ \subseteq \downarrow & & \downarrow \subseteq \\ (S,F+X,P) & \xrightarrow[\varphi']{} & (S',F'+X^{\varphi},P') \end{array}$$

Consider the following canonical bijection

$$\begin{array}{ll} I_{\varphi,X,M'}: & \{M'' \in |Mod(S',F'+X^{\varphi},P')| \mid M''\restriction_{(S',F',P')} = M'\} \to \\ & \{N \in |Mod(S,F+X,P)| \mid N\restriction_{(S,F,P)} = M'\restriction_{\varphi}\} \end{array}$$

defined for each variable $(x,s,(S,F,P))$ in X by

$$(I_{\varphi,X,M'}\; M'')_{(x,s,(S,F,P))} = M'_{(x,\varphi^{\mathrm{st}} s,(S',F',P'))}.$$

Then we have the following succession of equivalent relations:

$M' \models (Sen\ \varphi)(\forall X)\rho$	
$M' \models (\forall X^{\varphi})(Sen\ \varphi')\rho$	definition of $Sen\ \varphi'$
$M'' \models (Sen\ \varphi')\rho$ for each expansion M'' of M'	definition of the satisfaction relation
$M''\restriction_{\varphi'} \models \rho$ for each expansion M'' of M'	induction hypothesis
$N \models \rho$ for each expansion N of $M'\restriction_{\varphi}$	by $I_{\varphi,X,M'}$
$M'\restriction_{\varphi} \models (\forall X)\rho$	definition of satisfaction relation.

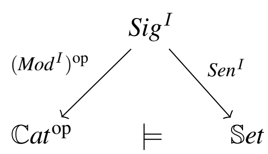
$\square$

Non-empty sorts. Some results rely on models involved not interpreting sorts as empty sets. The best way to guarantee this is at the level of signatures. If we want that a certain sort s gets necessarily interpreted as a non-empty set it suffices to have a term of sort s. Such sorts are called *non-empty sorts*, otherwise are called *empty sorts*. Of course, we should not be misled by this terminology, an empty sort does not necessarily have to be interpreted as an empty set, but it might be. On the other hand, a non-empty sort is always necessarily interpreted as a non-empty set. This concept can be extended to other model theories rather than first order model theory, whenever we have sorts and operations like in $\mathcal{FOL}$ we can talk about empty / non-empty sorts.

Institutions

We can understand that $\mathcal{FOL}$ represents an aggregation of many complex structures and definitions, also involving complex notations. One of the benefits of the concept of institution is that we can get rid off all these complexities, we need not live with them even when investigating properties of $\mathcal{FOL}$. This is achieved by abstracting the concrete structures, and consequently considering the properties of these structures as axioms. The result of this abstraction is as follows. An *institution* $I = (Sig^I, Sen^I, Mod^I, \models^I)$ consists of

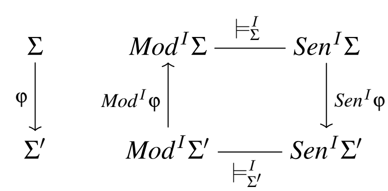
1. a category Sig^I, whose objects are called *signatures*,
2. a functor $Sen^I : Sig^I \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
3. a functor $Mod^I : (Sig^I)^{\mathrm{op}} \to \mathbb{C}at$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-(*model*) *homomorphisms*, and
4. a relation $\models_\Sigma \subseteq |Mod^I\Sigma| \times (Sen^I\Sigma)$ for each $\Sigma \in |Sig^I|$, called Σ-*satisfaction*,



such that for each morphism $\varphi : \Sigma \to \Sigma'$ in Sig^I, the *Satisfaction Condition*

$$M' \models_{\Sigma'} (Sen^I\varphi)e \text{ if and only if } (Mod^I\varphi)M' \models_\Sigma e$$

holds for each $M' \in |Mod^I\Sigma'|$ and $e \in Sen^I\Sigma$. The Satisfaction Condition can be graphically represented by the following commutative diagram:



The meaning of the Satisfaction Condition of institutions is that

> Truth is invariant under change of notation (and under extension of the context).

We may denote the reduct functor $Mod^I\varphi$ by $\_\restriction_\varphi$ and the sentence translation $Sen^I\varphi$ simply by $\varphi(-)$. When $M = M'\restriction_\varphi$ we say that M is a φ-*reduct* of M and that M' is an φ-*expansion of* M. When φ is clear (such as an inclusion), we may even write $M\restriction_\Sigma$ rather than $M\restriction_\varphi$. Also, when there is no danger of ambiguity, we may skip the superscripts from the notations of the entities of the institution; for example Sig^I may be simply denoted Sig.

Facts 3.1, 3.3, 3.4 together with Prop. 3.2 can now be formulated as follows.

Corollary 3.6. *$\mathcal{FOL}$ is an institution.*

Closure under isomorphisms. In this book, we assume that all institutions are *closed under isomorphisms*, meaning that the satisfaction relation is invariant with respect to model isomorphisms, i.e., for any isomorphic Σ-models $M \cong N$ and for any Σ-sentence ρ,

$$M \models_\Sigma \rho \text{ if and only if } N \models_\Sigma \rho.$$

Although this is a very natural property from a model-theoretic perspective, it evidently should not be expected in general at the level of abstract institutions.

Semantic consequence. The satisfaction relation between models and sentences determines a *semantic consequence* relation between sets of sentences: for E and E' sets of Σ-sentences, $E \models_\Sigma E'$ if and only if each Σ-model satisfying all sentences in E also satisfies all sentences in E'. The following list of properties of semantic consequence that hold in any institution is straightforward to establish but they are important as they represent the axioms defining the abstract concept of consequence.

Proposition 3.7. *Let I be any institution, Σ be any of its signatures, and E, E', E'' be any sets of Σ-sentences. Then*

$E \models E'$	reflexivity
$E' \models E$ *when* $E \subseteq E'$	monotonicity
$E \models E' \cup E''$ *when* $E \models E'$ *and* $E \models E''$	union
$E \models E''$ *when* $E \models E'$ *and* $E' \models E''$	transitivity.
If $\varphi : \Sigma \to \Sigma'$ *is signature morphism and* $\varphi E \models_{\Sigma'} \varphi E'$ *then* $E \models_\Sigma E'$	translation.

If φ enjoys the reverse of the 'translation' property then we say that φ is *conservative*. This is not a general property of signature morphisms, like 'translation' is, but it is an important property to have in situations when by translating sentences we should not be able to deduce more.

Exercises

3.1. Give an example of an institution that is *not* closed under isomorphisms.

3.2 Examples of institutions

This section is devoted to examples of institutions other than $\mathcal{FOL}$. All of them represent some concrete logical systems that occur in logic or computing science. The reader is invited to complete the missing details, including a proof of the Satisfaction Condition for each of the examples presented (that in general mimics the proof of Prop.), as these constitute the first set of relevant exercises in this book. We are also already aware that the very abstract nature of the concept of institution allows for interesting examples that do not have a 'logical' nature like the examples presented below. We will see some of that in the proposed exercises.

Sub-institutions. Many examples of institutions are obtained as 'sub-institutions' of given institutions. A *sub-institution* $I' = (Sig', Sen', Mod', \models')$ of $I = (Sig, Sen, Mod, \models)$ is obtained by narrowing either the category of the signatures, the sentences, and / or the class of models of I. We may express this formally as follows:

- Sig' is a sub-category of Sig,
- for each signature $\Sigma \in |Sig'|$ we have that
  - $Sen'\Sigma \subseteq Sen\Sigma$, and
  - $Mod'\Sigma$ is a sub-category of $Mod\Sigma$,
- for each signature morphism $\varphi \in Sig'$ we have that
  - $Sen'\varphi$ is the restriction of $Sen\varphi$, and
  - $Mod'\varphi$ is the restriction of $Mod\varphi$,
- for each signature $\Sigma \in |Sig'|$, the satisfaction relation $\models'_\Sigma$ is the restriction of the satisfaction relation $\models_\Sigma$.

Below we give several rather well-known examples of 'sub-institutions' of $\mathcal{FOL}$.

Single-sorted logic ($\mathcal{FOL}^1$). This is the 'sub-institution' of $\mathcal{FOL}$ determined by the single-sorted signatures for a fixed sort. The name of this sort does not matter since different choices give rise to 'isomorphic sub-institutions'.

Note that in $\mathcal{FOL}^1$ the arities of the operation and of the relation symbols are essentially natural numbers rather than strings of sort symbols. Also the set of sorts S may be omitted from the notation of signatures, therefore the $\mathcal{FOL}^1$-signatures are pairs (F, P) of families F of sets of operation symbols and of families P of sets of relation symbols.

$\mathcal{FOL}^1$ is the version of first-order logic used mainly in conventional logic, while the more general many-sorted version $\mathcal{FOL}$ is used mainly in computing science.

Propositional logic ($\mathcal{PL}$). This can be seen as the 'sub-institution' of $\mathcal{FOL}$ obtained by restricting the signatures to those with an empty set of sort symbols. This means that $\mathcal{PL}$ signatures consist only of sets (of zero arity relation symbols), therefore $Sig^{\mathcal{PL}}$ is just $\mathbb{S}et$, for each set P the set of P-sentences consists of the Boolean expressions formed with variables from P, and the model functor is the contravariant power set functor $\mathcal{P}: \mathbb{S}et \to \mathbb{C}at^{\mathrm{op}}$ (the category of P-models is the partial order $(\mathcal{P}P, \subseteq)$ regarded as a category). Note that for any $\pi \in P$, a P-model $M \subseteq P$ satisfies π when $\pi \in M$.

Note also that P-models M can alternatively be regarded as valuations $M: P \to \{01\}$ to the standard Boolean algebra with two elements (where 0 denotes the bottom element, i.e. 'false', and 1 the top element, i.e. 'true'). Then $M \models_P \rho$ if and only if $M\rho = 1$ where $M\rho$ is the evaluation of ρ in the standard Boolean algebra.

While $\mathcal{PL}$ can be seen as a 'sub-institution' of $\mathcal{FOL}$, evidently it cannot be seen as a 'sub-institution' of the single-sorted version $\mathcal{FOL}^1$.

Positive first-order logic ($\mathcal{FOL}^+$). Sentences are restricted only to those constructed using $\wedge, \vee, \forall, \exists$, but not negation. Here $\vee$ and $\exists$ are no longer reducible to $\wedge$ and $\forall$ and vice versa.

Universal sentences in $\mathcal{FOL}$ ($\mathcal{UNIV}$). A *universal sentence* for a $\mathcal{FOL}$ signature (S,F,P) is a sentence of the form $(\forall X)\rho$ where ρ is a sentence formed without quantifiers.

Horn clause logic ($\mathcal{HCL}$). A *(universal) Horn sentence* for a $\mathcal{FOL}$ signature (S,F,P) is a (universal) sentence of the form $(\forall X)(H \Rightarrow C)$, where H is a finite conjunction of (relational or equational) atoms and C is a (relational or equational) atom, and $H \Rightarrow C$ is the implication of C by H. In the tradition of logic programming, universal Horn sentences are known as *Horn clauses*. We may often write Horn clauses as $(\forall X)H \Rightarrow C$ by omitting the brackets around $H \Rightarrow C$. Thus $\mathcal{HCL}$ has the same signatures and models as $\mathcal{FOL}$ but only the universal Horn clauses as sentences.

Equational logic. The institution $\mathcal{FOEQL}$ of *first order equational logic* is obtained from $\mathcal{FOL}$ by discarding both the relation symbols and their interpretation in models.

The institution $\mathcal{EQL}$ of *equational logic* is obtained by restricting the sentences of $\mathcal{FOEQL}$ only to universally quantified equations.

The institution $\mathcal{CEQL}$ of *conditional equational logic* is obtained as the 'intersection' between $\mathcal{FOEQL}$ and $\mathcal{HCL}$.

$\mathcal{EQLN}$ is the minimal extension of $\mathcal{EQL}$ with negation, allowing sentences obtained from atoms and negations of atoms through only one round of quantification, either universal or existential. More precisely, all sentences have the form $(QX)t_1\pi t_2$ where $Q \in \{\forall, \exists\}$, $\pi \in \{=, \neq\}$, and t_1 and t_2 are terms with variables X.

Relational logic ($\mathcal{REL}$). This is obtained as the sub-institution of $\mathcal{FOL}$ determined by those signatures without non-constant operation symbols. Many older works have developed conventional classical logic in $\mathcal{REL}$ rather than $\mathcal{FOL}$.

$(\Pi \cup \Sigma)^0_n$. This is the fragment of $\mathcal{FOL}$ containing only sentences of the form $Q\rho$ where Q consists of (at most) n alternated quantifiers (universal and existential) and ρ is atomic.

Second order logic ($\mathcal{SOL}$). This is obtained as the extension of $\mathcal{FOL}$ which allows quantification over sorts, operations, and relation symbols. This differs slightly from the usual presentations of second order logic in the literature which do not consider quantifications over the sorts (because it is usually considered in a single-sorted form).

Infinitary logic ($\mathcal{FOL}_{\infty,\omega}$, $\mathcal{FOL}_{\alpha,\omega}$, $\mathcal{FOL}_{\alpha,\beta}$). These are infinitary extensions of $\mathcal{FOL}$. $\mathcal{FOL}_{\infty,\omega}$ allows conjunctions of arbitrary sets of sentences, while $\mathcal{FOL}_{\alpha,\omega}$ admits conjunction of sets of sentences with cardinal smaller than α. Both of them allow only finitary quantifications. But we can even go that restriction by considering $\mathcal{FOL}_{\alpha,\beta}$, which is an extension of $\mathcal{FOL}$ allowing conjunctions of sets of sentences with cardinal smaller than α and quantifications with blocks of variables with cardinal smaller that β. In this case the set of variables names should be β. Note that $\mathcal{FOL} = \mathcal{FOL}_{\omega,\omega}$.

Infinitary Horn clause logic ($\mathcal{HCL}_{\infty,\omega}$, $\mathcal{HCL}_{\alpha,\beta}$). These are infinitary extensions of $\mathcal{HCL}$ obtained in the style of $\mathcal{FOL}_{\infty,\omega}$ / $\mathcal{FOL}_{\alpha,\beta}$ where the conjunctions refer to H being a conjunction of a set of atoms. Note that $\mathcal{HCL} = \mathcal{HCL}_{\omega,\omega}$.

Partial algebra ($\mathcal{PA}$)

A *partial* algebraic signature is a tuple (S,TF,PF) such that both (S,TF) and (S,PF) are algebraic signatures. Then TF are the *total* operations and PF are the *partial* operations. A *morphism of $\mathcal{PA}$ signatures* $\varphi\colon\ (S,TF,PF) \to (S',TF',PF')$ is just a morphism of algebraic signatures $(S,TF+PF) \to (S',TF'+PF')$ such that $\varphi(TF) \subseteq TF'$.

A *partial algebra* A for a $\mathcal{PA}$ signature (S,TF,PF) is just like an ordinary algebra but interpreting the operations of PF as partial rather than total functions, which means that A_σ might be *undefined* for some arguments. A *partial algebra homomorphism* $h\colon\ A \to B$ is a family of (total) functions $(h_s\colon\ A_s \to B_s)_{s\in S}$ indexed by the set of sorts S of the signature such that $h_w(A_\sigma a) = B_\sigma(h_s a)$ for each operation $\sigma\colon\ w \to s$ and each string of arguments $a \in A_w$ for which $A_\sigma a$ is defined.

The sentences have three kinds of atoms: *definedness* $\mathsf{def}(\_)$, *strong* equality $\stackrel{s}{=}$, and *existence* equality $\stackrel{e}{=}$. The definedness $\mathsf{def}(t)$ of a term t holds in a partial algebra A when the interpretation A_t of t is defined. The strong equality $t \stackrel{s}{=} t'$ holds when both terms are undefined or both of them are defined *and* are equal. The existence equality $t \stackrel{e}{=} t'$ holds when both terms are defined and equal. The sentences are formed from these atoms using Boolean connectives and quantification over *total* (first order) variables.

$QE(\mathcal{PA})$**.** A *(universal) quasi-existence equation* is an infinitary Horn sentence of the form

$$(\forall X)\bigwedge_{i\in I}(t_i \stackrel{e}{=} t'_i) \Rightarrow (t \stackrel{e}{=} t').$$

in the infinitary extension $\mathcal{PA}_{\infty,\omega}$ of $\mathcal{PA}$.

Let $QE(\mathcal{PA})$ be the sub-institution of the infinitary extension $\mathcal{PA}_{\infty,\omega}$ of $\mathcal{PA}$ which restricts the sentences only to quasi-existence equations, $QE_1(\mathcal{PA})$ the institution of the quasi-existence equations $(\forall X)\bigwedge_{i\in I}(t_i \stackrel{e}{=} t'_i) \Rightarrow (t \stackrel{e}{=} t')$ that have *either t or t'* 'already defined' (i.e., they occur as subterms of the terms of the equations in the premise or are formed only from total operation symbols), and $QE_2(\mathcal{PA})$ the institution of the quasi-existence equations that have *both t and t'* 'already defined'.

(First-order) Modal Logic ($\mathcal{MFOL}$)

In Chapter 12 we will undertake a deeper institution-independent study of modal logics institutions, while here we present only the standard extension of $\mathcal{FOL}$ with modalities and Kripke semantics.

The $\mathcal{MFOL}$ *signatures* are tuples (S, S_0, F, F_0, P, P_0) where

- (S, F, P) is a $\mathcal{FOL}$ signature, and
- (S_0, F_0, P_0) is a sub-signature of (S, F, P) of *rigid* symbols.

Signature morphisms $\varphi\colon (S, S_0, F, F_0, P, P_0) \to (S', S'_0, F', F'_0, P', P'_0)$ are just $\mathcal{FOL}$ signature morphisms $\varphi\colon (S, F, P) \to (S', F', P')$ which preserve the rigid symbols, i.e., $\varphi S_0 \subseteq S'_0$, $\varphi F_0 \subseteq F'_0$, $\varphi P_0 \subseteq P'_0$.

An $\mathcal{MFOL}$ *Kripke model* for a signature (S, S_0, F, F_0, P, P_0) is a pair (W, M) consisting of

- a *Kripke frame* $W = (|W|, W_\lambda)$ where $|W|$ is the set of 'possible worlds', and $W_\lambda \subseteq |W| \times |W|$ an 'accessibility' binary relation; and
- a family $M = (M^i)_{i \in |W|}$ of interpretations of the 'possible worlds' as (S, F, P)-models in $\mathcal{FOL}$, such that for all rigid symbols x, $M^i_x = M^j_x$ for all $i, j \in |W|$.

A *pointed Kripke model* is just a pair $((W, M), w)$ consisting of a Kripke model (W, M) and a possible world $w \in |W|$.

The *reduct* $(W', M')\restriction_\varphi$ of a Kripke model along a signature morphism φ is defined as (W', M) where $M^i = (M')^i\restriction_\varphi$ for each $i \in |W|$. This definition extends obviously to pointed Kripke models.

A Kripke model (W, R) is *T* when R is reflexive, *S4* when it is T and R is transitive, and is *S5* when it is *S4* and R is symmetric.

Homomorphisms between Kripke models preserve their mathematical structure. Thus a *Kripke model homomorphism* $h\colon (W, M) \to (W', M')$ consists of

- a Kripke frame homomorphism $h_0\colon W \to W'$ which is a homomorphism of binary relations, i.e. $h_0\colon |W| \to |W'|$ is function such that it preserves the accessibility relation ($\langle i, j\rangle \in W_\lambda$ implies $\langle h_0(i), h_0(j)\rangle \in W'_\lambda$), and
- for each $i \in |W|$ a $\mathcal{FOL}$ (S, F, P)-model homomorphism $h^i_1\colon M^i \to M'^{h_0 i}$ such that for each rigid sort s_0 we have that $(h^i_1)_{s_0} = (h^j_1)_{s_0}$ for all $i, j \in |W|$.

This definition can be extended to a homomorphism of pointed Kripke models $h\colon ((W, M), w) \to ((W', M'), w')$ by imposing that $h_0 w = w'$.

The (S, S_0, F, F_0, P, P_0)-*sentences* are expressions formed from $\mathcal{FOL}$ (S, F, P)-atoms by closing under usual Boolean connectives, universal and existential first-order quantifications by rigid variables, and unary modal connectives $\Box$ (necessity) and $\Diamond$ (possibility).

The satisfaction of $\mathcal{MFOL}$ sentences by the Kripke models is a ternary relation that is parameterised by the possible worlds w as follows:

- $(W, M) \models^w \rho$ iff $M^w \models^{\mathcal{FOL}} \rho$ for each atom ρ and each $w \in |W|$,

- $(W,M) \models^w \rho_1 \wedge \rho_2$ iff $(W,M) \models^w \rho_1$ and $(W,M) \models^w \rho_2$; and similarly for the other Boolean connectives,
- $(W,M) \models^i \Box\rho$ iff $(W,M) \models^j \rho$ for each j such that $\langle i, j\rangle \in W_\lambda$,
- $\Diamond\rho$ abbreviates $\neg\Box\neg\rho$,
- $(W,M) \models^w (\forall X)\rho$ when $(W,M') \models^w \rho$ for each expansion (W,M') of (W,M) to a Kripke $(S,F+X,P)$-model and $(W,M) \models^w (\exists X)\rho$ if and only if $(W,M) \models^w \neg(\forall X)\neg\rho$.

This yields two institutions of first order modal logic:

1. *local* first order modal logic , denoted $\mathcal{MFOL}^\sharp$, where the models are pointed Kripke models and satisfaction is defined by

$$((W,M),w) \models \rho \text{ if and only if } (W,M) \models^w \rho, \text{ and}$$

2. *global* first order modal logic , denoted $\mathcal{MFOL}^*$, where the models are Kripke models and satisfaction is defined by

$$(W,M) \models \rho \text{ if and only if } (W,M) \models^w \rho \text{ for all } w \in |W|.$$

Modal propositional logic ($\mathcal{MPL}$). Both the local ($\mathcal{MPL}^\sharp$) and the global ($\mathcal{MPL}^*$) versions of this institution arise by considering the local and the global sub-institutions of $\mathcal{MFOL}$ determined by the signatures with an empty set of sort symbols (and therefore empty sets of operation symbols) and empty sets of rigid relation symbols. Much of the conventional modal logic studies are concerned with these two institutions.

Hybrid logics ($\mathcal{HFOL}$, $\mathcal{HPL}$). These represent an enhancement of ordinary modal logic with explicit syntax for possible worlds semantics that allows direct access to the possible worlds. This means that the $\mathcal{MFOL}$ or $\mathcal{MPL}$ syntax is extended

- with a new kind of atoms, called *nominals*, that are explicit part of the signatures; so for example a $\mathcal{HFOL}$ signature is a pair (Nom,Σ) where Nom is such a set of nominals and Σ is an $\mathcal{MFOL}$ signature,
- for each $i \in \mathrm{Nom}$, a unary connective $@_i$ called *satisfaction at i*, and
- quantifications over finite blocks of nominals are also allowed.

The models of hybrid logics extend the usual Kripke model with interpretations for the nominals at the level of the Kripke frames, which means that the W part of the Kripke model contains also an interpretation $W_i \in |W|$ for each nominal $i \in \mathrm{Nom}$. The satisfaction relation is extended for the new syntax as follows:

- $((W,M) \models^w i) = (w = W_i)$ for each $i \in \mathrm{Nom}$;
- $((W,M) \models^w @_i\rho) = ((W,M) \models^{W_i} \rho)$; and
- the satisfaction of quantified sentences is based upon expansions of Kripke models with interpretations for the nominal variables.

Intuitionistic logic

Heyting algebras. A *Heyting algebra* A is a bounded lattice which is cartesian closed as category. In other words A is a partial order $(A, \leq)$ with a greatest element $\top$ and a least one $\perp$ and such that any two elements $a, b \in A$

- have a greatest lower bound (meet) $a \wedge b$ and a least upper bound (join) $a \vee b$, and
- there exists a greatest element x such that $a \wedge x \leq b$; this element is denoted $a \Rightarrow b$.

From these axioms, a series of important properties can be derived, such as the distributivity of the lattice. Also each element a has a *pseudo-complement* $\neg a$ defined as $a \Rightarrow \perp$. However, while it is possible to show that $a \leq \neg\neg a$, in general we do not have that $a = \neg\neg a$. When this equality holds for all elements a, the algebra A is called *Boolean algebra*. This means that Heyting algebras are more general than Boolean algebras. A famous class of examples of Heyting algebras that are not necessarily Boolean algebras come from general topology; the set of open sets of any topological space ordered by (sub)set inclusion forms a Heyting algebra.

Heyting algebras can be alternatively defined as a variety of universal algebras by a set of equations that extend the equational definition of bounded lattices with a new binary operation $\Rightarrow$ and axioms $(x \Rightarrow y) \wedge x \leq y$ and $y \leq x \Rightarrow (y \wedge x)$ (corresponding to the co-unit and the unit of the adjunction defining Heyting algebras as cartesian closed categories). These are equations indeed if we write $a \leq b$ as $a = a \wedge b$. It can be shown that as lattices, the Heyting algebras are distributive. In light of the perspective of Heyting algebras as universal algebras, a homomorphism $h\colon\ A \to B$ of Heyting algebras is a homomorphism of bounded lattices that in addition preserve the interpretation of $\Rightarrow$, i.e. $h(a \Rightarrow b) \ = \ ha \Rightarrow hb$.

Intuitionistic propositional logic ($\mathcal{IPL}$). The institution of intuitionistic propositional logic generalizes (classical) propositional logic ($\mathcal{PL}$) by considering models to be valuations of the propositional variables to arbitrary Heyting algebras rather than the binary Boolean algebra $\{0, 1\}$. More precisely, $\mathcal{IPL}$ has the same signatures as $\mathcal{PL}$, i.e., plain sets P, and for any set P, a P-model M is just a function $M\colon\ P \to A$ where A is any Heyting algebra. A model homomorphism $h\colon\ (M\colon\ P \to A) \to (N\colon\ P \to B)$ is a Heyting algebra homomorphism $h\colon\ A \to B$ such that $M;h \leq N$. The composition of $\mathcal{IPL}$ model homomorphism is defined by the composition of the underlying Heyting algebra homomorphisms. If $f\colon\ P \to P'$ is a signature morphism, then the reduct of any P'-model M' is just $f;M'$. $\mathcal{IPL}$ and $\mathcal{PL}$ share the same sentences.

The function M can be extended from P to SenP by $M(\rho_1 \wedge \rho_2) = M\rho_1 \wedge M\rho_2$, $M(\rho_1 \vee \rho_2) = M\rho_1 \vee M\rho_2$, $M(\neg\rho) = \neg M\rho$, $M(\rho_1 \Rightarrow \rho_2) = M\rho_1 \Rightarrow M\rho_2$, etc. The satisfaction relation is defined by

$$M \models_P \rho \ \text{ if and only if } \ M\rho = \top.$$

Preorder algebra ($\mathcal{POA}$)

The $\mathcal{POA}$ signatures are just ordinary algebraic signatures. The $\mathcal{POA}$ models are *preordered algebras* which are interpretations of signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder and each operation as a preorder functor, which means a preorder-preserving (i.e., monotone) function. A *preordered algebra homomorphism* is just a family of preorder functors (monotone functions) which is also an algebra homomorphism.

The sentences have two kinds of atoms: equations and *preorder atoms*. A preorder atom $t \leq t'$ is satisfied by a preorder algebra M when the interpretations of the terms are in the preorder relation of the carrier, i.e., $M_t \leq M_{t'}$. Full sentences are constructed from equational and preorder atoms by using Boolean connectives and first order quantification.

Horn preordered algebra ($\mathcal{HPOA}$). This is the sub-institution of $\mathcal{POA}$ whose sentences are the universal Horn sentences $(\forall X)H \Rightarrow C$ formed over equational and preorder atoms.

Multialgebras ($\mathcal{MA}$)

The category of the $\mathcal{MA}$ signatures is just that of the algebraic signatures. Multialgebras generalize algebras by nondeterministic operations returning a *set* of all possible outputs for the operation rather than a single value. Hence multialgebra operations are interpreted as functions from the carrier to the powerset of the carrier. Therefore each term $t = \sigma(t_1 \ldots t_n)$ is interpreted in any multialgebra M by $M_t = \bigcup\{M_\sigma(m_1 \ldots m_n) \mid m_1 \in M_{t_1}, \ldots, m_n \in M_{t_n}\}$.

Given a signature (S,F), a *multialgebra homomorphism* $h\colon\ M \to N$ consists of an S-indexed family of functions $\{h_s\colon\ M_s \to N_s \mid s \in S\}$ such that for each operation symbol $\sigma \in F_{s_1 \ldots s_n \to s}$ and each $m_k \in M_{s_k}$ for $1 \leq k \leq n$ we have

$$h_s M_\sigma(m_1, \ldots, m_n) \subseteq N_\sigma(h_{s_1} m_1, \ldots, h_{s_n} m_n).$$

The sentences have two kinds of atoms: set inclusion $\prec$ and (deterministic) element equality $\doteq$. The set inclusion $t \prec t'$ holds in a multialgebra M if and only if $M_t \subseteq M_{t'}$, i.e., the term t is "more deterministic" than t'. The element equality $t \doteq t'$ states that the terms t and t' are deterministic and must return the same element. (This means that M_t and $M_{t'}$ are both singleton sets containing the same element.) Full sentences are built from these atoms by using Boolean connectives and first order quantification in the manner of $\mathcal{FOL}$.

Membership algebra ($\mathcal{MBA}$)

A $\mathcal{MBA}$ signature is a tuple (S,K,F,kind) where S is a set of *sorts*, K is a set of *kinds*, (K,F) is an algebraic signature, and $\mathrm{kind}\colon\ S \to K$ is a function. A *morphism of $\mathcal{MBA}$*

signatures $\varphi : (S, K, F, \mathrm{kind}) \to (S', K', F', \mathrm{kind}')$ consists of functions $\varphi^{\mathrm{st}} : S \to S'$, $\varphi^{\mathrm{k}} : K \to K'$ such that the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{\mathrm{kind}} & K \\ {\scriptstyle \varphi^{\mathrm{st}}}\downarrow & & \downarrow{\scriptstyle \varphi^{\mathrm{k}}} \\ S' & \xrightarrow[\mathrm{kind}']{} & K' \end{array}$$

and a family of functions $\{\varphi^{\mathrm{op}}_{w \to s} \mid w \in K^*, s \in K\}$ such that $(\varphi^{\mathrm{k}}, \varphi^{\mathrm{op}}) : (K, F) \to (K', F')$ is an algebraic signature morphism.

Given a membership algebraic signature (S, K, F, kind), an (S, K, F, kind)-*algebra* A is a (K, F)-algebra together with a set $A_s \subseteq A_{\mathrm{kind}(s)}$ for each sort $s \in S$ such that $A_s \subseteq A_{\mathrm{kind}(s)}$ for each sort s. A (S, K, F, kind)-*algebra homomorphism* $A \to B$ is a (K, F)-algebra homomorphism such that $h_{\mathrm{kind}(s)} A_s \subseteq B_s$ for each sort s.

Sentences for membership algebra have two types of atoms, atomic equations $t = t'$ for t, t' any terms of the same kind, and *atomic membership* $t : s$ where s is a sort and t is an term of $\mathrm{kind}(s)$. A membership algebra A satisfies an equation $t = t'$ when $A_t = A_{t'}$ and satisfies a membership atom $t : s$ when $A_t \in A_s$. Full sentences are formed from atoms by iteration of Boolean connectives and first order quantification.

Higher Order Logic ($\mathcal{HOL}$)

For any set S of *sorts*, let $\overrightarrow{S}$ be the set of S-*types* defined as the least set such that $S \subseteq \overrightarrow{S}$ and $s_1 \to s_2 \in \overrightarrow{S}$ when $s_1, s_2 \in \overrightarrow{S}$. A $\mathcal{HOL}$ *signature* (S, F) consists of a set S of sorts and a family of sets of constants $F = \{F_s \mid s \in \overrightarrow{S}\}$. A *morphism of* $\mathcal{HOL}$ *signatures* $\varphi : (S, F) \to (S', F')$ consists of a function $\varphi^{\mathrm{st}} : S \to S'$ and a family of functions $\{\varphi^{\mathrm{op}}_s : F_s \to F'_{\varphi^{\mathrm{type}}(s)} \mid s \in \overrightarrow{S}\}$ where $\varphi^{\mathrm{type}} : \overrightarrow{S} \to \overrightarrow{S'}$ is the canonical extension of φ^{st} such that $\varphi^{\mathrm{type}}(s_1 \to s_2) = \varphi^{\mathrm{type}}(s_1) \to \varphi^{\mathrm{type}}(s_2)$.

Given a signature (S, F), an (S, F)-*model* interprets each sort $s \in S$ as a set M_s and each operation symbol $\sigma \in F_s$ as an element $M_\sigma \in M_s$, where for each type s_1, s_2, $M_{s_1 \to s_2} = [M_{s_1} \to M_{s_2}] = \{f \text{ function } \mid f : M_{s_1} \to M_{s_2}\}$. An (S, F)-*model homomorphism* $h : M \to N$ interprets each S-type s as a function $h_s : M_s \to N_s$ such that $h M_\sigma = N_\sigma$ for each $\sigma \in F$ and such that the diagram

$$\begin{array}{ccc} M_s & \xrightarrow{f} & M_{s'} \\ {\scriptstyle h_s}\downarrow & & \downarrow{\scriptstyle h_{s'}} \\ N_s & \xrightarrow[h_{s \to s'} f]{} & N_{s'} \end{array}$$

commutes for all types s and s' and each $f \in M_{s \to s'}$.

For any $\mathcal{HOL}$ signature (S, F), each operation symbol σ of type s is a term of type s, and (tt') is a term of type s_2 whenever t is a term of type $s_1 \to s_2$ and t' is a term of type

s_1. A $\mathcal{HOL}$ (S,F)-*equation* consists of a pair $t_1 = t_2$ of terms of the same type. A $\mathcal{HOL}$ (S,F)-sentence is obtained from equations by iteration of the usual Boolean connectives and of *higher order (universal or existential) quantification* which is defined similarly to the quantification in $\mathcal{FOL}$. Note however that because of the 'higher order' types, the constants in $\mathcal{HOL}$ denote higher order rather than first-order entities.

The interpretation of operation symbols by models can be extended to terms by defining $M_{(tt')} = M_t(M_{t'})$ for each term t of type $s_1 \to s_2$ and each term t' of type s_1. A model M satisfies the equation $t = t'$ when $M_t = M_{t'}$. This satisfaction relation can be extended in an obvious manner from equations to any sentences.

Henkin semantics. The institution of higher order logic with Henkin semantics, denoted $\mathcal{HNK}$, extends the $\mathcal{HOL}$ models by relaxing the condition $M_{s \to s'} = [M_s \to M_{s'}]$ to $M_{s \to s'} \subseteq [M_s \to M_{s'}]$.

Many-valued logic ($\mathcal{MVL}^\sharp$)

This institution generalizes ordinary logic based upon the two Boolean truth values, *true* and *false*, to larger sets of truth values that are structured by the concept of *residuated lattices*. This approach is different from intuitionistic logic.

Residuated lattices. A *residuated lattice* L is a bounded lattice (with $\leq$ denoting the underlying partial order that has infimum (meets) $\wedge$, supremum (joins) $\vee$, greatest 1 and lowest 0 elements) and which comes equipped with an additional commutative and associative binary operation $*$ which has 1 as identity and such that for all elements x, y and z

- $(x * y) \leq (x * z)$ if $y \leq z$, and
- there exists an element $x \Rightarrow z$ such that $y \leq (x \Rightarrow z)$ if and only if $y * x \leq z$.

The first condition above just means that $x * -$ is a functor on the partial order $(L, \leq)$, and the second condition means that it has a left adjoint $x \Rightarrow -$.

The ordinary two-valued situation can be recovered when L is the two values Boolean algebra with $*$ being the conjunction. Then $\Rightarrow$ is the ordinary Boolean implication. Heyting algebras come as an example of a residuated lattice in the same way, which shows that in some sense multiple valued logics cover intuitionistic logic. There is a myriad of interesting examples of residuated lattices used for multiple valued logics for which $*$ gets an interpretation rather different from the ordinary conjunction. One famous such example is the so-called *Łukasiewicz arithmetic conjunction* on the closed interval $[0,1]$ defined by $x * y = \max\{0, x + y - 1\}$. In this example $x \Rightarrow y = \min\{1, 1 - x + y\}$.

The $\mathcal{MVL}^\sharp$ institution. Let us fix a residuated lattice L that is also complete, i.e. it has infimum and supremum for any sets of elements.

$\mathcal{MVL}^\sharp$ has the same signatures as $\mathcal{REL}$, i.e. triples (S,C,P) with S set of sort symbols, C an S-sorted family of constants, and P and S^*-sorted family of relation symbols.

The (S,C,P)-sentences are pairs (ρ,x) where ρ is a *pre-sentence* and x is any element of L. The (S,C,P)-pre-sentences are very much like the $\mathcal{REL}$-sentences, they are constructed from relational atoms $\pi(c_1,\ldots,c_n)$ by the connectives $\perp, \top, \wedge, \vee, *, \Rightarrow$ and by universal $(\forall X)$ and existential $(\exists X)$ quantifications for finite sets X of variables.

An $\mathcal{MVL}$ (S,C,P)-model M interprets each sort $s \in S$ as a set M_s and each relation symbol $\pi \in P_w$ as an *L-fuzzy relation*, i.e. a function $M_\pi : M_w \to L$. A model homomorphism $h : M \to N$ consists of a function $h_s : M_s \to N_s$ for each sort $s \in S$ such that the interpretations of constants are preserved, i.e., $h_s M_c = N_c$ for each $c \in C_s$, and such that $M_\pi m \leq N_\pi(h_w m)$ for each $\pi \in P_w$ and each $m \in M_w$.

For each (S,C,P)-model M and each (S,C,P)-pre-sentence ρ we define a value $M \models \rho$ in L as follows:

- $(M \models \pi(c_1,\ldots,c_n)) = M_\pi(M_{c_1},\ldots,M_{c_n})$ for relational atoms,
- $(M \models \rho_1 \wedge \rho_2) = (M \models \rho_1) \wedge (M \models \rho_2)$ and similarly for the other connectives $\vee$, $*$ and $\Rightarrow$,
- $M \models \perp = 0$ if and only if $M_\perp = 0$ and $M \models \top$ if and only if $M_\top = 1$,
- $(M \models (\forall X)\rho) = \bigwedge\{M' \models \rho \mid M'\restriction_{(S,C,P)} = M\}$, and
- $(M \models (\exists X)\rho) = \bigvee\{M' \models \rho \mid M'\restriction_{(S,C,P)} = M\}$.

Then the $\mathcal{MVL}^\sharp$ satisfaction relation is defined by

$$M \models^{\mathcal{MVL}^\sharp}_{(S,C,P)} (\rho,x) \text{ if and only if } x \leq (M \models \rho).$$

Automata ($\mathcal{AUT}$)

Given a set V (of 'input symbols'), a *V-automaton A* consists of

- a set A_{state} of 'states' with some 'initial' state $A_0 \in A_{\text{state}}$ and with some 'final' states, and
- a *transition function* $A_t : V \times A_{\text{state}} \to A_{\text{state}}$.

A *homomorphism of V-automata* $h : A \to B$ consists of a function $h : A_{\text{state}} \to B_{\text{state}}$ such that $hA_0 = B_0$, ha is final whenever a is final, and

$$\begin{array}{ccc} V \times A_{\text{state}} & \xrightarrow{A_t} & A_{\text{state}} \\ {\scriptstyle 1_V \times h}\downarrow & & \downarrow{\scriptstyle h} \\ V \times B_{\text{state}} & \xrightarrow[B_t]{} & B_{\text{state}} \end{array}$$

commutes. The transition function extends canonically by iteration to $A_t^* : V^* \times A_{\text{state}} \to A_{\text{state}}$. A word (or string) $w \in V^*$ is *recognized* by a V-automaton if and only if $A_t^*(w,A_0)$ is a final state.

The institution $\mathcal{AUT}$ of automata has $\mathbb{S}et$ as its category of signatures (a signature being thus a set V of 'input symbols'), automata as models, and strings of input symbols as sentences. A string w is satisfied by an automaton A when A recognizes w.

Exercises

3.2. Extend the definition of $\mathcal{IPL}$ to an institution of 'intuitionistic first order logic'.

3.3. [56] **Contraction Algebras**
A *contraction algebraic signature* (S,F,q) consists of an algebraic signature (S,F) and a real number $0 < q < 1$. $\varphi\colon (S,F,q) \to (S',F',q')$ is a *morphism of contraction algebraic signatures* if $\varphi\colon (S,F) \to (S',F')$ is an algebraic signature morphism and $q' \leq q$.

(A,d) is a (S,F,q)-*contraction algebra* when A is an (S,F)-algebra, d gives a complete metric space (A_s,d_s) for each sort $s \in S$ such that d_s is bounded by 1, and

$$d(A_\sigma(a_1 \dots a_n), A_\sigma(b_1 \dots b_n)) \leq q \cdot max\{d(a_k,b_k) \mid k \in \{1,\dots,n\}\}.$$

A *homomorphism of contraction algebras* $h\colon (A,d) \to (A',d')$ is just an (S,F)-algebra homomorphism $A \to A'$ such that $d'(ha,hb) \leq d(a,b)$ for all elements $a,b \in A$.

For each algebraic signature (S,F) let $T^\omega_{(S,F)}$ be the S-sorted set of (possibly) infinite (S,F)-terms. Show that for any contraction algebra (A,d) there exists a unique mapping $T^\omega_{(S,F)} \to A$ mapping each (possibly) infinite term t to an element A_t of A such that $A_{\sigma(t_1,\dots,t_n)} = A_\sigma(A_{t_1},\dots,A_{t_n})$ for each infinite term $\sigma(t_1,\dots,t_n)$.

An (S,F,q)-*approximation equation* $t \approx_\varepsilon t'$ consists of a pair of (possibly) infinite terms t and t' and a real number $0 \leq \varepsilon < 1$. A contraction algebra A satisfies $t \approx_\varepsilon t'$ if and only if $d(A_t, A_{t'}) \leq \varepsilon$. Full 'approximation' sentences are formed from atomic approximation equalities by iteration of Boolean connectives and quantification. These data define an institution $\mathcal{CA}$ of contraction algebras and approximation sentences.

3.4. Linear Algebra
The institution $\mathcal{LA}$ has the category $\mathbb{C}Rng$ of commutative rings as the category of signatures such that for each commutative ring R the category of R-models is R-Mod the category of R-modules, an R-sentence is a linear system of equations with coefficients from R and the satisfaction relation is defined by the existence of solutions for the system of equations.

3.5. $\mathcal{HOL}$ **with** λ**-abstraction**
The institution $\mathcal{HOL}_\lambda$ has become quite popular for computer-assisted theorem proving. It adds λ-abstraction and products to $\mathcal{HOL}$. Signatures and signature morphisms are similar to those of $\mathcal{HOL}$. The only difference is in the definition of the set of higher types: a type Ω of truth values and products are added. Thus $\overrightarrow{S}$ is defined to be the least set such that

- $S \uplus \{\Omega\} \subseteq \overrightarrow{S}$,
- $s_1 \to s_2 \in \overrightarrow{S}$ and $s_1 \times s_2 \in \overrightarrow{S}$ when $s_1, s_2 \in \overrightarrow{S}$.

For each $\mathcal{HOL}_\lambda$-signature (S,F),

- each operation symbol σ of type s is a term of type s,
- $(t\ t')$ is a term of type s_2 whenever t is a term of type $s_1 \to s_2$ and t' is a term of type s_1,
- $\langle t_1, t_2 \rangle$ is a term of type $s_1 \times s_2$ when t_1 is a term of type s_1 and t_2 is a term of type s_2,

- for any finite list $X = \langle x_1{:}s_1, \ldots, x_n{:}s_n\rangle$ of typed variables and any $(S, F+X)$-term t of type s, $\lambda X.t$ is an (S,F)-term of type $(((s_1 \times s_2) \times \ldots) \times s_n) \to s$,
- $t_1 = t_2$ is a term of type Ω for terms t_1, t_2 of the same type.

A $\mathcal{HOL}_\lambda$-model (also called *standard model*) interprets Ω as a two-element set $\{\bot, \top\}$, $\_\times\_$ as a cartesian product, and is otherwise like a $\mathcal{HOL}$-model. The interpretation M_t of a term t in a model M is defined as in $\mathcal{HOL}$ for the cases σ and $(t\ t')$. $M_{\langle t_1,t_2\rangle}$ is just $\langle M_{t_1}, M_{t_2}\rangle$. $M_{\lambda X.t}$ is the function that, for any $(S, F+X)$-expansion M' of M, maps the tuple $\langle\langle\langle M'_{x_1}, M'_{x_2}\rangle, \ldots\rangle, M'_{x_n}\rangle$ to M'_t. $M_{t_1=t_2}$ is $\top$, if $M_{t_1} = M_{t_2}$, and $\bot$ otherwise.

A (S,F)-*sentence* ρ is just a (S,F)-term of type Ω. It holds in a model M if $M_\rho = \top$.

3.6. $\mathcal{HNK}$ with λ-abstraction
$\mathcal{HNK}_\lambda$ is a generalization of $\mathcal{HOL}_\lambda$, much in the same way as $\mathcal{HNK}$ is a generalization of $\mathcal{HOL}$. However, there is an additional requirement for models. Let a Σ-*frame* be like a $\mathcal{HOL}_\lambda$-model of signature Σ, but with the relaxed condition that $M_{s_1\to s_2}$ may be any subset of $[M_{s_1} \to M_{s_2}]$. A Σ-frame is a Σ-*general model*, if every Σ-term has an interpretation in it (note that the interpretations of λ-abstractions require the existence of certain functions in the model). The model functor of $\mathcal{HNK}_\lambda$ uses general models instead of standard models.

3.7. Categorical Equational Logic
For any category $\mathbb{A}$, an (unconditional) $\mathbb{A}$-*equation* $(\forall B)l = r$ is a pair of parallel arrows $l, r : C \to B$ in $\mathbb{A}$. An $\mathbb{A}$-*model* is simply any object of $\mathbb{A}$, and a homomorphism of $\mathbb{A}$-models is an arrow of $\mathbb{A}$. An $\mathbb{A}$-model A *satisfies* the equation $(\forall B)l = r$ when $l;h = r;h$ for each arrow $h : B \to A$.

For each right adjoint $\mathcal{U} : \mathbb{A}' \to \mathbb{A}$ with $\mathcal{F} : \mathbb{A} \to \mathbb{A}'$ as left adjoint, the following Satisfaction Condition holds:

$$\mathcal{U}A' \models (\forall B)l = r \text{ if and only if } A' \models (\forall \mathcal{F}B)\mathcal{F}l = \mathcal{F}r$$

for each $\mathbb{A}'$-model A' and each $\mathbb{A}$-equation $(\forall B)l = r$.

This defines the institution $\mathit{Cat}\mathcal{EQL}$ of *categorical equational logic* with categories as signatures and adjunctions as signature morphisms.

3.8. Institution of the signature morphisms
For any institution $(\mathit{Sig}, \mathit{Sen}, \mathit{Mod}, \models)$ we define

- $\mathit{Sig}^{\to}$ to be the category of functors $(\bullet \to \bullet) \to \mathit{Sig}$,
- $\mathit{Sen}^{\to}\varphi = \mathit{Sen}\Sigma$ for each signature morphism $\varphi \in \mathit{Sig}(\Sigma, \Sigma')$,
- $\mathit{Mod}^{\to}\varphi = \mathit{Mod}\Sigma'$ for each signature morphism $\varphi \in \mathit{Sig}(\Sigma, \Sigma')$, and
- for each signature morphism $\varphi : \Sigma \to \Sigma'$, for each Σ'-model M', and each Σ-sentence ρ, $M' \models^{\to}_{\varphi} \rho$ if and only if $M' \models_{\Sigma'} \varphi\rho$.

Then $(\mathit{Sig}^{\to}, \mathit{Sen}^{\to}, \mathit{Mod}^{\to}, \models^{\to})$ is an institution.

3.9. [220] **Extended models**
A *pre-institution* [217] I consists of the same data as an institution, but without the requirement of the Satisfaction Condition. An *extended model* of a signature Σ_1 is a pair (φ, N), where $\varphi : \Sigma_1 \to \Sigma_2$ is a signature morphism and N is a Σ_2-model. The extended model (φ, N) satisfies a Σ_1-sentence ρ if and only if $N \models^{I}_{\Sigma_2} \varphi\rho$. The extended models, together with the I-signatures and I-sentences form an institution.

3.10. [89] **Localisations**
For any institution $I = (\mathit{Sig}, \mathit{Sen}, \mathit{Mod}, \models)$ and each Σ-model M we define the *M-localisation of I*, denoted I/M, by

- $Sig^{I/M} = Sig/M$;
- $Sen^{I/M}(\Omega,\varphi) = Sen\Omega$;
- $Mod^{I/M}(\Omega,\varphi)$ is the singleton category containing only $(Mod\varphi)M$;
- the satisfaction relation of I/M is inherited from I.

Then $I/M = (Sig^{I/M}, Sen^{I/M}, Mod^{I/M}, \models)$ is an institution.

3.11. [123] **Charters**
A *charter* consists of

- an adjunction $(\mathcal{U},\mathcal{F},\eta,\varepsilon)$ between a category of signatures *Sig* and a category *Syn* of "syntactic systems", with $\mathcal{U}\colon\ Syn \to Sig$ the right adjoint and $\mathcal{F}$ the left adjoint,
- a "ground object" $G \in |Syn|$ (in which other syntactic systems are interpreted), and
- a "base" functor $B\colon\ Syn \to \mathbb{S}et$ (extracting the sentence component from the syntactic system) such that $B(G) = \{\mathbf{true},\mathbf{false}\}$.

An institution $(Sig, Sen, Mod, \models)$ is *chartable* when there exists a charter $(Sig, Syn, \mathcal{U}, \mathcal{F}, B, G)$ such that

- $|Mod\Sigma| = Syn(\Sigma, \mathcal{U}G)$ and $(Mod\varphi)M' = \varphi;M'$,
- $Sen = \mathcal{F};B$, and
- for each Σ-model $M\colon\ \Sigma \to \mathcal{U}G$

$$M \models_\Sigma e \ \text{ if and only if } \ (BM^\sharp)e = \mathbf{true}$$

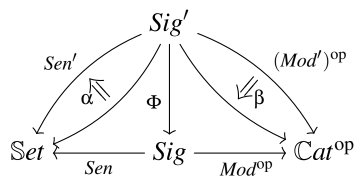
  where $M^\sharp$ is the unique arrow $\mathcal{F}\Sigma \to G$ such that $M = \eta_\Sigma; \mathcal{U}M^\sharp$.

$\mathcal{PL}$ is chartable by taking *Syn* as the category of (unsorted) $(\neg,\wedge,\vee,\Rightarrow)$-algebras (where $\neg$ is an unary operation and $\wedge$,$\vee$, and $\Rightarrow$ are binary operation symbols), $\mathcal{U} = B$ is the forgetful functor $Mod(\neg,\wedge,\vee,\Rightarrow) \to \mathbb{S}et$, and G is the canonical $(\neg,\wedge,\vee,\Rightarrow)$-algebraic structure on $\{\mathbf{true},\mathbf{false}\}$ interpreting the Boolean connectors as usual. Show that other institutions are chartable too.

3.3 Morphisms and comorphisms

Let us look into the way the institution $\mathcal{EQL}$ can be obtained by forgetting the relational part and discarding all sentences but equations in $\mathcal{FOL}$. This is a three-fold process. Firstly, there is a forgetful functor between the categories of signatures "forgetting" the relations, i.e., mapping each $\mathcal{FOL}$ signature (S,F,P) to the algebraic signature (S,F). On the sentences side, each (S,F)-equation can be regarded as an (S,F,P)-sentence; this gives a family of (trivial) translation functions between sets of sentences. On the models side, each (S,F,P)-model can be regarded as an (S,F)-algebra by forgetting the interpretations of the relation symbols; this gives a family of functors between categories of models. Notice that the satisfaction of sentences by models is invariant with respect to this mapping $\mathcal{FOL} \to \mathcal{EQL}$.

Institution morphisms. Such structure preserving mappings from a more complex to a simpler institution can be formalized by the general concept of *institution morphism* $(\Phi,\alpha,\beta): I' \to I$ consisting of



1. a functor $\Phi: Sig' \to Sig$, called the *signature functor*,
2. a natural transformation $\alpha: \Phi;Sen \Rightarrow Sen'$, called the *sentence transformation*, and
3. a natural transformation $\beta: Mod' \Rightarrow \Phi^{\mathrm{op}};Mod$, called the *model transformation*

such that the following *satisfaction condition* holds:

$$M' \models'_{\Sigma'} \alpha_{\Sigma'} e \text{ if and only if } \beta_{\Sigma'} M' \models_{\Phi\Sigma'} e$$

for any signature $\Sigma' \in |Sig'|$, for any Σ'-model M', and any $\Phi\Sigma'$-sentence e.

Although institution morphisms are suitable to formalize 'forgetful' mappings between more complex institutions to simpler ones, there are also other kinds of examples of institution morphisms. Some of them can be found in the exercises.

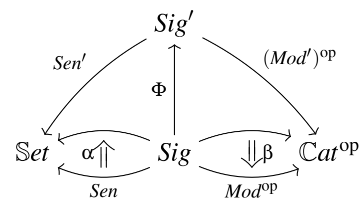
The composition of institution morphisms $(\Phi',\alpha',\beta'): I'' \to I'$ and $(\Phi,\alpha,\beta): I' \to I$ is $(\Phi';\Phi,\ \Phi'\alpha;\alpha',\ \beta';\Phi'^{\mathrm{op}}\beta): I'' \to I$. Under this composition, institutions and institution morphisms form the *category* $\mathbb{I}ns$ *of institution morphisms* . This can be established by routine calculations which are left as an exercise to the reader.

Institution modifications. The category $\mathbb{I}ns$ has a 2-dimension too, given by the institution *modifications*. An institution modification between institution morphisms $(\Phi,\alpha,\beta) \Rightarrow (\Phi',\alpha',\beta')$ consists of

1. a natural transformation $\tau: \Phi \Rightarrow \Phi'$, called the *signature transformation*,
2. a modification $\omega: \beta \Rightarrow \beta';\tau Mod$, called the *model transformation*, i.e., for each $\Sigma' \in |Sig'|$, a natural transformation $\omega_{\Sigma'}: \beta_{\Sigma'} \Rightarrow \beta'_{\Sigma'};Mod\tau_{\Sigma'}$.

This makes $\mathbb{I}ns$ a 2-category with institutions as 0-cells, institution morphisms as 1-cells, and their modifications as 2-cells. Routine calculations, left as an exercise to the reader, show that the horizontal composition of institution morphisms and the vertical composition of modifications satisfy the 2-category Interchange Laws (see Section 2.4).

Comorphisms. This relationship between $\mathcal{FOL}$ and $\mathcal{EQL}$ can be also looked at from the opposite direction, by emphasizing the "embedding" rather than the "forgetful" aspect. Each algebraic signature (S,F) can be regarded as a $\mathcal{FOL}$ signature $(S,F,\emptyset)$ without relation symbols. This determines an "embedding" functor from the category of algebraic signatures to the category of $\mathcal{FOL}$ signatures. On the sentence side, each (S,F)-equation is an $(S,F,\emptyset)$-sentence, and each $(S,F,\emptyset)$-model is just an (S,F)-algebra. The satisfaction of sentences by models is invariant with respect to this embedding of $\mathcal{EQL}$ into $\mathcal{FOL}$. Such an embedding relationship between institutions is formalized by the concept of *institution comorphism* $(\Phi,\alpha,\beta)\colon\ I \to I'$ consisting of



1. a functor $\Phi\colon\ Sig \to Sig'$,
2. a natural transformation $\alpha\colon\ Sen \Rightarrow \Phi;Sen'$, and
3. a natural transformation $\beta\colon\ \Phi^{\mathrm{op}};Mod' \Rightarrow Mod$

such that the following *satisfaction condition* holds:

$$M' \models'_{\Phi\Sigma} \alpha_\Sigma e \ \text{ if and only if }\ \beta_\Sigma M' \models_\Sigma e$$

for any signature $\Sigma \in |Sig|$, for any $\Phi(\Sigma)$-model M', and any Σ-sentence e. The category of institutions and their comorphisms is denoted by $co\mathbb{I}ns$.

Category theoretic thinking promotes the idea that the arrows are the primary concept rather than the objects. It is even possible to define the concept of category only using arrows, the objects being assimilated to the identity arrows. Institutions serve as a clear example of this since both $\mathbb{I}ns$ and $co\mathbb{I}ns$ have institutions as objects but have different classes of arrows, both classes having the same level of preservation of institutional structure. Therefore in principle, we cannot refer to the ‘category of institutions’, which does not make sense, instead we should refer to the ‘category of institution morphisms’ or to the ‘category of institution comorphisms’.

Kolmogorov’s embedding of ‘classical’ logic into intuitionistic logic. This non-trivial example of “embedding” comorphism has a special importance in logic: it shows that intuitionistic logic is more expressive than classical logic.

The semantic side of $\mathcal{PL}$ can be widened by considering models as valuations into arbitrary Boolean algebras rather than in the binary Boolean algebra. The resulting institution, which shares with $\mathcal{PL}$ the signatures and the sentences is denoted $\mathcal{BPL}$and called *Boolean propositional logic*. It can be shown (proof omitted) that for P any set, and for E and E' any sets of P-sentences, $E \models^{\mathcal{PL}} E'$ if and only if $E \models^{\mathcal{BPL}} E'$, meaning that both the standard and the arbitrary Boolean algebra semantics of propositional logic yield the same semantic consequence relation.

The “embedding” comorphism $(\Phi,\alpha,\beta)\colon\ \mathcal{BPL} \to \mathcal{IPL}$ is defined as follows:

– Φ is the identity, i.e. $\Phi = 1_{\mathbb{S}et}$.

– For each set P, α_P adds a double negation to each sub-sentence, i.e.

– $\alpha_P p = \neg\neg p$ for each $p \in P$,

– $\alpha_P(\neg\rho) = \neg\alpha_P\rho$ for each P-sentence ρ, and

– $\alpha_P(\rho_1 \otimes \rho_2) = \neg\neg(\alpha_P\rho_1 \otimes \alpha_P\rho_2)$ for any P-sentences ρ_1 and ρ_2 and any connective $\otimes \in \{\wedge, \vee, \Rightarrow\}$.

Note that α_P are injective.

– For each set P, $\beta_P(M : P \to A) = M; r_A$ for any Heyting algebra A and any function $M : P \to A$, where $r_A : A \to R(A)$ defined by $r_A a = \neg\neg a$ is the canonical map from A to the Boolean algebra $R(A)$ of the regular elements of A. Recall that $a \in A$ is *regular* if and only if $a = \neg\neg a$. According to the theory of Heyting algebras, $R(A)$ are Boolean algebras with $a \cap b = a \wedge b$, $a \cup b = \neg\neg(a \vee b)$, $(a \Rightarrow' b) = (a \Rightarrow b)$, $\neg' a = \neg a$, where $\cap, \cup, \Rightarrow', \neg'$ and $\wedge, \vee, \Rightarrow, \neg$ are the conjunctions, disjunctions, implications, and negations, in $R(A)$ and A, respectively.

The proof of the Satisfaction Condition for this comorphism relies upon the fact that $M(\alpha_P\rho) = (M; r_A)[\rho]$ for each P-sentence ρ, where $M(\alpha_P\rho)$ is the evaluation of $\alpha_P\rho$ in A and $(M; r_A)[\rho]$ is the evaluation of ρ in the *Boolean algebra* $R(A)$. Here it is important to note that in general $(M; r_A)[\rho] \neq r_A(M\rho)$. The equality $M(\alpha_P\rho) = (M; r_A)[\rho]$ can be shown by induction on the structure of ρ (details of the proof left to the reader), by relying crucially upon another important fact about regular elements in Heyting algebras, namely that r_A is a homomorphism of Heyting algebras.

Conservative comorphisms. Conservative comorphisms are important because in addition to preserving the semantic consequence between sentences (property valid for *any* comorphism), they also reflect the semantic consequence. A comorphism $(\Phi, \alpha, \beta) : I \to I'$ is *conservative* when for any $E, E' \subseteq Sen\Sigma$, $E \models_\Sigma E'$ if and only if $\alpha_\Sigma E \models'_{\Phi\Sigma} \alpha_\Sigma E'$. Very often in the applications this property follows from the following property which is much easier to establish. A comorphism $(\Phi, \alpha, \beta) : I \to I'$ has the *model expansion property* when for each I-signature Σ, β_Σ is surjective on the models, i.e., for each Σ-models M there exists a $\Phi\Sigma$-model M' such that $M = \beta_\Sigma M'$. For example both "embedding" comorphisms $\mathcal{EQL} \to \mathcal{FOL}$ and $\mathcal{BPL} \to \mathcal{IPL}$ presented above are conservative. In the former example β_Σ's are isomorphisms, while in the latter example we have that $R(A) = A$ for each Boolean algebra A.

Proposition 3.8. *Let $(\Phi, \alpha, \beta) : I \to I'$ be any comorphism. For any I-signature Σ and any sets of Σ-sentences E and E' we have that*

1. *$E \models_\Sigma E'$ implies $\alpha_\Sigma E \models'_{\Phi\Sigma} \alpha_\Sigma E'$.*
2. *If (Φ, α, β) has the model expansion property then it is conservative.*

Proof. 1. For any $\Phi\Sigma$-model M' such that $M' \models' \alpha_\Sigma E$, by the Satisfaction Condition of the comorphism we have that $\beta_\Sigma M' \models_\Sigma E$. By the hypothesis, it follows that $\beta_\Sigma M' \models_\Sigma E'$. By the Satisfaction Condition of the comorphism again (this time in the other direction) we obtain that $M' \models'_{\Phi\Sigma} E'$.

2. Let us assume that $\alpha_\Sigma E \models'_{\Phi\Sigma} \alpha_\Sigma E'$ and consider any Σ-model M of E. Since (Φ, α, β) has the model expansion property, there exists a $\Phi\Sigma$-model M' such that $M = \beta_\Sigma M'$. By the Satisfaction Condition for the comorphism, we have that $M' \models'_{\Phi\Sigma} \alpha_\Sigma E$. By the hypothesis it follows that $M' \models'_{\Phi\Sigma} \alpha_\Sigma E'$. By the Satisfaction Condition again we get that $M = \beta_\Sigma M' \models_\Sigma E'$. □

The adjoint relationship. Often the forgetful nature of many functors can be captured formally by the concept of right adjoint functor. For example, the embedding of the algebraic signatures into the $\mathcal{FOL}$ signatures is a left adjoint to the forgetful functor from the $\mathcal{FOL}$ signatures to the algebraic signatures. The following general theorem shows that the 'embedding' comorphism $\mathcal{EQL} \to \mathcal{FOL}$ and the 'forgetful' morphism $\mathcal{FOL} \to \mathcal{EQL}$ determine each other, their interdependency being caused by the adjunction between their categories of signatures.

Theorem 3.9. *An adjunction $(\Phi, \overline{\Phi}, \zeta, \overline{\zeta})$ between the categories of signatures*[3] *of institutions I and I' determines a canonical bijection between institution morphisms $(\Phi, \alpha, \beta) : I' \to I$ and institution comorphisms $(\overline{\Phi}, \overline{\alpha}, \overline{\beta}) : I \to I'$ given by the following equalities:*

– $\overline{\alpha} = \zeta Sen; \overline{\Phi}\alpha$ *and* $\overline{\beta} = \overline{\Phi}^{\mathrm{op}}\beta; \zeta^{\mathrm{op}} Mod$*, and*

– $\alpha = \Phi\overline{\alpha}; \overline{\zeta} Sen'$ *and* $\beta = \overline{\zeta}^{\mathrm{op}} Mod'; \Phi^{\mathrm{op}}\overline{\beta}$.

The proof of this theorem follows by routine calculations, which are left as an exercise for the reader.

An institution morphism or comorphism is called *adjoint* when this is part of a morphism-comorphism duality determined by an adjunction between the categories of signatures. Notice that the composition of institution adjoints is still an adjoint. Let $e\mathbb{I}ns$ denote the category of pairs of institution adjoint morphism-comorphism.

Equivalence of institutions. As in the case of categories, the equivalence concept for institutions captures the fact that the institutions are the 'same', while being weaker than isomorphism. This concept is also an example of an adjoint institution morphism. An institution morphism (Φ, α, β) is an *equivalence of institutions* when

- Φ is an equivalence of categories,
- α_Σ has an inverse up to semantic equivalence, denoted α'_Σ, such that α' is a natural transformation, and
- β_Σ is an equivalence of categories, such that its inverse up to isomorphism and the corresponding isomorphism natural transformations are natural in Σ.

[3] $\Phi : Sig' \to Sig$ is the right adjoint, $\overline{\Phi}$ is the left adjoint, ζ is the unit, and $\overline{\zeta}$ is the co-unit of the adjunction.

Institution encodings

There is a class of comorphisms, very useful in applications, which are generally not adjoints. Rather than giving the flavour of an 'embedding', they are in fact 'encodings' of more complex institutions into simpler ones. We give now a couple of examples.

Encoding relations as operations in $\mathcal{FOL}$. This example formalizes the basic intuition in logic that relations can be simulated by pseudo-Boolean-valued operations. We may map each $\mathcal{FOL}$ signature (S,F,P) to an algebraic signature $(S+\{\mathbf{b}\},F+\overline{P}+\{\mathbf{true}\})$ where $\mathbf{b}$ is a (new) sort, $\mathbf{true}$ is a (new) constant of sort $\mathbf{b}$, and for each arity $w \in S^*$, $\overline{P}_{w\to s} = P_w$ if $s = \mathbf{b}$ and $\overline{P}_{w\to s} = \emptyset$ otherwise. This determines an institution comorphism $\mathcal{FOL} \to \mathcal{FOEQL}$ which

– maps each relational atom πt to the equation $\pi t = \mathbf{true}$, and

– maps each $(S+\{\mathbf{b}\},F+\overline{P}+\{\mathbf{true}\})$-algebra A to the (S,F,P)-model βA maintaining the interpretations of the sorts and F-operations of A but $(\beta A)_\pi = A_\pi^{-1}(A_{\mathbf{true}})$ for each relation symbol π.

We leave to the reader the task of developing the details of the definition of this comorphism and its Satisfaction Condition.

Encoding modalities in relational logic. Let $\mathcal{REL}^1$ be the single-sorted variant of $\mathcal{REL}$. We may build a comorphism $(\Phi,\alpha,\beta): \mathcal{MPL}^* \to \mathcal{REL}^1$ as follows:

- Each $\mathcal{MPL}$-signature, i.e., set P, gets mapped to the single-sorted relational signature without constants $(\emptyset,\overline{P})$ where
  - $\overline{P}_1 = P$, $\overline{P}_2 = \{r\}$, and $\overline{P}_n = \emptyset$ for $n \notin \{1,2\}$,
- each $(\emptyset,\overline{P})$-model M gets mapped to the Kripke P-model $\beta M = (W,\overline{M})$ with $|W|$ being the carrier set of M, $W_\lambda = M_r$, and $\overline{M}^w = \{\pi \mid w \in M_\pi\}$ for each $w \in |W|$, and
- for each P-sentence ρ, $\alpha\rho = (\forall x)\alpha_x\rho$ where $\alpha_x: Sen^{\mathcal{MPL}}P \to Sen^{\mathcal{REL}^1}(\{x\},\overline{P})$ is such that
  - $\alpha_x\pi = \pi x$ for each $\pi \in P$,
  - α_x commutes with the Boolean connectives, i.e., $\alpha_x(\rho_1 \wedge \rho_2) = \alpha_x\rho_1 \wedge \alpha_x\rho_2$, etc., and
  - $\alpha_x(\Box\rho) = (\forall y)(r(x,y) \Rightarrow \alpha_y\rho)$.

This example shows that modal propositional logic is a 'fragment' of ordinary (single-sorted) first-order logic.

Exercises

3.12. Borrowing the Satisfaction Condition along comorphisms
Given the following data:

1. a tuple $(Sig, Sen, Mod, \models)$ that besides the Satisfaction Condition satisfies all the other axioms of an institution,
2. an institution $(Sig', Sen', Mod', \models')$, and
3. a triple (Φ, α, β) that satisfies all axioms of a comorphism $(Sig, Sen, Mod, \models) \to (Sig', Sen', Mod', \models'$ $)$ with the model expansion property.

we have that $(Sig, Sen, Mod, \models)$ is an institution.

3.13. Morphism $\mathcal{FOL} \to \mathcal{REL}$
Each $\mathcal{FOL}$ signature (S, F, P) can be mapped to a $\mathcal{FOL}$ signature $(S, C(F), \overline{F} \cup P)$ without non-constant operation symbols, where $C(F)$ is the set of constants of F, $\overline{F}_s = \emptyset$ for each sort $s \in S$, and $\overline{F}_{ws} = F_{w \to s}$ when w is non-empty.
This determines a non-adjoint institution morphism $\mathcal{FOL} \to \mathcal{REL}$. (*Hint:* For each (S, F, P)-model M and each $\sigma \in \overline{F}_{ws}$ where w is non-empty, $(\beta M)_\sigma = \{\langle m, M_\sigma m \rangle \mid m \in M_w\}$ and $\alpha(\sigma(x, y)) = (\sigma x = y)$ for each $\sigma \in \overline{F}$.)

3.14. Morphism $\mathcal{PA} \to \mathcal{FOL}$
There exists a forgetful institution morphism $\mathcal{PA} \to \mathcal{FOL}$ which forgets the partial operations. Is this an adjoint morphism?

3.15. Morphisms $\mathcal{FOL} \to \mathcal{PL}$
The embedding of $\mathcal{PL}$ into $\mathcal{FOL}$ that regards each $\mathcal{PL}$ signature as a $\mathcal{FOL}$ signature without sorts is an adjoint comorphism. Describe its dual institution morphism $\mathcal{FOL} \to \mathcal{PL}$.
Another institution morphism $\mathcal{FOL} \to \mathcal{PL}$ maps any $\mathcal{FOL}$ signature to its set of sentences regarded as a $\mathcal{PL}$ signature. Develop its full definition and show that this is not an adjoint morphism. (*Hint:* the signature functor does not preserve products.)

3.16. Morphism $\mathcal{FOL} \to \mathcal{MFOL}$
There exists an institution morphism $\mathcal{FOL} \to \mathcal{MFOL}$ which maps any $\mathcal{FOL}$ signature (S, F, P) to the $\mathcal{MFOL}$ signature (S, S, F, F, P, P), such that α erases the modalities $\Box$ and $\Diamond$ from the sentences, and $\beta M = (W, R)$ such that $I_W = \{*\}$, $W^* = M$, $R = \{\langle *, * \rangle\}$.

3.17. Morphism $\mathcal{POA} \to \mathcal{FOL}$
There exists a forgetful institution morphism $\mathcal{POA} \to \mathcal{FOL}$ which forgets the preorder structure both syntactically and semantically.

3.18. Morphism $\mathcal{PL} \to \mathcal{IPL}$
There exists a trivial adjoint institution morphism $\mathcal{PL} \to \mathcal{IPL}$ which regards the standard two-element Boolean algebra as a Heyting algebra.

3.19. Morphism $\mathcal{MA} \to \mathcal{POA}$
Each multialgebra operation determines an ordinary algebra operation on the powerset of the carrier of the multialgebra. This determines a preordered algebra.

1. Adjust the concept of homomorphism of multialgebras such that the mapping of multialgebras to preorder algebras is functorial.

2. By mapping each preorder atom $t \leq t'$ to its corresponding inclusion sentence $t \prec t'$ and each equation $t = t'$ to the conjunction of the inclusions $t \prec t'$ and $t' \prec t$ we obtain an adjoint morphism of institutions $\mathcal{MA} \to \mathcal{POA}$.

3.20. Morphism $\mathcal{FOL} \to \mathcal{MA}$
Each $\mathcal{FOL}$-model can be canonically regarded as a 'deterministic' multialgebra, i.e., in which all operations are deterministic. This determines an institution morphism $\mathcal{FOL} \to \mathcal{MA}$ which at the level of sentences maps both deterministic equations $t \doteq t'$ and inclusions $t \prec t'$ to equations $t = t'$.

3.21. Morphism $\mathcal{CA} \to \mathcal{FOL}$
There exists a forgetful institution morphism $\mathcal{CA} \to \mathcal{FOL}$ mapping each contraction algebra to its underlying algebra and each equation $t = t'$ to the approximation equation $t \approx_0 t'$.

3.22. Morphism and comorphism $\mathcal{MBA} \to \mathcal{FOL}$
Each membership algebraic signature (S, K, F, kind) determines a $\mathcal{FOL}$ signature (K, F, P) where $P = \{\_ : s \mid s \in S\}$ such that $\_ : s \in P_{\text{kind}(s)}$ for each sort s. This determines both an institution morphism and an institution comorphism $\mathcal{MBA} \to \mathcal{FOL}$. (*Hint: $Mod^{\mathcal{MBA}}$*(S, K, F, kind) is canonically isomorphic to $Mod^{\mathcal{FOL}}(K, F, P)$ by mapping each (S, K, F, kind)-algebra A to the (K, F, P)-model with $A_{(\_ : s)} = A_s$, and $Sen^{\mathcal{FOL}}(K, F, P)$ is canonically isomorphic to $Sen^{\mathcal{MBA}}(S, K, F, \text{kind})$ by mapping each atomic relation $t : s$ to the atomic membership $t : s$ and by mapping equations to themselves.)

3.23. Comorphism $\mathcal{AUT} \to \mathcal{FOL}^1$
Any set V determines a $\mathcal{FOL}^1$ signature $(F = V + \{0\}, P = \{final\})$ such that $F_0 = \{0\}$, $F_1 = V$, and $P_1 = \{final\}$. This can be extended to a functor $\mathbb{S}et \to Sig^{\mathcal{FOL}^1}$ which constitutes the signature functor Φ for a comorphism $\mathcal{AUT} \to \mathcal{FOL}^1$.

3.24. Comorphism $\mathcal{HNK} \to \mathcal{HOL}$
The inclusion of model categories $Mod^{\mathcal{HOL}}(S, F) \subseteq Mod^{\mathcal{HNK}}(S, F)$ determines a canonical comorphism $\mathcal{HNK} \to \mathcal{HOL}$. Note this does not have the flavour either of an 'embedding' or of an 'encoding'.

3.25. Comorphism $\mathcal{FOEQL} \to \mathcal{HNK}$
Each algebraic signature (S, F) can be regarded as a $\mathcal{HOL}$-signature by defining the type of σ as $s_1 \to (s_2 \to \ldots (s_n \to s) \ldots)$ for each operation symbol $\sigma \in F_{s_1 \ldots s_n \to s}$. Then each (S, F)-term $\sigma(t_1, \ldots, t_n)$ can be mapped to its 'Polish prefix translation', the $\mathcal{HOL}$ (S, F)-term $\alpha\, \sigma(t_1, \ldots, t_n) = (\ldots(\sigma\ \alpha t_1 \ldots \alpha t_n))$. This determines a canonical institution comorphism $\mathcal{FOEQL} \to \mathcal{HNK}$. By using the encoding of relations as operations, this can be extended to an institution comorphism $\mathcal{FOL} \to \mathcal{HNK}$.

3.26. Comorphism $\mathcal{HOL} \to \mathcal{HOL}_\lambda$
There is a 'natural embedding' comorphism from $\mathcal{HOL}$ to $\mathcal{HOL}_\lambda$, and also a similar comorphism from $\mathcal{HNK}$ to $\mathcal{HNK}_\lambda$, such that the translation α on the sentences is defined as follows:

- any equation $t = t'$ is mapped to the term $t = t'$ of Ω,
- $(\forall X)\rho$ is mapped to $\lambda X.\rho = \lambda X.true$,
- $\neg e$ is mapped to $e = false$,
- $e_1 \wedge e_2$ is mapped to $\langle e_1, e_2 \rangle = \langle true, true \rangle$,

where *true* abbreviates $\lambda x{:}\Omega.x = \lambda x{:}\Omega.x$ and *false* abbreviates $(\forall x{:}\Omega)x$.

3.27. [22] **Comorphism $\mathcal{PL} \to \mathcal{WPL}$**
Let *weak propositional logic* (denoted $\mathcal{WPL}$) designate a variant of $\mathcal{PL}$, where the sentences are the same as in $\mathcal{PL}$, but the models are valuations $M\colon\ Sen(P) \to \{0,1\}$ of *all* sentences that respect the usual truth table semantics of all the Boolean connectives except for negation, for which they respect only one-half of the usual condition:

- $M(\rho_1 \wedge \rho_2) = 1$ if and only if both $M\rho_1 = 1$ and $M\rho_2 = 1$, $M(\rho_1 \vee \rho_2) = 0$ if and only if both $M\rho_1 = 0$ and $M\rho_2 = 0$, $M(\rho_1 \Rightarrow \rho_2) = 1$ if and only if $M\rho_1 = 0$ or $M\rho_2 = 1$, and
- $M(\neg\rho) = 0$ if $M\rho = 1$.

There exists a comorphism $\mathcal{PL} \to \mathcal{WPL}$ such that the sentence translations are defined by $\alpha_P\pi = \pi$ for $\pi \in P$, $\alpha_P(\rho_1 \otimes \rho_2) = \alpha_P\rho_1 \otimes \alpha_P\rho_1$ for $\otimes \in \{\wedge,\vee,\Rightarrow\}$, and $\alpha_P(\neg\rho) = \alpha_P\rho \Rightarrow \neg\alpha_P\rho$ and such that the models are translated by $\beta_P M' = \{\pi \mid M'\pi = 1\}$.

3.28. ***S*-sorted $\mathcal{FOL}$**
For any fixed set S, let $\mathcal{FOL}^S = (Sig^S, Sen^S, Mod^S, \models)$ be the institution of *S-sorted first-order logic* defined as the sub-institution of $\mathcal{FOL}$ determined by the subcategory Sig^S of the signatures with S-sorted operation and relation symbols. (A signature in Sig^S is just a $\mathcal{FOL}$ signature (S,F,P), and a signature morphism φ in Sig^S is identity on the sort symbols, i.e., $\varphi^{\mathrm{st}} = 1_S$.)

1. Each function $u\colon\ S \to S'$ determines a canonical 'forgetful' adjoint institution morphism $(\Phi^u,\alpha^u,\beta^u)\colon\ \mathcal{FOL}^{S'} \to \mathcal{FOL}^S$ such that for each signature (S',F',P') of S'-sorted operation and relation symbols, $\Phi^u(S',F',P') = (S,F,P)$ with $F_{w\to s} = F'_{uw\to u(s)}$ and $P_w = P'_{uw}$ for each arity $w \in S^*$ and each sort symbol $s \in S$.
2. Describe the institution comorphism $(\overline{\Phi^u},\overline{\alpha^u},\overline{\beta^u})\colon\ \mathcal{FOL}^S \to \mathcal{FOL}^{S'}$ adjoint to $(\Phi^u,\alpha^u,\beta^u)$. Show that $\overline{\alpha^u}$ is a bijection when u is injective.

3.29. Exercise 3.8 continued
For each institution $(Sig,Sen,Mod,\models)$ there exists a forgetful adjoint institution morphism $(Sig^{\to}, Sen^{\to}, Mod^{\to}, \models^{\to}) \to (Sig,Sen,Mod,\models)$ which maps each signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ to its domain signature Σ.

3.30. [181] **Charter morphisms (Ex. 3.11 continued)**
Define the concept of *charter morphism* and show that there is a functor from the category of charters to $\mathbb{I}ns$. Does this have a left adjoint?

3.31. For any adjoint pair of institution morphism (Φ,α,β) and institution comorphism $(\overline{\Phi},\overline{\alpha},\overline{\beta})$ between the institutions I and I' corresponding to an adjunction $(\Phi,\overline{\Phi},\zeta,\overline{\zeta})$ between their categories of signatures, the following squares commute:

$$
\begin{array}{ccc}
Sen\Sigma & \xrightarrow{Sen\varphi} & Sen(\Phi\Sigma') \\
{\scriptstyle \overline{\alpha}_\Sigma}\downarrow & & \downarrow{\scriptstyle \alpha_{\Sigma'}} \\
Sen'(\overline{\Phi}\Sigma) & \xrightarrow[Sen'\overline{\varphi}]{} & Sen'\Sigma'
\end{array}
\qquad\qquad
\begin{array}{ccc}
Mod\Sigma & \xleftarrow{Mod\varphi} & Mod(\Phi\Sigma') \\
{\scriptstyle \overline{\beta}_\Sigma}\uparrow & & \uparrow{\scriptstyle \beta_{\Sigma'}} \\
Mod'(\overline{\Phi}\Sigma) & \xleftarrow[Mod'\overline{\varphi}]{} & Mod'\Sigma'
\end{array}
$$

for each signature morphism $\varphi\colon\ \Sigma \to \Phi\Sigma'$ and $\overline{\varphi}\colon\ \overline{\Phi}\Sigma \to \Sigma'$ such that $\varphi = \zeta_\Sigma;\Phi\overline{\varphi}$.

3.4 Institutions as functors

The definition of the concept of institution given so far supports very well model-theoretic intuitions. In this section, we give an alternative more categorical definition for institutions.

Rooms. A *room* is a triple (S,M,R) such that S is a set, M is a category, and R is a function $|M| \to [S \to 2]$ where (as usual) $|M|$ is the class of the objects of M and $[S \to 2] = \mathbb{S}et(S,2) = \{f : S \to 2 \mid f \text{ function}\}$ with 2 denoting the set $\{0,1\}$.

A *rooms morphism* $(s,m) : (S',M',R') \to (S,M,R)$ consists of a function $s : S \to S'$ and a functor $m : M' \to M$ such that the diagram below commutes:

$$\begin{array}{ccc} |M'| & \xrightarrow{R'} & [S' \to 2] \\ {\scriptstyle |m|}\downarrow & & \downarrow{\scriptstyle (s;-)} \\ |M| & \xrightarrow[R]{} & [S \to 2] \end{array}$$

where $|m|$ is the 'discretization' of m, i.e., the mapping on the objects given by m, and $(s;-)f = s;f$ for each function $f : S' \to 2$.

Let $\mathbb{R}oom$ be the category of rooms and their morphisms.

Proposition 3.10. *$\mathbb{R}oom$ has all small limits.*

Proof. Because $\mathbb{S}et^{\mathrm{op}}$ has all small limits (since $\mathbb{S}et$ has all small co-limits) and the (contravariant) homomorphism-functor $\mathbb{S}et(-,2) : \mathbb{S}et^{\mathrm{op}} \to \mathbb{S}et$ preserves them, by Prop. 2.3 we obtain that the comma-category $A/Sen(-,2)$ has all small limits. Moreover, it is easy to see that for each function $f : A \to B$, the induced functor $B/\mathbb{S}et(-,2) \to A/\mathbb{S}et(-,2)$ preserves these limits.

This means that the indexed category $(|-|)/\mathbb{S}et(-,2) : \mathbb{C}at^{\mathrm{op}} \to \mathbb{C}at$ mapping each category M to $|M|/\mathbb{S}et(-,2)$ satisfies the hypotheses of the limit part of Thm. 2.10. It follows that its Grothendieck category $((|-|)/\mathbb{S}et(-,2))^{\sharp}$, which is just $\mathbb{R}oom$, has all small limits. □

Institutions as functors. Let Sig be any category and $I : Sig^{\mathrm{op}} \to \mathbb{R}oom$ a functor. If we write

- $I\Sigma = (Sen\Sigma, Mod\Sigma, \models_{\Sigma})$ for each object $\Sigma \in |Sig|$, and
- $I\varphi = (Sen\varphi, Mod\varphi)$ for each arrow $\varphi \in Sig$,

then it is easy to see that $(Sig, Sen, Mod, \models)$ is an institution. The converse is also true, institutions are exactly the functors $Sig^{\mathrm{op}} \to \mathbb{R}oom$.

Any functor $\Phi : Sig' \to Sig$ induces a canonical functor

$$(\Phi^{\mathrm{op}};-) : \mathbb{C}at((Sig')^{\mathrm{op}}, \mathbb{R}oom) \to \mathbb{C}at(Sig^{\mathrm{op}}, \mathbb{R}oom).$$

This gives an indexed category $\mathbb{C}at((-)^{\mathrm{op}}, \mathbb{R}oom) : \mathbb{C}at^{\mathrm{op}} \to \mathbb{C}at$.

Fact 3.11. *The category $\mathbb{I}ns$ of the institution morphisms is the Grothendieck category $\mathbb{C}at((-)^{\mathrm{op}}, \mathbb{R}oom)^{\sharp}$.*

Now we have already collected all necessary ingredients to showing easily the completeness property of $\mathbb{I}ns$.

Corollary 3.12. *$\mathbb{I}ns$ has all small limits.*

Proof. Because $\mathbb{R}oom$ has small limits (cf. Prop. 3.10), by Prop. 2.2 we have that each $\mathbb{C}at(Sig^{\mathrm{op}}, \mathbb{R}oom)$ has all small limits. Note that each functor $(\Phi^{\mathrm{op}};-)$ preserves these limits. Since $\mathbb{C}at$ has all small limits we can apply now the limit part of Thm. 2.10. □

Exercises

3.32. The Satisfaction Condition for institutions is equivalent to the satisfaction relation $\models_{\Sigma} : |Mod\Sigma| \to [Sen\Sigma \to 2]$ being a natural transformation:

$$
\begin{array}{ccc}
\Sigma & \qquad & Mod\Sigma \xrightarrow{\models_{\Sigma}} [Sen\Sigma \to 2] \\
\Big\downarrow{\scriptstyle \varphi} & & {\scriptstyle Mod\varphi}\Big\uparrow \qquad\qquad \Big\uparrow{\scriptstyle (Sen\varphi)^{-1}} \\
\Sigma' & & Mod\Sigma' \xrightarrow[\models_{\Sigma'}]{} [Sen\Sigma' \to 2]
\end{array}
$$

3.33. [230, 130]

1. $\mathbb{R}oom$ has small co-limits.
2. $co\mathbb{I}ns$ has small limits.

3.34. [62] $\mathbb{R}oom$ can be enriched with a 2-categorical structure given by the natural transformations $m_1 \Rightarrow m_2$ between the 'model components' of room homomorphisms $(s_1, m_1), (s_2, m_2) : (S', M', R') \to (S, M, R)$. Then the 2-categorical structure on $\mathbb{I}ns$ given by the institution modifications arises as a Grothendieck 2-categorical construction.

3.35. [187] Each institution comorphism $(\Phi, \alpha, \beta) : (Sig, Sen, Mod, \models) \to (Sig', Sen', Mod', \models')$ determines a *span* of institution morphisms

$$(Sig, Sen, Mod, \models) \longleftarrow (Sig,\ \Phi;Sen',\ \Phi^{\mathrm{op}};Mod',\ \models) \longrightarrow (Sig', Sen', Mod', \models').$$

In a category $\mathbb{C}$, two spans $A \xleftarrow{f_1} B_1 \xrightarrow{g_1} A'$ and $A \xleftarrow{f_2} B_2 \xrightarrow{g_2} A'$ are *equivalent* when there exists an isomorphism $i : B_1 \to B_2$ such that $f_1 = i; f_2$ and $g_1 = i; g_2$. In any category with pullbacks, equivalence classes of spans (denoted $[\_]$) can be composed as follows:

$$[A \xleftarrow{f} B \xrightarrow{g} A'];[A' \xleftarrow{f'} B' \xrightarrow{g'} A''] = [A \xleftarrow{h;f} B_0 \xrightarrow{h';g'} A'']$$

where $B \xleftarrow{h} B_0 \xrightarrow{h'} B'$ is a pullback of $B \xrightarrow{g} A' \xleftarrow{f'} B'$. This yields a category $span(\mathbb{C})$ having the same objects as $\mathbb{C}$ but (equivalence class of) spans as arrows. Show that the construction of a span of morphisms from an institution comorphism is *functorial*, i.e., it yields a functor $co\mathbb{I}ns \to span(\mathbb{I}ns)$.

Notes. The origins of institution theory are within the theory of algebraic specification, the seminal work being [124].

$\mathcal{FOL}$ was first presented as an institution in [124]. That early capture of $\mathcal{FOL}$ as an institution differs from ours especially in how the variables are considered, their approach is global while ours is local. The global treatment of variables owes to the logic tradition and had led to mathematically incorrect definitions of the $\mathcal{FOL}$ institution due to the possibility of clashes between variables and constants in signatures, a situation that may result in a series of undesirable consequences, such as the failure of the Satisfaction Condition. This style of defining the $\mathcal{FOL}$ institution was all pervasive in the rather vast institution theory literature until the publication [76]. However, even after [76] there is still a lack of awareness about the incompatibility between the 'naive'[4] approach to variables from traditional logic and the purpose to define institutions rigorously. It is interesting that the local approach to variables employed by our definition of $\mathcal{FOL}$ matches very well the way variables are implemented in a series of algebraic specification and programming systems. But the message we get from this is that the capture of logical systems rigorously as institutions is non-trivial and in the process some aspects that are traditionally treated 'naively' in logic require an intricate mathematical treatment in line with the rigour characteristic to the axiomatic style of institution theory. Moreover, our $\mathcal{FOL}$ definition puts forward the perspective of variables-as-constants, which is contrary to the traditional view of variables whose separation from the constants is imposed globally (i.e., across all signatures) by designation. With variables-as-constants the semantics of the quantifiers can be defined only in terms of model reducts, which is essential for the development of institution-independent model theory. We will see later in the book what this means and also how this approach greatly benefits the applications of institution-independent model theory to concrete logics.

There are many approaches to partial algebra, two classical references being [36, 208], however, it has been organized as the institution $\mathcal{PA}$ presented here in [186]. Preorder algebras ($\mathcal{POA}$) are used for formal specification and verifications of algorithms [94], for automatic generation of case analysis [94], and in general for reasoning about transitions between states of systems. $\mathcal{POA}$ constitutes an unlabeled form of Meseguer's rewriting logic [176], but the latter fails to be an institution. The institution of multialgebras has been studied in [159]. Our multialgebra homomorphisms are called 'weak' homomorphisms in the literature, for alternative notions of multialgebra homomorphisms see [242]. Membership algebra has been introduced by [177]. For a historical overview of modal logics we suggest [133], while [29] gives rather a complete presentation of modern modal propositional logic. The hybrid versions of modal logics originate with the work in [207], other important references being [28, 35]. Standard modal logic was first captured as an institution in [225]. Higher-order logic with Henkin semantics has been introduced and studied in [43, 145], a recent book on the topic being [11]. Here, in the main text, we consider a simplified variant close to the work of [178], while some exercises consider a more sophisticated version containing products and λ-abstraction in a variant very close to the original one. Contraction algebras have been introduced in [56] in the context of the extension of logic programming to infinite terms. The institution $Cat\mathcal{EQL}$ of categorical equational logic is a slightly more abstract version of the institution of the so-called 'category-based equational logic' of [117, 58]. Formal intuitionistic logic was developed by seminal works [113, 148, 156] while Heyting algebras emerged from the so-called 'closure algebras' or 'Browerian algebras' investigated by McKinsey and Tarski. A categorical approach to intuitionistic logic can be found in [158]. Investigations of many-valued logics have a long history [114, 164, 206] and still constitute an active area of research. Fuzzy sets have been introduced in

[4] In the same sense like 'naive' set theory [142] is opposed to axiomatic set theory.

[245] (see also [115, 119]) as a formalism for dealing with vagueness in systems engineering and they constitute the origin of the so-called fuzzy logic approach. Many-valued logic has been presented for the first time as an institution in [2] in a slightly different way than here. A very brief list of logics from formal specification theory that have been captured as institutions but have not been presented here includes polymorphic [220], temporal [107], process [107], behavioral [26], coalgebraic [45], object-oriented [125] logics.

Due to its abstract definition, institutions may accommodate examples which might appear as 'non-logical', at least in the conventional sense. While some of them are only mildly 'non-logical' (automata, linear algebra), much less conventional examples appear in myriad ways including abstract constructions on (already existing) institutions. In the book [219] we can find some interesting 'non-logical' institutions for programming, such as an institution for functional programming, and another one for imperative programming.

Institution morphisms were introduced in [124], while comorphisms were studied later under the name of "plain map" in [175] or "representation" in [229, 231]. The literature studies many other types of mappings between institutions, each of them playing a specific role in applications. The name "comorphism" was introduced by [130]. The duality between institution morphisms and comorphisms was established in [13]. In [60, 62, 95] institution adjoints are called "embeddings". Notice also that institution adjoint morphism or comorphism are *not* adjunctions in the 2-categorical sense. Equivalences of institutions have been introduced in [191].

Kolmogorov's translation of 'classical' propositional logic into intuitionistic propositional logic by adding a double negation to each sub-sentence was originally introduced in [156], and in the form of a comorphism was presented in [190]. The encoding of modal logic into $\mathcal{FOL}$ is due to van Bentham [238]. The encoding of relations as operations was introduced in [55] and was used in [57] for reducing ordinary logic programming to equational logic programming.

The presentation of institutions as functors was given already in [124] and the 2-categorical structure of the category of institutions has been studied in [62]. Completeness of $\mathbb{I}ns$ was first obtained by Tarlecki in [226] and of $co\mathbb{I}ns$ in [230]. Both results have been re-done by Roşu using Kan extensions [211]. Cocompleteness fails for both $\mathbb{I}ns$ and $co\mathbb{I}ns$ due to foundational issues (see [130] for a counterexample originally due to Tarlecki) but it can still be recovered under the condition that the categories of the signatures are small.

Chapter 4

Theories and Models

In this chapter, we will continue with the development of some fundamental institution theoretic concepts that play an important role for our institution-independent approach to model theory.

The concepts of (logical) theory and its semantic closure play an important role in the semantics of formal / algebraic specification but also in institution encodings supporting the transfer of model-theoretic properties between institutions. These applications are conceptually facilitated by an abstract construction of an 'institution of theories' over an arbitrary institution which relies on the crucial concept of morphisms between theories. The foundations of these developments lie in the Galois connection between the syntax and the semantics of institutions, a property that can be presented at the fully abstract level without any additional assumptions.

Theory co-limits are especially useful in formal specification theory since they provide support for advanced modularization techniques of software systems. They are also required within the context of some institution encodings.

Model amalgamation, here introduced as a limit preservation property of the model functor, is the institutional property which is required by almost all institution-independent model-theoretic developments. Even the Satisfaction Condition of institutions capturing logics with quantifiers may rely upon a form of model amalgamation. In fact when we proved the Satisfaction Condition of $\mathcal{FOL}$ we have already used that implicitly, but now we will become aware of this property explicitly and at the abstract level. The institution theoretic concept of model amalgamation is a rather basic property of institutions which should not be confused with the existence of a common (elementary) extension of models, a much harder property playing an important role in conventional model theory.

The method of diagrams pervades a large part of the conventional model theory and, in its abstract institution-independent form, also of the results presented in this book. At the level of abstract institutions this appears as a coherence property between the syntactic and semantic sides of the institution, which gets a simple categorical formulation. As a consequence of the method of diagrams, we develop a general result about the existence of co-limits of models.

The concepts of 'sub-model' and 'quotient model' are handled by the so-called 'inclusion systems', which constitute a categorical abstraction of their basic factorization property. Another rather different application domain for inclusion systems are the signature morphisms, which is especially relevant for the studies of modularization properties of formal specification.

The last topic of this chapter is that of the free constructions of models along theory morphisms, called 'liberality' in institution theory. Liberality is intimately related to good computational properties of the actual institutions and it plays a crucial role in the semantics of abstract data types and logic programming. In its simple form, liberality means the existence of initial models for theories, a property which holds for Horn theories.

4.1 Theories and their morphisms

A Galois connection between syntax and semantics. Let Σ be a signature in an institution $(Sig, Sen, Mod, \models)$. Then, for a signature Σ, each set E of Σ-sentences is called a Σ-*theory* and may be denoted (Σ, E). We define:

– For each theory (Σ, E), $E^* = \{M \in Mod\Sigma \mid M \models_\Sigma e \text{ for each } e \in E\}$.

– For each class $\mathbb{M}$ of Σ-models, $\mathbb{M}^* = \{e \in Sen\Sigma \mid M \models_\Sigma e \text{ for each } M \in \mathbb{M}\}$.

For any individual sentence or model X, by X^* we mean $\{X\}^*$. These two functions, denoted "$(-)^*$", form what is known as a *Galois connection* (see Sect. 2.3). This means that they satisfy the following easy-to-check properties for any collections E, E' of Σ-sentences and collections $\mathbb{M}, \mathbb{M}'$ of Σ-models:

1. $E \subseteq E'$ implies $E'^* \subseteq E^*$.
2. $\mathbb{M} \subseteq \mathbb{M}'$ implies $\mathbb{M}'^* \subseteq \mathbb{M}^*$.
3. $E \subseteq E^{**}$.
4. $\mathbb{M} \subseteq \mathbb{M}^{**}$.

Closed classes of models $\mathbb{M} = \mathbb{M}^{**}$ are called *elementary* and E is a *closed theory* when $E = E^{**}$.

The above properties 1–4 imply quite immediately the following properties:

5. $E^* = E^{***}$.
6. $\mathbb{M}^* = \mathbb{M}^{***}$.
7. There is a dual (i.e., inclusion reversing) isomorphism between closed theories and elementary classes of models.

Note that for E and E' sets of sentences, $E^* \subseteq E'^*$ means $E \models E'$. Two sentences e and e' of the same signature are *semantically equivalent* (denoted as $e \models\!\mid e'$) if they are satisfied by the same class of models, i.e., $\{e\} \models \{e'\}$ and $\{e'\} \models \{e\}$.

Two models M and M' of the same signature are *elementarily equivalent* (denoted as $M \equiv M'$) if they satisfy the same set of sentences, i.e., $M^* = M'^*$.

Presentations of theories. A theory (Σ, E) is *presented by a theory* (Σ, Γ) when $\Gamma^* = E^*$.

Theory morphisms. A *theory morphism* is a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $\varphi E \subseteq E'^{**}$.

Proposition 4.1. *In any institution I, the theory morphisms form a category – denoted Th^I – with the composition inherited from the category of the signatures Sig^I.*

Proof. That composition of theory morphisms is a theory morphism follows by simple calculations using the observation that $\varphi E^{**} \subseteq (\varphi E)^{**}$ for each signature morphism $\varphi : \Sigma \to \Sigma'$ and theory (Σ, E). $\square$

Let *CTh* denote the full subcategory of *Th* determined by the closed theories. As we will see further in the book, in a few cases *CTh* is more meaningful than *Th*.

The institution of the theories. The model functor *Mod* of an institution can be extended from the category of its signatures *Sig* to a model functor from the category of its theories *Th*, by mapping a theory (Σ, E) to the full subcategory $Mod^{\text{th}}(\Sigma, E)$ of $Mod\Sigma$ consisting of all Σ-models satisfying E. The correctness of the definition of Mod^{th} is guaranteed by the Satisfaction Condition of the base institution; this is easy to check. This leads to the *institution of theories*

$$I^{\text{th}} = (Sig^{\text{th}}, Sen^{\text{th}}, Mod^{\text{th}}, \models^{\text{th}})$$

over the base institution $I = (Sig, Sen, Mod, \models)$ where

- Sig^{th} is the category *Th* of theories of I,
- $Sen^{\text{th}}(\Sigma, E) = Sen\Sigma$, and
- for each (Σ, E)-model M and Σ-sentence e, $M \models^{\text{th}}_{(\Sigma,E)} e$ if and only if $M \models_\Sigma e$.

This construction is very useful for institution encodings. Often, comorphisms encoding 'complex' institutions into 'simpler' ones map signatures of the 'complex' institution to *theories* of the 'simpler' institution. As comorphisms usually correspond to embeddings, from the point of view of the structural complexity of institutions this is quite an expected cost, since such a difference of complexity has to show up somewhere. The rest of this section is devoted to examples of such encodings. The reader is invited to complete the definitions given and to check all the details of each of these examples, including their Satisfaction Condition.

Encoding many-sorted logic into single-sorted logic

This is a comorphism $(\Phi, \alpha, \beta) : \mathcal{FOL} \to (\mathcal{FOL}^1)^{\text{th}}$ defined as follows:

- A many-sorted signature (S, F, P) gets mapped to the single-sorted theory $((\overline{F}, \overline{P} + \{(- : s) \mid s \in S\}), \Gamma_{(S,F,P)})$ where

- for each natural number n, $\overline{F}_n = \{\sigma \in F_{w\to s} \mid |w| = n\}$ and $\overline{P}_n = \{\sigma \in P_w \mid |w| = n\}$ (here by $|w|$ we denote the length of the string w),
- $\Gamma_{(S,F,P)} = \{(\forall x_1 \dots x_n) \bigwedge_{i\leq n}(x_i : \; s_i) \Rightarrow (\sigma(x_1 \dots x_n) : \; s) \mid \sigma \in F_{s_1 \dots s_n \to s}\}$.

• On the sentence side:
  - any equation $t = t'$ gets mapped to itself,
  - α commutes with the Boolean connectives, i.e., $\alpha(\rho_1 \wedge \rho_2) = \alpha\rho_1 \wedge \alpha\rho_2$, etc.,
  - any sentence of the form $(\forall x)\rho$ gets mapped to $(\forall x)(x : \; s) \Rightarrow \alpha\rho$ with s being the sort of the variable x.

• On the models side, for each (S,F,P)-model M
  - $(\beta M)_s = M_{(- : \; s)}$ for each sort s,
  - for each operation symbol $\sigma \in F_{w\to s}$, $(\beta M)_\sigma$ is the restriction of M_σ to $(\beta M)_w$, and
  - $(\beta M)_\pi = (\beta M)_w \cap M_\pi$ for each relation symbol $\pi \in P_w$.

This comorphism may give an insight into why and how the single-sorted approach of conventional mathematical practice works despite the fact that mathematical realities constitute a many-sorted heterogeneous rather than a single-sorted homogeneous framework.

Encoding operations as relations in $\mathcal{FOL}$

This is a comorphism $(\Phi, \alpha, \beta) : \; \mathcal{FOL} \to \mathcal{REL}^{\text{th}}$ defined as follows:

• Each $\mathcal{FOL}$ signature (S,F,P) gets mapped to a $\mathcal{REL}$-theory $((S, C(F), \overline{F} + P), rel_{(S,F,P)})$ where
  - $C(F)$ is the set of the constants of F,
  - $\overline{F}_s = \emptyset$ for each sort $s \in S$ and $\overline{F}_{ws} = F_{w\to s}$ when w is non-empty, and
  - $rel_{(S,F,P)} =$
    $\{((\forall X)(\exists y)\sigma(X,y)) \wedge ((\forall X,y,y')\sigma(X,y) \wedge \sigma(X,y') \Rightarrow (y = y')) \mid \sigma \in \overline{F}_{ws}\}$.

• On the sentence side:
  - $x = y$ gets mapped to itself when both x and y are constants,
  - $\alpha(\sigma(t_1,\dots,t_n) = y) = (\exists\{x_1,\dots,x_n\})(\sigma(x_1,\dots,x_n,y) \wedge \bigwedge_{1\leq i\leq n} \alpha(t_i = x_i))$ for each operation symbol σ, appropriate list of terms $t_1,\dots,t_n$ and $x_1,\dots,x_n$ variables (i.e., new constants),
  - $\alpha(t_1 = t_2) = (\exists y)(\alpha(t_1 = y) \wedge \alpha(t_2 = y))$ for any terms t_1 and t_2 of the same sort and y variable,
  - $\alpha\,\pi(t_1,\dots,t_n) = (\exists\{x_1,\dots,x_n\})(\pi(x_1,\dots,x_n) \wedge \bigwedge_{i\leq n} \alpha(t_i = x_i))$ for each relational atom $\pi(t_1,\dots,t_n)$, and

– α commutes with the Boolean connectives and with the quantifiers.

- On the models side, for each $((S, C(F), \overline{F} + P), rel_{(S,F,P)})$-model M,
  – for each relation symbol $\sigma \in \overline{F}$, $(\beta M)_\sigma m = y$ if and only if $\langle m, y\rangle \in M_\sigma$.

This encoding goes in a sense opposite to the encoding of the relations as operations presented in Sect. 3.3, that one being quite exceptional since it does map the signatures to signatures rather than to proper theories. In that case, the difference in structural complexity, which is rather slight, still shows up at the level of the signatures.

Encoding partial operations as total operations

The so-called '*operational encoding*' of $\mathcal{PA}$ into $\mathcal{FOL}$ is a comorphism $\mathcal{PA} \to \mathcal{FOL}^{\text{th}}$ defined as follows:

- Each $\mathcal{PA}$-signature (S, TF, PF) gets mapped to the $\mathcal{FOL}$-theory

$$((S, TF + PF, (D_s)_{s\in S}), \Gamma_{(S,TF,PF)})$$

  where

  – for each sort $s \in S$, D_s is a relation symbol of arity s,

  and $\Gamma_{(S,TF,PF)}$ consists of the Horn sentences

  – $(\forall X) D_s(\sigma X) \Rightarrow D_w X$ for each $\sigma \in (TF + PF)_{w\to s}$, and

  – $(\forall X) D_w X \Rightarrow D_s(\sigma X)$ for each $\sigma \in TF_{w\to s}$

  (where $D_w X$ denotes $\bigwedge_{(x:s)\in X} D_s x$).

- On the sentence side:

  – $\alpha(t \stackrel{e}{=} t') = (D_s t \wedge (t = t'))$,

  – α commutes with the Boolean connectives, and

  – $\alpha((\forall X)\rho) = (\forall X)(D_w X \Rightarrow \alpha\rho)$ for each sentence ρ.

- Each (total) $((S, TF + PF, D), \Gamma_{(S,TF,PF)})$-model M gets mapped to the partial (S, TF, PF)-algebra βM such that

  – $(\beta M)_s = M_{D_s}$ for each sort s, and

  – for each operation $\sigma : s_1 \dots s_n \to s$, $(\beta M)_\sigma$ is the 'restriction' of M_σ to $M_{D_{s_1}} \times \cdots \times M_{D_{s_n}}$ and 'co-restriction' to M_{D_s}. (Note that if $\sigma \in PF$ this restriction may be partial in order to give results in M_{D_s}.)

Encoding partial operations as relations

Another comorphism $\mathcal{PA} \to \mathcal{FOL}^{\text{th}}$, which may be called the '*relational encoding*' of $\mathcal{PA}$ into $\mathcal{FOL}$, encodes the partial operations as relations as follows:

- Each $\mathcal{PA}$-signature (S, TF, PF) gets mapped to the $\mathcal{FOL}$-theory $((S, TF, \overline{PF}), \Gamma_{(S,TF,PF)})$ such that $\overline{PF}_{ws} = PF_{w \to s}$ for each $w \in S^*$ and $s \in S$, and

$$\Gamma_{(S,TF,PF)} = \{(\forall X, y, z)\sigma(X, y) \wedge \sigma(X, z) \Rightarrow (y = z) \mid \sigma \in PF\}.$$

- Each $(S, TF, \overline{PF})$-model M gets mapped to the partial (S, TF, PF)-algebra βM such that
  - $(\beta M)_x = M_x$ for each $x \in S$ or $x \in TF$,
  - for each $\sigma \in PF$, if $(m, m_0) \in M_\sigma$ then $(\beta M)_\sigma m = m_0$.
- α commutes with the quantifiers and the Boolean connectives, and

$$\alpha(t \stackrel{e}{=} t') = (\exists X, x_0) bind(t, x_0) \wedge bind(t', x_0)$$

  where for each (S, TF, PF)-term t and variable x, $bind(t, x)$ is a (finite) conjunction of atoms defined by

$$bind(\sigma(t_1 \dots t_n), x) = \bigwedge_{1 \le i \le n} bind(t_i, x_i) \wedge \begin{cases} \sigma(x_1, \dots, x_n) = x & \text{when } \sigma \in TF \\ \sigma(x_1, \dots, x_n, x) & \text{when } \sigma \in PF \end{cases}$$

  and X is the set of the variables introduced by $bind(t, x_0)$ and $bind(t', x_0)$.

The proof of the Satisfaction Condition relies upon the fact that $M \models (\exists X, x_0) bind(t, x_0)$ if and only if $(\beta M)_t$ is defined. Moreover in such case we have that $(\beta M)_t = M'_{x_0}$, where M' is the unique expansion of M that satisfies $bind(t, x_0)$.

Exercises

4.1. In any institution, for any signature Σ

- $(\bigcup_{i \in I} E_i)^* = \bigcap_{i \in I} E_i^*$ for each family $(E_i)_{i \in I}$ of sets of Σ-sentences, and
- $(\bigcup_{i \in I} \mathbb{M}_i)^* = \bigcap_{i \in I} \mathbb{M}_i^*$ for each family $(\mathbb{M}_i)_{i \in I}$ of classes of Σ-models.

4.2. For a fixed signature, any (possibly infinite) intersection of closed theories is a closed theory too.

4.3. Strong theory morphisms
A theory morphism $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ in *CTh* is *strong* when $E' = (\varphi E)^{**}$. Show that strong theory morphisms are closed under composition.

4.4. Given a signature morphism $\varphi\colon \Sigma \to \Sigma'$ in any institution

- for each E_1 and E_2 sets of Σ-sentences, $E_1 \models_\Sigma E_2$ implies $\varphi E_1 \models_{\Sigma'} \varphi E_2$,

- for each set E of Σ-sentences, $(\varphi E)^* = (Mod\varphi)^{-1}E^*$, and
- $(\Sigma, \varphi^{-1}E')$ is closed when (Σ', E') is closed.

4.5. [161] **Semantic topology**
Recall that a *topology* (X, τ) consists of a set X and a set τ of subsets of X such that $\emptyset, X \in \tau$ and τ is closed under finite intersections and (possibly infinite) unions. Then for each signature Σ of any institution, the class of $|Mod\Sigma|$ of all Σ-models admits a natural *semantic topology*

$$\tau_\Sigma = \{\bigcup_{i\in I} E_i^* \mid (E_i)_{i\in I} \text{ family of finite sets of } \Sigma\text{-sentences}\}.$$

Recall also that given two topologies (X, τ) and (X', τ') a function $f\colon X \to X'$ is *continuous* when $f^{-1}U' \in \tau$ for all $U' \in \tau'$. Then $Mod\varphi\colon (|Mod\Sigma'|, \tau_{\Sigma'}) \to (|Mod\Sigma|, \tau_\Sigma)$ is continuous for each signature morphism $\varphi\colon \Sigma \to \Sigma'$.

4.6. For any institution morphism $(\Phi, \alpha, \beta)\colon I' \to I$, $(\Phi\Sigma', \alpha_{\Sigma'}^{-1}E')$ is a closed theory of I when (Σ', E') is a closed theory of I'.

4.7. In $\mathcal{AUT}$ each theory is closed. (*Hint:* each language can be represented as a (possibly infinite) intersection of regular languages.)

4.8. In any institution I the forgetful functor $Th \to Sig$ determines a canonical institution adjoint morphism $I^{\text{th}} \to I$. Moreover

- $(I^{\text{th}})^{\text{th}} \to I^{\text{th}}$ is an equivalence of institutions, and
- $(-)^{\text{th}}\colon \mathbb{I}ns \to \mathbb{I}ns$ is a functor mapping each institution morphism (Φ, α, β) to the institution morphism $(\Phi^{\text{th}}, \alpha^{\text{th}}, \beta^{\text{th}})$ such that $\Phi^{\text{th}}(\Sigma', E') = (\Phi\Sigma', \alpha_{\Sigma'}^{-1}E'^{**})$ and α^{th} and β^{th} being the restrictions of α and β.

4.9. Comorphism $\mathcal{POA} \to \mathcal{FOL}^{\text{th}}$
There exists a comorphism $\mathcal{POA} \to \mathcal{FOL}^{\text{th}}$ mapping each algebraic signature (S, F) to the $\mathcal{FOL}$-theory $((S, F, (\leq_s)_{s\in S}), pre_{(S,F)})$ such that

- for each sort symbol $s \in S$ the arity of $\leq_s$ is ss, and
- $pre_{(S,F)}$ contains the preorder axioms corresponding to all $\leq_s$ plus the monotonicity axioms for $\leq$ corresponding to all $\sigma \in F$.

4.10. [71] **Comorphism $\mathcal{PA} \to \mathcal{FOEQL}^{\text{th}}$**
There exists a comorphism $(\Phi, \alpha, \beta)\colon \mathcal{PA} \to \mathcal{FOEQL}^{\text{th}}$ such that:

- Each $\mathcal{PA}$-signature (S, TF, PF) gets mapped to $\mathcal{FOEQL}$-theory $((S \cup \{\mathbf{b}\}, TF \oplus PF), \Gamma_{(S,TF,PF)})$ where $(TF \oplus PF)_{w\to s} = TF_{w\to s} \cup PF_{w\to s}$ when $s \neq \mathbf{b}$, $(TF \oplus PF)_{ss\to\mathbf{b}} = \{ⓔ_s\}$ for each $s \in S$, and $(TF \oplus PF)_{\to\mathbf{b}} = \{\mathsf{true}\}$, and $\Gamma_{(S,TF,PF)}$ contains the following conditional equations:
  1. $(\forall X)(X ⓔ X = \mathsf{true}) \Rightarrow (\sigma(X) ⓔ \sigma(X) = \mathsf{true})$ for any *total* operation symbols σ and X a string of variables matching the arity of σ.[1]
  2. $(\forall X, Y)(X ⓔ Y = \mathsf{true}) \Rightarrow (X ⓔ X = \mathsf{true})$.
  3. $(\forall X, Y)(X ⓔ Y = \mathsf{true}) \Rightarrow (X = Y)$.
  4. $(\forall X)(\sigma(X) ⓔ \sigma(X) = \mathsf{true}) \Rightarrow (X ⓔ X = \mathsf{true})$ for any *total* or *partial* operation symbols.

[1] If $X = \{x_1, \ldots, x_n\}$ then $X ⓔ X$ denotes the finite conjunction $(x_1 ⓔ x_1) \wedge \cdots \wedge (x_n ⓔ x_n)$.

- $\alpha_{(S,TF,PF)}(t \stackrel{e}{=} t') = (t \circledcirc t' = \mathsf{true})$, $\alpha_{(S,TF,PF)}$ preserves the Boolean connectives, and $\alpha_{(S,TF,PF)}(\forall X)\rho = (\forall X)((X \circledcirc X) \Rightarrow \alpha_{(S,TF\cup X,PF)}\rho)$.
- $(\beta A)_s = \{a \in A_s \mid A_{\circledcirc_s}(a,a) = A_{\mathsf{true}}\}$ for each $s \in S$, and
- $(\beta A)_\sigma a = \begin{cases} A_\sigma a, & \text{when } A_{\circledcirc_s}(A_\sigma a, A_\sigma a) = A_{\mathsf{true}}, \\ \text{undefined}, & \text{otherwise} \end{cases}$
  for each operation symbol $\sigma \in TF_{w\to s} \cup PF_{w\to s}$.

4.11. Comorphism $\mathcal{IPL} \to (\mathcal{FOEQL}^1)^{\mathrm{th}}$
Let $\mathcal{FOEQL}^1$ be the single-sorted variant of $\mathcal{FOEQL}$. There exists a comorphism $(\Phi,\alpha,\beta)\colon\ \mathcal{IPL} \to (\mathcal{FOEQL}^1)^{\mathrm{th}}$ such that:

- Let (H,E) be the single-sorted equational theory of the Heyting algebras with $H_0 = \{\top,\bot\}$, $H_1 = \{\neg\}$, and $H_2 = \{\wedge,\vee,\Rightarrow\}$ (otherwise $H_n = \emptyset$). Each set (= $\mathcal{IPL}$-signature) P gets mapped by Φ to the theory $(H+P,E)$ where P are added to H as constants.
- $\alpha_P\rho = (\rho = \top)$ for each $\mathcal{IPL}$-signature P and each P-sentence ρ.
- For each $\mathcal{IPL}$-signature P and each $(H+P,E)$-algebra A, $\beta_P A = M$ where $M\colon\ P \to A\restriction_H$ is defined by $M\pi = A_\pi$ for each $\pi \in P$.

4.12. Comorphism $\mathcal{HNK} \to \mathcal{FOEQL}^{\mathrm{th}}$
There exists a comorphism $(\Phi,\alpha,\beta)\colon\ \mathcal{HNK} \to \mathcal{FOEQL}^{\mathrm{th}}$ such that

- Each $\mathcal{HNK}$-signature (S,F) gets mapped to the theory $((\overrightarrow{S},\overrightarrow{F}),\Gamma_{(S,F)})$ where
  - $\overrightarrow{S}$ is the set of all S-types,
  - $\overrightarrow{F}_s = F_s$ for each $s \in \overrightarrow{S}$, $\overrightarrow{F}_{[(s\to s')s]\to s'} = \{\mathsf{app}_{s,s'}\}$ for all $s,s' \in \overrightarrow{S}$ and $\overrightarrow{F}_{w\to s} = \emptyset$ otherwise,
  - $\Gamma_{(S,F)} = \{(\forall f,g,x)\mathsf{app}_{s,s'}(f,x) = \mathsf{app}_{s,s'}(g,x)) \Rightarrow (f=g) \mid s,s' \in \overrightarrow{S}\}$.
- $(\beta_{(S,F)}M) = \overline{M}$ where $\overline{M}$ is the inductively (on the structure of the types) defined $\mathcal{HNK}$-model such that there exists an isomorphism $\mathrm{fun}^M\colon\ M \to \overline{M}$ (here $\overline{M}$ is canonically regarded as a $\mathcal{FOL}$ $((\overrightarrow{S},\overrightarrow{F}),\Gamma_{(S,F)})$-model with app interpreted as ordinary functional application) with fun^M_s being identities for $s \in S$.

  Then $\beta_{(S,F)}$ is an equivalence of categories with an 'inverse' $\overline{\beta}_{(S,F)}$ such that $\overline{\beta}_{(S,F)};\beta_{(S,F)} = 1$ and $\mathrm{fun}\colon\ 1 \stackrel{\cong}{\Rightarrow} \beta_{(S,F)};\overline{\beta}_{(S,F)}$.
- α is defined as the canonical extension of the mapping on the terms α^{tm} defined by $\alpha^{\mathrm{tm}}(tt') = \mathsf{app}(\alpha^{\mathrm{tm}}t,\alpha^{\mathrm{tm}}t')$.

4.13. Comorphism $\mathcal{HOL}_\lambda \to \mathcal{HOL}^{\mathrm{th}}$
There is an 'encoding' comorphism from $\mathcal{HOL}_\lambda$ to $\mathcal{HOL}^{\mathrm{th}}$. (*Hints*: A $\mathcal{HOL}_\lambda$-signature (S,F) is mapped to a $\mathcal{HOL}$-theory that extends (S,F) with an axiomatization of Ω, product types and pairing functions. λ-abstraction is coded in an innermost way by appropriate existential quantification over functions. $\lambda x{:}s.t$ is just coded as f, where $\exists f{:}s \to s'.\forall x{:}s.fx \;=\; t \wedge\cdots$ is added at an appropriate place.)

This 'encoding' comorphism can also be modified into a comorphism from $\mathcal{HNK}_\lambda$ to $\mathcal{HNK}^{\mathrm{th}}$. (*Hint*: It must additionally be ensured that all λ-terms have a denotation. This can be expressed by appropriate existential statements.)

4.14. Comorphism $\mathcal{LA} \to (\mathcal{FOEQL}^1)^{\text{th}}$
Let $\mathcal{FOEQL}^1$ be the single-sorted variant of $\mathcal{FOEQL}$. There exists an institution comorphism $\mathcal{LA} \to (\mathcal{FOEQL}^{\mathbf{1}})^{\text{th}}$ mapping each commutative ring $R = (|R|, +, -, \times, 0)$ to the theory (F_R, E_R) where

- $(F_R)_0 = \{0\}$, $(F_R)_1 = |R| \uplus \{-\}$, $(F_R)_2 = \{+\}$, and $(F_R)_n = \emptyset$ otherwise, and
- E_R consists of the axioms for the commutative group for $\{+,-,0\}$ and $(\forall x)\, r(r'x) = (r \times r')x$, $(\forall x)\, (r+r')x = rx + r'x$, $(\forall x)\, (-r)x = -rx$, and $(\forall x) 0x = 0$ for each $r, r' \in |R|$ elements of the ring R.

4.15. [190] **Comorphism $\mathcal{WPL} \to \mathcal{PL}^{\text{th}}$ (see Ex. 3.27)**
For each set P (of propositional variables) let us consider $SenP$ as a $\mathcal{PL}$ signature and let Γ_P be the specification of the $\mathcal{WPL}$ semantics, i.e., $\Gamma_P = \{[\rho_1 \star \rho_2] \Leftrightarrow ([\rho_1] \star [\rho_2]) \mid \star \in \{\wedge, \vee, \Rightarrow\}$ and $\rho_1, \rho_2 \in SenP\} \cup \{[\rho] \Rightarrow \neg[\neg\rho] \mid \rho \in SenP\}$, where, in order to avoid confusion, by $[\rho]$ we denote the $\mathcal{WPL}$-sentence ρ regarded as a propositional variable of the $\mathcal{PL}$ signature $SenP$. The mapping of P to $(SenP, \Gamma_P)$ determines a comorphism $\mathcal{WPL} \to \mathcal{PL}^{\text{th}}$.

4.16. [159] **Comorphism $\mathcal{PA} \to \mathcal{MA}^{\text{th}}$**
This comorphism is defined by mapping each $\mathcal{PA}$ signature (S, TF, PF) to a $\mathcal{MA}$-theory $((S, TF + PF), \Gamma_{(S,TF,PF)})$ such that

- $\Gamma_{(S,TF,PF)} = \{(\forall y, X)(y \doteq y) \wedge (y \prec \sigma X \Rightarrow \sigma X \doteq \sigma X) \mid \sigma \in TF + PF\}$,
- $\alpha(t \overset{e}{=} t') = (t \doteq t')$, α commutes with the Boolean connectives and $\alpha((\forall X)\rho) = (\forall X)((X \doteq X) \wedge \alpha\rho)$,
- $(\beta A)_s = A_s$ for each sort $s \in S$ and $(\beta A)_\sigma(a_1, \ldots, a_n) = a$ when $A_\sigma(a_1, \ldots, a_n) = \{a\}$, otherwise it is undefined.

4.17. [159] **Comorphism $\mathcal{MBA} \to \mathcal{MA}^{\text{th}}$**
Each $\mathcal{MBA}$ signature (S, K, F, kind) can be mapped to the $\mathcal{MA}$-theory $((K, F \cup \{p_s \mid s \in S\}), \Gamma_{(S,K,F,\text{kind})})$, where p_s are constants of sort $\text{kind}(s)$ and

$$\Gamma_{(S,K,F,\text{kind})} = \{(\forall X)(X \doteq X) \Rightarrow (\sigma X \doteq \sigma X) \mid \sigma \in F\}.$$

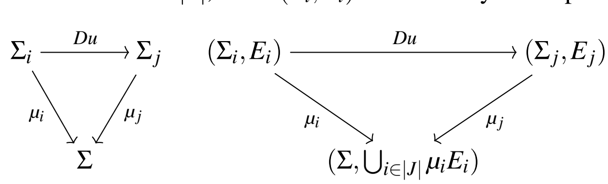
This determines a comorphism $\mathcal{MBA} \to \mathcal{MA}^{\text{th}}$.

4.2 Theory (co-)limits

The following simple result shows that the limits and co-limits of theories exist in dependence on limits and co-limits of signatures.

Proposition 4.2. *In any institution, the forgetful functor $U : Th \to Sig$ lifts limits and co-limits. Moreover, the forgetful functor $CTh \to Sig$ lifts them uniquely.*

Proof. Consider a functor $D : J \to Th$. When $\mu : D;U \Rightarrow \Sigma$ is a co-limit co-cone in Sig. By a simple checking, we obtain that $\mu : D \Rightarrow (\Sigma, (\bigcup_{i \in |J|} \mu_i E_i))$ is a co-limit co-cone in Th, where for each $i \in |J|$, $Di = (\Sigma_i, E_i)$ is the theory corresponding to i.



Similarly, when $\mu : \Sigma \Rightarrow D;U$ is a limit cone in *Sig*, then $\mu : (\Sigma, \bigcap_{i\in|J|} \mu_i^{-1} E_i^{**}) \Rightarrow D$ is a limit co-cone in *Th*. □

Corollary 4.3. *In any institution, the category Th of its theories, respectively CTh of its closed theories, have whatever limits or co-limits its category Sig of the signatures has.*

Limits and co-limits of $\mathcal{FOL}$ signatures

We can apply Prop. 4.2 through Cor. 4.3 to show that $\mathcal{FOL}$ has small limits and co-limits of (closed) theories by proving that the category $Sig^{\mathcal{FOL}}$ of $\mathcal{FOL}$-signatures has small limits and co-limits. The arguments of the proof of the result below can be repeated with some adjustments in the form to many other concrete institutions, especially when they are in a many-sorted format.

Proposition 4.4. *The category of $\mathcal{FOL}$ signatures has small limits and co-limits.*

Proof. Let us use $\uplus$ / $\prod$ for denoting disjoint unions / cartesian products of sets, respectively. Given a any set S, because $\mathbb{S}et$ has all small limits and co-limits, cf. Prop. 2.2, the functor categories $\mathbb{C}at(S^* \times S, \mathbb{S}et)$ and $\mathbb{C}at(S^*, \mathbb{S}et)$ have small limits and co-limits too. So do their products $\mathbb{C}at(S^* \times S, \mathbb{S}et) \times \mathbb{C}at(S^*, \mathbb{S}et)$ (by calculating (co-)limits componentwise).

Each function $f : S \to S'$ determines a functor $\mathbb{C}at(S'^* \times S', \mathbb{S}et) \times \mathbb{C}at(S'^*, \mathbb{S}et) \to \mathbb{C}at(S^* \times S, \mathbb{S}et) \times \mathbb{C}at(S^*, \mathbb{S}et)$ by composition to the left with $(f^* \times f, f^*)$. This functor has

- a left adjoint mapping each (F,P) to (F',P') such that

$$F'_{w' \to s'} = \biguplus \{F_{w \to s} \mid f^*(ws) = w's'\},\ P'_{w'} = \biguplus \{P_w \mid f^* w = w'\}, \text{ and}$$

- a right adjoint mapping each (F,P) to (F'',P'') such that

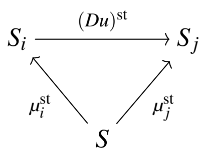
$$F''_{w' \to s'} = \prod \{F_{w \to s} \mid f^*(ws) = w's'\},\ P''_{w'} = \prod \{P_w \mid f^* w = w'\}.$$

From Prop. 2.6 we know that a right adjoint preserves all limits, thus the hypotheses of Thm. 2.10 are fulfilled for the indexed category $\mathbb{S}et^{\mathrm{op}} \to \mathbb{C}at$ mapping each set S to $\mathbb{C}at(S'^* \times S', \mathbb{S}et) \times \mathbb{C}at(S'^*, \mathbb{S}et)$. It follows that its Grothendieck category, which is exactly $Sig^{\mathcal{FOL}}$, has small limits and co-limits. □

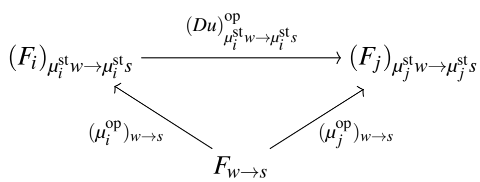
The above proof involves some non-trivial category theoretic machinery. It is also useful to understand in more concrete terms how the limits and the co-limits of $\mathcal{FOL}$ signatures are constructed.

Limits. For any small category J, the limit cone $\mu\colon (S,F,P) \Rightarrow (D\colon J \to Sig^{\mathcal{FOL}})$ (where $Du\colon (S_i,F_i,P_i) \to (S_j,F_j,P_j)$ for each $u \in J(i,j)$) is defined by

1. μ^{st} is the limit of $D;(-)^{\mathrm{st}}\colon J \to \mathbb{S}et$



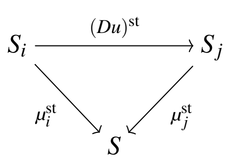
2. Each arity $w \in S^*$ and each sort $s \in S$ determine a functor $J \to \mathbb{S}et$ mapping each arrow $u \in J(i,j)$ to $(Du)^{\mathrm{op}}_{\mu_i^{\mathrm{st}} w \to \mu_i^{\mathrm{st}} s}\colon (F_i)_{\mu_i^{\mathrm{st}} w \to \mu_i^{\mathrm{st}} s} \to (F_j)_{\mu_j^{\mathrm{st}} w \to \mu_j^{\mathrm{st}} s}$. Let $((\mu_i^{\mathrm{op}})_{w \to s})_{i \in |J|}$ be the limit cone of this functor.



3. For each arity $w \in S^*$, $((\mu_i^{\mathrm{rl}})_w\colon P_w \to (P_i)_{\mu_i^{\mathrm{st}} w})_{i \in |J|}$ is the limit cone for the functor $J \to \mathbb{S}et$ mapping each arrow $u \in J(i,j)$ to $(Du)^{\mathrm{rl}}_{\mu_i^{\mathrm{st}} w}\colon (P_i)_{\mu_i^{\mathrm{st}} w} \to (P_j)_{\mu_j^{\mathrm{st}} w}$.

Co-limits. For any small category J, the co-limit co-cone $\mu\colon (D\colon J \to Sig^{\mathcal{FOL}}) \Rightarrow (S,F,P)$ (where $Du\colon (S_i,F_i,P_i) \to (S_j,F_j,P_j)$ for each $u \in J(i,j)$) is defined by

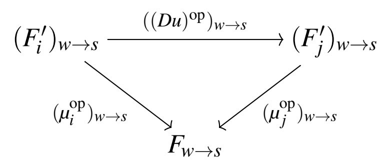
1. μ^{st} is the co-limit of $D;(-)^{\mathrm{st}}\colon J \to \mathbb{S}et$



2. For each arity $w \in S^*$ and each sort $s \in S$, let $(F_i')_{w \to s} = \biguplus_{\mu_i^{\mathrm{st}} w_i s_i = ws} (F_i)_{w_i \to s_i}$. For each arrow $u \in J(i,j)$ let $(Du)^{\mathrm{op}}_{w \to s}\colon (F_i')_{w \to s} \to (F_j')_{w \to s}$ be the 'disjoint union' of all functions

$$(F_i)_{w_i \to s_i} \xrightarrow{(Du)^{\mathrm{op}}_{w_i \to s_i}} (F_j)_{(Du)^{\mathrm{st}} w_i \to (Du)^{\mathrm{st}} s_i} \longrightarrow \biguplus_{\mu_j^{\mathrm{st}}(w_j s_j) = ws} (F_j)_{w_j \to s_j}.$$

Then we define $((\mu_i)^{\mathrm{op}}_{w\to s})_{i\in|J|}$ to be the co-limit co-cone for the functor $J \to \mathbb{S}et$ mapping each u to $(Du)^{\mathrm{op}}_{w\to s}$.



For each w_i and s_i we define $(\mu_i^{\mathrm{op}})_{w_i\to s_i}$ as the restriction of $(\mu_i^{\mathrm{op}})_{w\to s}$ to $(F_i)_{w_i\to s_i}$.

3. For each $i \in |J|$ and arity $w_i \in S_i^*$ we define $(\mu_i^{\mathrm{rl}})_{w_i}$ in the same way we have defined $(\mu_i^{\mathrm{op}})_{w_i\to s_i}$ in the item above.

Exercises

4.18. The category of $\mathcal{CA}$ signatures has small co-limits but only finite limits.

4.19. The category of $\mathcal{HOL}$ / $\mathcal{HNK}$ signatures does have pushouts and small co-products.

4.20. Weak co-amalgamation for sentences
In $\mathcal{FOL}$ the sentence functor weakly preserves pullbacks, i.e., any pullback of signature morphisms gets mapped by $Sen^{\mathcal{FOL}}$ to a weak pullback in $\mathbb{S}et$.

4.21. [70] **Finitely presented signatures**
A $\mathcal{FOL}$ signature (S,F,P) is finitely presented (as an object of $Sig^{\mathcal{FOL}}$) if and only if S, F, and P are finite. (F 'finite' means that $\{(w,s) \mid F_{w\to s} \neq \emptyset\}$ is finite and each non-empty $F_{w\to s}$ is also finite and the same for P.)

4.22. [98] **Finitary sentences**
A sentence ρ for a signature Σ of an institution is *finitary* when there exists a signature morphism $\varphi\colon\ \Sigma_0 \to \Sigma$ such that Σ_0 is finitely presented and there exists a Σ_0-sentence ρ_0 such that $\rho = \varphi\rho_0$. Then any $\mathcal{FOL}$-sentence is finitary. Give an example of a $\mathcal{FOL}_{\infty,\omega}$-sentence which is *not* finitary.

4.23. [70] **Finitely presented closed theories**
Assume an institution with finitary sentences. Then for each finitely presented closed theory (Σ,E) (i.e., it is a finitely presented object in the category CTh of theories),

- Σ is a finitely presented signature, and
- E can be presented by a finite set of sentences, i.e., there exists a finite set E_0 of Σ-sentences such that $E = (E_0)^{**}$.

4.3 Model amalgamation

Model amalgamation in institutions. In any institution, a commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is an *amalgamation square* if and only if for each Σ_1-model M_1 and a Σ_2-model M_2 such that $M_1\upharpoonright_{\varphi_1} = M_2\upharpoonright_{\varphi_2}$, there exists a unique Σ'-model M', called the *amalgamation* of M_1 and M_2, such that $M'\upharpoonright_{\theta_1} = M_1$ and $M'\upharpoonright_{\theta_2} = M_2$. When $\varphi_1, \varphi_2, \theta_1, \theta_2$ are clear, the amalgamation M' may be denoted as $M_1 \otimes M_2$.

Without the uniqueness requirement on the amalgamation M', we say that this is a *weak amalgamation* square.

Note that from a categorical viewpoint, the model amalgamation property means that

$$\begin{array}{ccc} |Mod\Sigma| & \xleftarrow{Mod\varphi_1} & |Mod\Sigma_1| \\ {\scriptstyle Mod\varphi_2}\uparrow & & \uparrow{\scriptstyle Mod\theta_1} \\ |Mod\Sigma_2| & \xleftarrow[Mod\theta_2]{} & |Mod\Sigma'| \end{array}$$

is a pullback in $\mathbb{C}lass$, the 'category' of classes.[2]

To have model amalgamation, it is necessary that the corresponding square of signature morphisms does not collapse entities of Σ_1 and Σ_2 which do not come from Σ (via φ_1 and φ_2). On the other hand, for ensuring the uniqueness of the amalgamation it is necessary that Σ' does not contain entities which do not come from either Σ_1 or Σ_2. Therefore the primary candidates for model amalgamation are the pushout squares of signature morphisms. We may say that an institution has *model amalgamation* if and only if each pushout of signatures is an amalgamation square.

Model amalgamation in $\mathcal{FOL}$

Modulo some adjustments the result below can be replicated to a multitude of actual institutions.

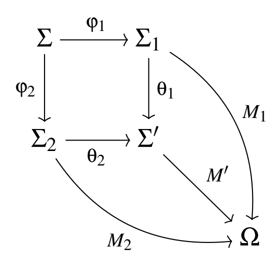
Proposition 4.5. *$\mathcal{FOL}$ has model amalgamation.*

Proof. We consider any pushout (θ_1, θ_2) of a span of signature morphisms (φ_1, φ_2) and Σ_k-models M_k, $k = 1,2$, such that $M_1\upharpoonright_{\varphi_1} = M_2\upharpoonright_{\varphi_2}$. Let U be the union of all the carrier sets of M_1 and M_2. We define the following signature $\Omega = (S^\Omega, F^\Omega, P^\Omega)$:

[2] This is the 'extension' of $\mathbb{S}et$ having classes as objects.

- $S^\Omega = \mathcal{P}U$, i.e., the set of all subsets of U,
- for any sets $s_1,\ldots,s_n,s \subseteq U$, $F^\Omega_{s_1\ldots s_n\to s} = \mathbb{S}et(s_1\times\cdots\times s_n, s)$, i.e., the set of all functions $s_1\times\cdots\times s_n \to s$, and
- for any sets $s_1,\ldots,s_n$, $P^\Omega_{s_1\ldots s_n} = \mathcal{P}(s_1\times\cdots\times s_n)$, i.e., the set of all subsets of $s_1\times\cdots\times s_n$.

We note that for any $\mathcal{FOL}$-signature (S,F,P), the (S,F,P)-models that have all carrier sets included in U are exactly the signature morphisms $(S,F,P)\to\Omega$. Moreover, under this perspective the reduct $M'\upharpoonright_\varphi$ of any model M' appears as $\varphi;M'$.



Therefore $M_1\upharpoonright_{\varphi_1} = M_2\upharpoonright_{\varphi_2}$ just means that $\varphi_1;M_1 = \varphi_2;M_2$. Let $M' : \Sigma'\to\Omega$ be the unique signature morphism such that $\theta_k;M' = M_k$ for $k=1,2$. Then M' is the unique amalgamation of M_1 and M_2. □

Note that in the proof of Prop. 4.5 the assumption that all carrier sets are included in U does not affect the validity our argument because of two reasons. On the one hand, the pushout square of $\mathcal{FOL}$-signatures does not allow in Σ' for syntactic entities beyond those that come from Σ_1 and Σ_2. On the other hand, the $\mathcal{FOL}$-model reducts preserve the carrier sets. This means that all candidates for the amalgamation M' cannot go beyond this assumption. Alternatively, we can shortcut this argument by going beyond sets by allowing classes and functions between classes. Under this approach we let U be the class of all sets. Then a pushout in $\mathbb{S}et$ is also a pushout in the 'category' $\mathbb{C}lass$ of classes.

The proof of Prop. 4.5 is rather smart, but it may not bring the best insight into the structure of the amalgamation M'. Let us explain this more directly. Since Σ' is the vertex of a pushout, according to the construction of co-limits of $\mathcal{FOL}$-signatures discussed above, for each sort s' of Σ' there exists $k\in\{1,2\}$ and s_k a sort of Σ_k such that $s' = \theta^{\rm st}_k s_k$. Then $M'_{s'} = (M_k)_{s_k}$. But what if we had $s' = \theta^{\rm st}_1 s_1 = \theta^{\rm st}_2 s_2$? In this situation $(M_1)_{s_1} = (M_2)_{s_2} = M_s$ where s is a sort of Σ such that $s_k = \varphi^{\rm st}s$, so $M'_{s'}$ is well-defined. The existence of s is given by the way pushouts exist in $\mathbb{S}et$. Similar considerations apply also to the interpretations or operation and relation symbols in M', just by following the details of the construction of pushouts of $\mathcal{FOL}$-signatures.

Extended model amalgamation. The concept of model amalgamation is most often used in the form presented above, for squares of signature morphisms. However, sometimes other forms of model amalgamation are necessary, e.g., for co-cones over other types of diagrams of signature morphisms.

Given a diagram $D: J \to Sig$, a *D-model* is a family $(M_i)_{i\in|J|}$ such that

- M_i is a Di-model for each $i \in |J|$, and
- $M_j\upharpoonright_{Du} = M_i$ for each arrow $u \in J(i,j)$.

We say that a co-cone μ over a diagram $D: J \to Sig$ of signature morphisms has *model amalgamation* when for each D-model $(M_i)_{i\in|J|}$ there exists a unique model M such that $M\upharpoonright_{\mu_i} = M_i$ for each $i \in |J|$. When we drop the uniqueness requirement, we say that μ has *weak model amalgamation.*

An institution has J*-model amalgamation* for a category J when all co-limits of all diagrams $J \to Sig$ have model amalgamation. Ordinary model amalgamation, as originally introduced in this section, is thus J-model amalgamation for J being a span of arrows $\bullet \longleftarrow \bullet \longrightarrow \bullet$.

This terminology can be also extended to classes $\mathbb{J}$ of categories J. For example, when $\mathbb{J}$ consists of all directed, respectively total posets, we talk about *directed*, respectively *inductive*, *model amalgamation*.

The proof of Prop. 4.5 can be extended without any problem to all small co-limits of signatures (which exist by Prop. 4.4).

Proposition 4.6. *$\mathcal{FOL}$ has J-model amalgamation for all small categories J.*

Exact institutions

In most situations the kind of amalgamation which is needed is at the level of the models only, however, there are results which rely upon a form of amalgamation for model homomorphisms.

Amalgamation for model homomorphisms means that *Mod* maps any pushout of signatures to a pullback of categories (of models) rather than to a pullback of classes (of models). We called this property the *semi-exactness* of the institution. The terminology introduced for model amalgamation can be extended to exactness. Thus an institution $(Sig, Sen, Mod, \models)$ is

- *semi-exact* when the model functor $Mod: Sig^{\mathrm{op}} \to \mathbb{C}at$ preserves pullbacks,
- *directed / inductive-exact* when *Mod* preserves directed / inductive limits,
- *(J)-exact* when *Mod* preserves all (J)(small) limits, and
- *weakly* J-exact when *Mod* preserves weak J-limits.[3]

$\mathcal{FOL}$ exactness. Prop. 4.5 can be refined to model homomorphisms.

Proposition 4.7. *$\mathcal{FOL}$ is exact.*

[3] A weak universal property, such as adjunction, limits, etc., is the same as the ordinary universal property except that only the existence part is required while uniqueness is not required.

Proof. We have to show that $Mod^{\mathcal{FOL}}: (Sig^{\mathcal{FOL}})^{\mathrm{op}} \to \mathbb{C}at$ preserves all small limits. Let us consider the case of the pullbacks (which in $Sig^{\mathcal{FOL}}$ appear as pushouts), other limits being handled similarly.

We re-use the idea underlying the proof of Prop. 4.5 by changing the signature Ω in order to capture model homomorphisms as follows. Let $h_k: M_k \to N_k$ be Σ_k-model homomorphisms, $k = 1,2$, where $h_1\restriction_{\varphi_1} = h_2\restriction_{\varphi_2}$.

- $S^{\Omega} = \{s \text{ function} \mid dom(s), cod(s) \subseteq U\}$ where U is the union of all carrier sets of M_1, M_2, N_1, N_2.
- for all functions $s_1, \dots, s_n, s$, $F^{\Omega}_{s_1 \dots s_n \to s}$ is the subset of

  $$\mathbb{S}et(dom(s_1) \times \cdots \times dom(s_n), dom(s)) \times \mathbb{S}et(cod(s_1) \times \cdots \times cod(s_n), cod(s))$$

  of all pairs of functions which satisfy the homomorphism property for operations, i.e., $\langle \mu, \nu \rangle \in F^{\Omega}_{s_1 \dots s_n \to s}$ if and only if

  $$\mu; s = (s_1 \times \cdots \times s_n); \nu$$

- for any functions $s_1, \dots, s_n$, $P^{\Omega}_{s_1 \dots s_n}$ is the subset of

  $$\mathcal{P}(dom(s_1) \times \cdots \times dom(s_n)) \times \mathcal{P}(cod(s_1) \times \cdots \times cod(s_n))$$

  of all pairs that satisfy the homomorphism property for relations, i.e., $\langle \mu, \nu \rangle \in P^{\Omega}_{s_1 \dots s_n}$ if and only if

  $$(s_1 \times \cdots \times s_n)\mu \subseteq \nu.$$

Then for any $\mathcal{FOL}$-signature (S,F,P), an (S,F,P)-model homomorphism $h: M \to N$ such that all carrier sets of of M and N are included in U arise as a signature morphism $h: (S,F,P) \to \Omega$ as follows. For any $s \in S$, $M_s = dom(h_s)$, $N_s = cod(h_s)$, for any $\sigma \in F_{w \to s}$, $h_\sigma = \langle M_\sigma, N_\sigma \rangle$, etc. $\square$

Model amalgamation for theories

Given a weak amalgamation square in an institution

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

if $M_1 \models_{\Sigma_1} E_1$, $M_2 \models_{\Sigma_2} E_2$ and $M_1\restriction_{\varphi_1} = M_2\restriction_{\varphi_2}$ let M' be an amalgamation of M_1 and M_2. Then, by the Satisfaction Condition, $M' \models_{\Sigma'} \theta_1 E_1 \cup \theta_2 E_2$.

This argument works also for all co-cones of diagrams of presentation morphisms. By recalling how co-limits of theories are constructed on top of co-limits of signatures (Prop. 4.2), the above argument shows that any model amalgamation property of an institution can be lifted from the level of the signatures to the level of the (closed) theories. Moreover, this can be extended easily to model homomorphisms too. These considerations are collected by the following result.

Theorem 4.8. *If an institution I is J-exact, then the institution I^{th} of its theories is J-exact too.*

As an application, from Prop. 4.7 and Thm. 4.8 it follows that:

Corollary 4.9. *The institution $\mathcal{FOL}^{\mathrm{th}}$ is exact.*

Model amalgamation for institution mappings

The transfer of institutional properties along institution mappings, usually comorphisms, relies sometimes upon a form of model amalgamation of the respective institution mapping.

Exact comorphisms. An institution comorphism $(\Phi, \alpha, \beta) : I \to I'$ is *exact* if for each I-signature morphism $\varphi : \Sigma_1 \to \Sigma_2$ the naturality square below

$$\begin{array}{ccc} Mod\Sigma_1 & \xleftarrow{\beta_{\Sigma_1}} & Mod'(\Phi\Sigma_1) \\ {\scriptstyle Mod\varphi}\uparrow & & \uparrow{\scriptstyle Mod'(\Phi\varphi)} \\ Mod\Sigma_2 & \xleftarrow[\beta_{\Sigma_2}]{} & Mod'(\Phi\Sigma_2) \end{array}$$

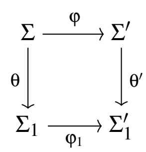
is a pullback. When discarding the model homomorphisms from the above (i.e., the diagram above is a pullback of classes of models rather than categories of models), we say that (Φ, α, β) *has model amalgamation*. This means that for any $\Phi\Sigma_1$-model M'_1 and any Σ_2-model M_2, if $\beta_{\Sigma_1} M'_1 = M_2\restriction_{\varphi}$, then there exists a unique $\Phi\Sigma_2$-model M'_2 such that $\beta_{\Sigma_2} M'_2 = M_2$ and $M'_2\restriction_{\Phi\varphi} = M'_1$. If we drop the uniqueness requirement on M'_2, then we say that (Φ, α, β) *has weak model amalgamation*.

As a simple example, notice that the exactness of the institution comorphism $\mathcal{EQL} \to \mathcal{FOL}$ holds trivially because the model translation functors $\beta_{(S,F)}$ are isomorphisms for all algebraic signatures (S, F).

Exact morphisms. A similar definition can be formulated for *exact institution morphisms*. However, in the actual institutions, comorphisms rather than morphisms interact better with model amalgamation. For example, while the comorphism $\mathcal{EQL} \to \mathcal{FOL}$ is trivially exact, its adjoint (forgetful) institution morphism $\mathcal{FOL} \to \mathcal{EQL}$ does *not* have model amalgamation. However, it does have weak model amalgamation. Why is that?

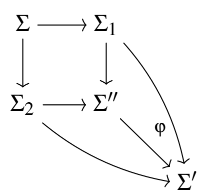
Exercises

4.24. A signature morphism $\varphi : \Sigma \to \Sigma'$ has the *model expansion property* when each Σ-model has at least a φ-expansion. Given a weak amalgamation square in an institution



show that φ_1 has the model expansion property when φ has it.

4.25. In the commuting diagram below



if $[\Sigma, \Sigma_1, \Sigma_2, \Sigma'']$ is an amalgamation square, then $[\Sigma, \Sigma_1, \Sigma_2, \Sigma']$ is a weak amalgamation square if and only if φ has the model expansion property.

4.26. Categorical equational logic $Cat\mathcal{EQL}$ (see Ex. 3.7) is trivially exact.

4.27. For any semi-exact institution I, the institution $I^{\to}$ of its signature morphisms (see Ex. 3.8) is semi-exact.

4.28. A method to prove model amalgamation properties of institutions I is to 'borrow' them from another institution I' via a comorphism $I \to I'$ with the following properties:

1. the signature translation functor preserves the respective co-limits of signatures, and
2. the model translation functor has a left inverse.

Apply this method to obtain model amalgamation properties for various institutions presented in this book.

4.29. The institution $\mathcal{IPL}$ is exact.

4.30. The institution $\mathcal{MVL}^\sharp$ is exact.

4.31. The institutions $\mathcal{FOL}^1$, $\mathcal{LA}$ and $\mathcal{AUT}$ are semi-exact but they are not exact.

4.32. The institution $\mathcal{CA}$ of contraction algebras does *not* have model amalgamation. Find out why. However, model amalgamation holds for the pushout squares for which the contraction parameter q is fixed for all signatures.

4.33. While $\mathcal{HOL}$ has model amalgamation, $\mathcal{HNK}$ has only weak model amalgamation.

4.34. The institution $\mathcal{WPL}$ of Ex. 3.27 has weak model amalgamation.

4.35. Any chartable institution has model amalgamation. (see Ex. 3.11)

4.36. The sub-institutions of $\mathcal{FOL}$ obtained by restricting the model homomorphisms to those which are injective, respectively surjective, are exact.

4.37. The comorphism $\mathcal{FOL} \to \mathcal{FOEQL}$ encoding relations as operations (see Sect. 3.3) does not have model amalgamation but it has weak model amalgamation.

4.38. Let us assume that for an institution morphism $(\Phi, \alpha, \beta) : I' \to I$, the signature translation functor $\Phi : Sig' \to Sig$ has a left adjoint $\overline{\Phi}$, is full and surjective on objects, and that I' is semi-exact. If (Φ, α, β) is exact, then its adjoint institution comorphism $(\overline{\Phi}, \overline{\alpha}, \overline{\beta})$ is exact too.

4.39. Study the model amalgamation properties of the following comorphisms which have been introduced above in the book (either in the main text or in exercises): $\mathcal{MPL}^* \to \mathcal{REL}^1$, $\mathcal{PA} \to \mathcal{FOL}^{\text{th}}$, $\mathcal{FOL} \to \mathcal{REL}^{\text{th}}$, $\mathcal{POA} \to \mathcal{FOL}^{\text{th}}$, $\mathcal{FOL} \to \mathcal{PA}^{\text{th}}$, $\mathcal{FOL} \to (\mathcal{FOL}^1)^{\text{th}}$, and $\mathcal{HNK} \to \mathcal{FOEQL}^{\text{th}}$.

4.40. [84] **Institutional seeds**
We can extend the idea underlying Prop. 4.5 to a develop a simple generic way to define institutions. An *institutional seed* consists of a 'sentence functor' $Sen : Sig \to \mathbb{S}et$, a fixed 'signature' $\Omega \in |Sig|$ and a 'truth function' $T : Sen\Omega \to \{0, 1\}$. Show that by defining

- for each 'signature' Σ, $Mod\Sigma = \{M \mid M : \Sigma \to \Omega\}$,
- for each 'signature morphism' $\varphi : \Sigma \to \Sigma'$, $Mod\varphi : Mod\Sigma' \to Mod\Sigma$ by $(Mod\varphi)M' = \varphi; M'$, and
- for each $M \in Mod\Sigma$ and $\rho \in Sen\Sigma$, $M \models_\Sigma \rho$ if and only if $(T(SenM))\rho = 1$,

the tuple $(Sig, Sen, Mod, \models)$ is an institution.
Show that $\mathcal{PL}$ and *localized* variants (in the sense that all carrier sets of models are subsets of a fixed set U) of other institutions such as $\mathcal{FOL}$, $\mathcal{PA}$ etc. can be defined in this way from institutional seeds.

4.4 The method of diagrams

This is one of the most useful methods of model theory in general. Note that 'diagrams' here are used with a different meaning than the categorical diagrams. In this section, we begin with the presentation of the concrete concept of model-theoretic diagrams in $\mathcal{FOL}$. Then we prove an institution-theoretic property of $\mathcal{FOL}$ diagrams which when taken as an axiom provides the concept of diagram in abstract institutions. Next, we introduce a couple of basic concepts around the diagrams. Finally, we develop an initial general application of diagrams. Many more applications will naturally follow as we advance through the book.

Diagrams in $\mathcal{FOL}$

Each model M of a signature (S, F, P) determines an extension of signatures
$\iota : (S, F, P) \hookrightarrow (S, F_M, P)$ where

- $(F_M)_{w \to s} = F_{w \to s}$ for any non-empty arity $w \in S^*$ and any sort $s \in S$, and
- $(F_M)_{\to s} = F_{\to s} \uplus M_s$ (disjoint union) for any sort $s \in S$.

Then note that M can be canonically expanded to an (S, F_M, P)-model M_M by interpreting the new constants of $(F_M)_{\to s}$ by the corresponding elements of M_s, i.e., $(M_M)_a = a$ for each $a \in M$. Let E_M be the set of all atoms (either equational or relational) satisfied by M_M.

The theory $((S, F_M, P), E_M)$, called the *diagram of* M, has the crucial categorical property that it *axiomatizes the class of homomorphisms from* M.

Proposition 4.10. *There exists a natural isomorphism*

$$i : \ Mod((S, F_M, P), E_M) \to M/Mod(S, F, P).$$

Proof. The isomorphism i maps each (S, F_M, P)-model N satisfying E_M to the (S, F, P)-model homomorphism $h_N : \ M \to N{\upharpoonright}_\iota$ such that $h_N a = N_a$ for each element $a \in M$. Let us check that h_N is indeed a model homomorphism.

– For each operation $\sigma \in F_{w \to s}$ and for each $m \in M_w$ we have that:

1	$(\sigma m = M_\sigma m) \in E_M$	$M_M \models \sigma m = M_\sigma m$
2	$N \models \sigma m = M_\sigma m$	1, $N \models E_M$
3	$N_{\sigma m} = N_{M_\sigma m}$	2, definition of $\models$
4	$h_N(M_\sigma m) = N_{M_\sigma m}$	definition of h
5	$h_N m = N_m$	definition of h
6	$N_{\sigma m} = N_\sigma(N_m) = (N{\upharpoonright}_\iota)_\sigma(h_N m)$	5
7	$h_N(M_\sigma m) = (N{\upharpoonright}_\iota)_\sigma(h_N m)$	3, 4, 6.

– We consider any relation symbol $\pi \in P_w$ and any $m \in M_\pi$. Then

1	$\pi m \in E_M$	$M_M \models \pi m$
2	$N \models \pi m$	$N \models E_M$
3	$N_m \in N_\pi$	2
4	$h_N m = N_m$	definition of h
5	$h_N m \in N_\pi$	3, 4

The inverse isomorphism i^{-1} maps any (S, F, P)-model homomorphism $h : \ M \to N$ to the (S, F_M, P)-model $i^{-1}h = N_h$ where $(N_h){\upharpoonright}_\iota = N$ and $(N_h)_a = ha$ for each $a \in M$. We have to check that $N_h \models E_M$. We first notice that h is also an (S, F_M, P)-model homomorphism $M_M \to N_h$.

– Consider an equation $t = t'$ in E_M. Then

1	$(N_h)_t = h((M_M)_t), \ (N_h)_{t'} = h((M_M)_{t'})$	by induction on the structure of t, t'
2	$(M_M)_t = (M_M)_{t'}$	$M_M \models E_M$
3	$(N_h)_t = (N_h)_{t'}$	1, 2
4	$N_h \models t = t'$	3, definition of $\models$.

– Now consider a relational atom $\pi t \in E_M$ (here t denotes an appropriate string of terms rather than a single term). Then

5	$M_M \models \pi t$	$M_M \models E_M$
6	$(M_M)_t \in (M_M)_\pi$	5, definition of $\models$
7	$(N_h)_t \in (N_h)_\pi$	1, 6, h homomorphism
8	$N_h \models \pi t$	7, definition of $\models$.

We have analyzed i and i^{-1} on models only. They also work as expected on model homomorphisms. □

Changing model homomorphisms. To maintain the isomorphic relationship between the category of homomorphisms $M/Mod(S,F,P)$ and the category of the models of the theory $((S,F_M,P),E_M)$, any change of the concept of model homomorphism induces a change in the concept of diagram. Note that considering other model homomorphisms between $\mathcal{FOL}$ models means working with another institution. For example, if we impose some condition which shrinks the class of model homomorphisms, then consequently the diagram should get bigger so that the class of its models shrinks too.

Below we give a list of several possibilities for model homomorphisms between $\mathcal{FOL}$ models obtained by imposing some additional conditions on the standard $\mathcal{FOL}$ model homomorphisms. In all cases, diagrams do exist as shown in the right-hand side column of the table below. All entries of the table can be checked similarly to the proof of Prop. 4.10. A $\mathcal{FOL}$-model homomorphism $h: M \to N$

– is *closed* when $M_\pi = h^{-1}N_\pi$ for each relation symbol π of the signature, and

– is an *elementary embedding* when $M_M \equiv N_h$ (where $N_h = i(h)$ like in the proof of Prop. 4.10). Note that because $M_M \models m \neq m'$ for all $m,m' \in M$ that differ, h is also injective.)

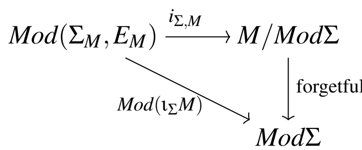
model homomorphisms	E_M
all	all atoms in $(M_M)^*$
injective	all atoms and negations of atomic equations in $(M_M)^*$
closed	all atoms and negations of atomic relations in $(M_M)^*$
closed and injective	all atoms and negations of atoms in $(M_M)^*$
elementary embeddings	$(M_M)^*$

In some other model-theoretic works, in the context of $\mathcal{FOL}$ models, the diagrams of the last entry in the table above are called ‘elementary diagrams’.

Institution-independent diagrams

We may note that the isomorphism between the category of model homomorphisms from M (for whatever concept of model homomorphism we employ) and the class of models of the ‘diagram’ of M is a purely categorical property which can be therefore formulated at an institution-independent level. An institution $(Sig, Sen, Mod, \models)$ has *diagrams* if and only if for each signature Σ and each Σ-model M

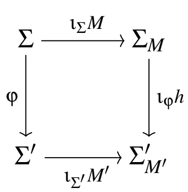
- there exists a signature Σ_M and a signature morphism $\iota_\Sigma M : \Sigma \to \Sigma_M$, that is functorial in Σ and M, and
- a set E_M of Σ_M-sentences such that $Mod(\Sigma_M, E_M)$ and the comma category $M/Mod\Sigma$ are naturally isomorphic, i.e., the following diagram commutes by the isomorphism $i_{\Sigma,M}$ natural in Σ and M:



(4.1)

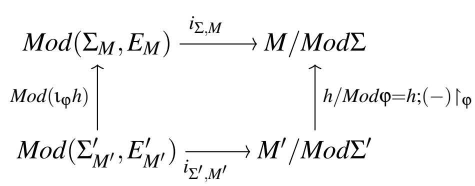
The signature morphism $\iota_\Sigma M : \Sigma \to \Sigma_M$ is called the *elementary extension of* Σ *via* M and the set E_M of Σ_M-sentences is called the *diagram* of the model M. For each model homomorphism $h : M \to N$ let N_h denote $i^{-1}_{\Sigma,M} h$.

The “functoriality” of ι means that for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each Σ-model homomorphism $h : M \to M' \upharpoonright_\varphi$, there exists a theory morphism
$\iota_\varphi h : (\Sigma_M, E_M) \to (\Sigma'_{M'}, E_{M'})$ such that



commutes and $\iota_\varphi h \,;\, \iota_{\varphi'} h' = \iota_{\varphi;\varphi'}(h; h' \upharpoonright_\varphi)$ and $\iota_{1_\Sigma} 1_M = 1_{\Sigma_M}$.

The “naturality” of i means that for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each Σ-model homomorphism $h : M \to M' \upharpoonright_\varphi$ the following diagram commutes:



The reader is invited to check the above functoriality and naturality properties of the diagrams for $\mathcal{FOL}$ and its sub-institutions presented above.

An institution with diagrams ι may be denoted by $(Sig, Sen, Mod, \models, \iota)$.

A(n even) more categorical formulation

The diagrams ι of an institution $(Sig, Sen, Mod, \models)$ can be expressed more compactly as a functor $\iota : Mod^\sharp \to Th^\to$ from the Grothendieck category $Mod^\sharp$ determined by the model functor Mod to the category Th of the theories of the institution such that

$$\begin{array}{ccc} Mod^\sharp & \xrightarrow{\ \iota\ } & Th^\to \\ {\scriptstyle (\text{fibration})\ \text{projection}}\downarrow & & \downarrow{\scriptstyle dom} \\ Sig & \xrightarrow[\text{left adjoint}]{} & Th \end{array}$$

commutes, where $Th^\to$ is the functor category of theories morphisms (i.e., the functors $(\bullet \longrightarrow \bullet) \to Th$), and dom is the functor projecting on the domain of the presentation morphisms, and such that the following functors $Mod^\sharp \to \mathbb{C}at^\to$ are isomorphic:

$$\begin{array}{ccc} Mod^\sharp & & Mod^\sharp \\ {\scriptstyle \iota}\downarrow & & \Big\downarrow \\ Th^\to & \cong & {\scriptstyle -/Mod(-)} \\ {\scriptstyle Mod^\to}\downarrow & & \Big\downarrow \\ \mathbb{C}at^\to & & \mathbb{C}at^\to \end{array}$$

where

- $Mod^\to(\,(\Sigma, E) \xrightarrow{\ \varphi\ } (\Sigma', E')\,) = Mod(\Sigma', E') \xrightarrow{Mod\varphi} Mod(\Sigma, E)$ and
- $(-/Mod(-))\langle \Sigma, M\rangle = M/Mod\Sigma \to Mod\Sigma$.

Elementary homomorphisms

Recall that a $\mathcal{FOL}$-model homomorphism $h : M \to N$ is by definition an *elementary embedding* if and only if M_M and N_h are elementarily equivalent (they satisfy exactly the same sentences). Note that this relies on the diagrams of $\mathcal{FOL}$. In the same way a concept of 'elementary homomorphism' can be defined in any abstract institution provided it has diagrams.

Fact 4.11. *In any institution with diagrams ι, the diagram of any model M has an initial model, denoted M_M.*

A model homomorphism $h : M \to N$ is ι-*elementary* when $N_h (= i^{-1}_{\Sigma,M} N) \models (M_M)^*$. By applying the Satisfaction Condition for $\iota_\Sigma M$ we have:

Fact 4.12. *For each ι-elementary homomorphism $h : M \to N$, $M^* \subseteq N^*$.*

When the ι-elementary homomorphisms are closed under compositions and under model reducts, by restricting the model homomorphisms only to those that are ι-elementary we get a sub-institution of the original institution. This happens in $\mathcal{FOL}$, although seeing why and how is not immediate. Much more difficult is to have this property at the general institution-independent level. In Sect. 5.6 we will solve this problem.

We say that an institution with diagrams ι is ι-*elementary* when each model homomorphism is elementary. For instance the sub-institution $E(\mathcal{FOL})$ determined by the $\mathcal{FOL}$ the elementary embeddings is ι'-elementary where ι' is the system of diagrams of the $\mathcal{FOL}$ elementary embeddings (it shares with the standard system of $\mathcal{FOL}$ diagrams the elementary extensions but the diagram of a model M is $(M_M)^*$).

Fact 4.13. *An institution with diagrams ι is ι-elementary if and only if $(M_M)^* = (E_M)^{**}$ for each model M.*

Morphisms of institutions with diagrams

Given a model M for a $\mathcal{FOL}$ signature (S,F,P), notice that the forgetful institution morphism $\mathcal{FOL} \to \mathcal{EQL}$

- maps the elementary extension $(S,F,P) \hookrightarrow (S,F_M,P)$ to the elementary extension $(S,F) \hookrightarrow (S,F_M)$ of algebraic signatures which corresponds to the (S,F)-algebra underlying M, and
- the diagram of the (S,F)-algebra underlying M is the restriction of the diagram of M to all equations.

This situation suggests that the forgetful institution morphism $\mathcal{FOL} \to \mathcal{EQL}$ acts as a 'morphism of diagrams' between the system of diagrams of $\mathcal{FOL}$ and that of $\mathcal{EQL}$.

In general, a *morphism of institutions with diagrams* $(\Phi,\alpha,\beta)\colon\ (I',\iota') \to (I,\iota)$ is an institution morphism such that

$$\begin{array}{ccc} Mod'^{\sharp} & \xrightarrow{\iota'} & Th'^{\to} \\ {\scriptstyle \beta^{\sharp}}\downarrow & & \downarrow{\scriptstyle \Phi^{\to}} \\ Mod^{\sharp} & \xrightarrow[\iota]{} & Th^{\to} \end{array}$$

commutes, where

- for each signature $\Sigma' \in |Sig'|$ and each Σ'-model M', the functor $\beta^{\sharp}$ maps $\langle \Sigma', M' \rangle$ to $\langle \Phi\Sigma', \beta'_{\Sigma} M' \rangle$, and
- the functor $\Phi^{\to}$ maps each theory morphism $\varphi\colon\ (\Sigma'_1, E'_1) \to (\Sigma'_2, E'_2)$ to $\Phi\varphi\colon\ (\Phi\Sigma'_1, \alpha^{-1}_{\Sigma'_1} E'^{**}_1) \to (\Phi\Sigma'_2, \alpha^{-1}_{\Sigma'_2} E'^{**}_2)$.

More concretely, this means that $\Phi(\iota'_{\Sigma'} M') = \iota_{\Phi\Sigma'}(\beta_{\Sigma'} M')$ (which implies $\Phi\Sigma'_{M'} = (\Phi\Sigma')_{\beta_{\Sigma'} M'}$) and $E_{\beta_{\Sigma'} M'} \models\!\mid \alpha^{-1}_{\Sigma'_{M'}} E^{**}_{M'}$ for each signature $\Sigma' \in |Sig'|$ and each Σ'-model M'.

The category of institutions with diagrams is denoted as $\mathbb{EDI}ns$.

A dual concept of 'comorphism of institutions with elementary diagrams' can be defined similarly.

Co-limits of models

In the presence of diagrams, co-limits of models can be obtained from corresponding co-limits of signatures. This is an important consequence of the existence of diagrams for at least two reasons. On the one hand in the actual institutions, co-limits of models are much more difficult to establish than co-limits of signatures. On the other hand, in general, in actual institutions, the co-limits of models are more complicated than the limits of models. For instance, in algebra the limits of algebraic structures are created from the limits of their carrier sets. Groups are an example. But on the other hand, if we think of co-products of groups it is quite clear that these are significantly more complicated than the products. And if we considered R-modules then the gap between limits and co-limits becomes even more clear since the co-products of R-modules (R commutative ring) are the tensor products.

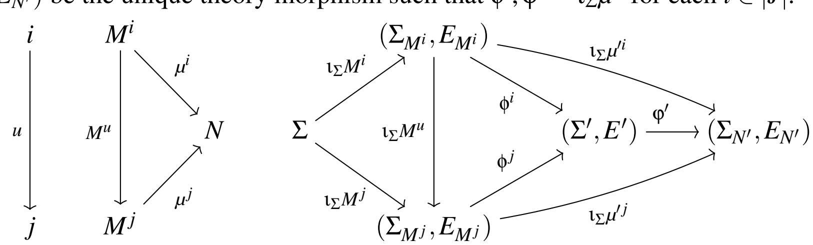
Theorem 4.14. *Consider an institution with diagrams and initial models of theories. If the category of signatures Sig has J-co-limits and the institution has J-model amalgamation then, for each signature Σ, the category of Σ-models has J-co-limits.*

Proof. Let J be a category such that Sig has J-co-limits, and consider a J-diagram $M : J \to Mod\Sigma$ of Σ-models. Let us denote Mi by M^i for each index $i \in |J|$, Mu by M^u for each index morphism $u \in J$, and let $\iota_\Sigma M^i : \Sigma \to (\Sigma_{M^i}, E_{M^i})$ be the diagram of M^i.

- We take the co-limit $(\phi^i : (\Sigma_{M^i}, E_{M^i}) \to (\Sigma', E'))_{i \in |J|}$ of $(\iota_\Sigma M^u : (\Sigma_{M^i}, E_{M^i}) \to (\Sigma_{M^j}, E_{M^j}))_{u \in J}$.
- Let $0_{\Sigma',E'}$ be the initial model in $Mod(\Sigma', E')$. Then we define $N = 0_{\Sigma',E'} \restriction_\varphi$ where $\varphi = \iota_\Sigma M^i ; \phi^i$.
- Then the co-limit $\mu : M \Rightarrow N$ is defined by $\mu_i = i_{\Sigma,M^i}(0_{\Sigma',E'} \restriction_{\phi^i})$ for each $i \in |J|$. That μ is a co-cone can be checked as follows. For each $u : i \to j$ we have that:

1	$i^{-1}_{\Sigma,M^i}(M^u;\mu^j) = i^{-1}_{\Sigma,M^j}(\mu^j)\restriction_{\iota_\Sigma M^u}$	naturality of i
2	$i^{-1}_{\Sigma,M^j}(\mu^j)\restriction_{\iota_\Sigma M^u} = 0_{\Sigma',E'}\restriction_{\phi^j}\restriction_{\iota_\Sigma M^u} = 0_{\Sigma',E'}\restriction_{\phi^i}$	definition of μ^j, $\phi^i = \iota_\Sigma M^u ; \phi^j$
3	$M^u;\mu^j = i_{\Sigma,M^i}(0_{\Sigma',E'}\restriction_{\phi^i}) = \mu^i$	1, 2, definition of μ^i.

- For proving the universal property of μ let us consider a co-cone $\mu' : M \Rightarrow N'$.
  - This determines a co-cone $((\Sigma_{M^i}, E_{M^i}) \xrightarrow{\iota_\Sigma \mu'^i} (\Sigma_{N'}, E_{N'}))_{i \in |J|}$ and let $\varphi' : (\Sigma', E') \to (\Sigma_{N'}, E_{N'})$ be the unique theory morphism such that $\phi^i;\varphi' = \iota_\Sigma \mu'^i$ for each $i \in |J|$.



Let $h: N \to N'$ be the φ-reduct of the unique model homomorphism $h': 0_{\Sigma',E'} \to (N'_{N'})\upharpoonright_{\varphi'}$. For each $i \in |J|$ we have that

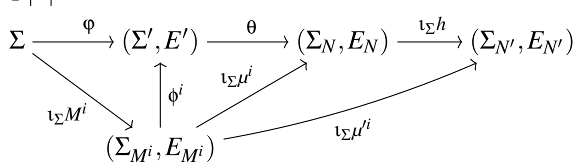
$$h'\upharpoonright_{\phi^i}: 0_{\Sigma',E'}\upharpoonright_{\phi^i} \to (N'_{N'})\upharpoonright_{\varphi'}\upharpoonright_{\phi^i} (= (N'_{N'})\upharpoonright_{\iota_\Sigma \mu'^i})$$

$$0_{\Sigma',E'}\upharpoonright_{\phi^i} = i^{-1}_{\Sigma,M^i}\mu^i \qquad \text{definition of } \mu^i$$

$$i^{-1}_{\Sigma,M^i}\mu'^i = i^{-1}_{\Sigma,M^i}(\mu'^i;1_{N'}) = (i^{-1}_{\Sigma,N'}1_{N'})\upharpoonright_{\iota_\Sigma \mu'^i} = (N'_{N'})\upharpoonright_{\iota_\Sigma \mu'^i} \qquad i \text{ natural, definition of } N'_{N'}.$$

Hence $h = i_{\Sigma,M^i}(h'\upharpoonright_{\phi^i}): \mu^i \to \mu'^i$ in the comma category $M^i/Mod\Sigma$. This shows that $\mu^i;h = \mu'^i$ for each $i \in |J|$.

– It remains to show that h is unique. Let us first establish that $N_N\upharpoonright_\theta = 0_{\Sigma',E'}$ where $\theta: (\Sigma',E') \to (\Sigma_N,E_N)$ is the unique theory morphism such that $\phi^i;\theta = \iota_\Sigma \mu^i$ for each $i \in |J|$.



For this, by the J-model amalgamation hypothesis is enough to establish that $N_N\upharpoonright_\theta\upharpoonright_{\phi^i} = 0_{\Sigma',E'}\upharpoonright_{\phi^i}$ for each $i \in |J|$. This holds because

$$\begin{aligned} i_{\Sigma,M^i}(N_N\upharpoonright_\theta\upharpoonright_{\phi^i}) &= i_{\Sigma,M^i}(N_N\upharpoonright_{\iota_\Sigma \mu^i}) && \phi^i;\theta = \iota_\Sigma \mu^i \\ &= \mu^i; i_{\Sigma,N}(N_N) && \text{naturality of } i \\ &= \mu^i; 1_N = \mu^i && \text{definition of } N_N. \end{aligned}$$

– Now we apply $i^{-1}_{\Sigma,N}$ to $h: 1_N \to h$; let denote the result by $h'': N_N \to N'_h = N'_{N'}\upharpoonright_{\iota_\Sigma h}$. Then

1 $\quad \phi^i ; \theta ; \iota_\Sigma h = \iota_\Sigma \mu^i ; \iota_\Sigma h = \iota_\Sigma \mu'^i = \phi^i ; \varphi'$ — definition of θ, ι functorial, definition of φ'

2 $\quad \theta ; \iota_\Sigma h = \varphi$ — 1, ϕ co-limit co-cone

3 $\quad h''\upharpoonright_\theta: N_N\upharpoonright_\theta \to N'_{N'}\upharpoonright_{\iota_\Sigma h}\upharpoonright_\theta = N'_{N'}\upharpoonright_\varphi$ — 2, definition of h''.

– The uniqueness of $h = h''\upharpoonright_\theta\upharpoonright_\varphi$ now follows from the uniqueness of $h''\upharpoonright_\theta$ which follows from the initiality property of $(N_N)\upharpoonright_\theta = 0_{\Sigma',E'}$.

□

Co-limits of $\mathcal{FOL}$ models. Let us apply Thm. 4.14 above to obtain the existence of co-limits of $\mathcal{FOL}$ models. The method illustrated by the proof of Cor. 4.15 may be also applied to other actual institutions.

Corollary 4.15. *The category of models of any $\mathcal{FOL}$ signature has small co-limits.*

Proof. Let us consider the sub-institution $\mathcal{AFOL}$ of the atoms of $\mathcal{FOL}$, which restricts the sentences to (equational or relational) atoms only. $\mathcal{AFOL}$ inherits the $\mathcal{FOL}$ diagrams, but unlike $\mathcal{FOL}$, it has initial models for all its theories (a result which we anticipate and is given by Cor. 4.28 below). The category of signatures has small co-limits (cf. Prop. 4.4) and the institution has J-model amalgamation for all small categories J (cf. Prop. 4.7) Therefore by Thm. 4.14 the category of models of any signature has small co-limits. □

Exercises

4.41. The standard diagrams of $\mathcal{FOL}$ can be defined slightly differently than the ordinary way, such that the elementary extension adds to the given signature only the elements which are not interpretations of constants.

4.42. Co-products of groups
Construct the co-products of any two groups G^1 and G^2 by following the steps in the proof Thm. 4.14. The more ambitious readers may also try to construct in the same way the tensor product of two R-modules.

4.43. A $\mathcal{FOL}$ model homomorphism $h\colon\ M \to N$ is *strong* when $N_\pi = h(M_\pi)$ for each relation symbol π of the signature. The sub-institution of infinitary first-order logic $\mathcal{FOL}_{\infty,\omega}$ where the model homomorphisms are restricted to the strong ones, has diagrams with the same elementary extensions as in $\mathcal{FOL}$ but such that for each (S,F,P)-model M the diagram E_M consists of the $\mathcal{FOL}$ diagram plus

- $\{\neg\pi m \in (M_M)^* \mid \pi \in P\}$, and
- all sentences of the form

$$(\forall X)(\pi X \Rightarrow \bigvee_{m\in M_w} (X = m))$$

for each relation symbol π of arity w, and where $X = m$ means $\bigwedge_{1\leq k\leq n}(x_k = m_k)$ for $X = x_1 \ldots x_n$ and $m = m_1 \ldots m_n$.

4.44. Borrowing diagrams
Let I' be an institution with diagrams ι' and let $(\Phi,\alpha,\beta)\colon\ I \to I'$ be an institution comorphism such that

1. Φ is full and faithful,
2. β_Σ are isomorphisms (for each model M let M' denote $\beta_\Sigma^{-1}M$),
3. for each Σ-model M in I:
   (a) there exists a signature Σ_M in I such that $(\Phi\Sigma)_{M'} = \Phi\Sigma_M$, and
   (b) for each sentence $\rho' \in E_{M'}$ there exists a Σ_M-sentence ρ such that $\rho' \models\!\mid \alpha_{\Sigma_M}\rho$.

Then the institution I has diagrams ι defined by

- $\iota_\Sigma M$ is the unique signature morphism such that $\Phi(\iota_\Sigma M) = \Phi\Sigma_M$, and
- $E_M = \{\rho \mid \text{there exists } \rho' \in E_{M'} \text{ such that } \alpha_{\Sigma_{M'}}\rho \models\!\mid \rho'\}$.

Apply this general result to the embedding comorphism $\mathcal{EQL} \to \mathcal{FOL}$.

4.45. The table below gives the diagrams of several institutions:

I	Σ	Σ_M	M_M	E_M
$\mathcal{PA}$	(S,TF,PF)	(S,TF_M,PF) with $(TF_M)_{\to s} = TF_{\to s} \uplus M_s$ for $s \in S$	$(M_M)_m = m$ for $m \in M$	$\{t \stackrel{e}{=} t' \mid M_M \models t \stackrel{e}{=} t'\}$
$\mathcal{POA}$	(S,F)	(S,F_M) with $(F_M)_{\to s} = F_{\to s} \uplus M_s$ for $s \in S$	$(M_M)_m = m$ for $m \in M$	$\{t = t' \mid M_M \models t = t'\} \cup$ $\{t \leq t' \mid M_M \models t \leq t'\}$
$\mathcal{MBA}$	(S,K,F,kind)	(S,K,F_M,kind) with $(F_M)_{\to k} = F_{\to k} \uplus M_k$ for $k \in K$	$(M_M)_m = m$ for $m \in M$	$\{t = t' \mid M_M \models t = t'\} \cup$ $\{(t : s) \mid M_M \models (t : s)\}$
$\mathcal{MA}$	(S,F)	(S,F_M) with $(F_M)_{\to s} = F_{\to s} \uplus M_s$ for $s \in S$	$(M_M)_m = \{m\}$ for $m \in M$	$\{m \doteq m \mid m \in M\} \cup$ $\{x \prec \sigma m \mid \sigma \in F_{w\to s}$ and $m \in M_w$ and $x \in M_\sigma m\}$
$\mathcal{CA}$	(S,F,q)	(S,F_M,q) with $(F_M)_{\to s} = F_{\to s} \uplus M_s$ for $s \in S$	$(M_M)_m = m$ for $m \in M$	$\{t \approx_\varepsilon t' \mid M_M \models t \approx_\varepsilon t'\}$
$\mathcal{HNK}$	(S,F)	(S,F_M) with $(F_M)_s = F_s \uplus M_s$ for $s \in S$	$(M_M)_m = m$ for $m \in M$	$\{t = t' \mid M_M \models t = t'\}$

4.46. Diagrams in $\mathcal{IPL}$
$\mathcal{IPL}$ has diagrams as follows. Let $M: P \to A$ be a P-model. Then Σ_M is the disjoint union $P \uplus A$ and we define the Σ_M-model $M_M: P \uplus A \to A$ by $(M_M)_\pi = M_\pi$, $\pi \in P$ and $(M_M)_a = a$, $a \in A$. Let Δ_A be the diagram of A in $\mathcal{FOL}$; this consists of equalities of the form $a \wedge a' = a\, A_\wedge\, a'$, $a \vee a' = a\, A_\vee\, a'$, $a \Rightarrow a' = a\, A_\Rightarrow\, a'$. Let Δ'_A consist of the translations of the equations in Δ_A as $\mathcal{IPL}$-sentences. (For instance $a \wedge a' = b$ gets translated as $(a \wedge a' \Rightarrow b) \wedge (b \Rightarrow a \wedge a')$.) Then $E_M = \Delta'_A \cup \{a \Rightarrow \pi \mid M_M \models a \Rightarrow \pi,\ a \in A,\ \pi \in P\}$.

4.47. [74] **Diagrams in $\mathcal{MVL}^\sharp$**
$\mathcal{MVL}^\sharp$ does not have diagrams, however, its extension with equalities between constants does have diagrams as follows. For each $\mathcal{MVL}^\sharp$ signature (S,C,P) and each (S,C,P)-model M, the elementary extension adds the elements of M as new constants and the diagram of M consists of $\{(\pi m, M_\pi m) \mid \pi \in P_w,\ m \in M_w\} \cup \{c = M_c \mid s \in S,\ c \in C_s\}$.

4.48. Diagrams in $\mathcal{WPL}$
For any given set P (of propositional symbols), there exists a partial order on the $\mathcal{WPL}$-models (see Ex. 3.27) given by $M \leq N$ if and only if for each $\pi \in P$, $M\pi = 1$ implies $N\pi = 1$. Let this partial order define the category $Mod^{\mathcal{WPL}}P$. Then $\mathcal{WPL}$ has diagrams.

4.49. Let $Cat_+\mathcal{EQL}$ be the sub-institution of $Cat\mathcal{EQL}$ determined by categories with binary co-products. Then $Cat_+\mathcal{EQL}$ has empty diagrams. (*Hint:* For any object A in a category $\mathbb{C}$ having binary co-products, the elementary extension of $\mathbb{C}$ via A is the left adjoint to the forgetful functor $A/\mathbb{C} \to \mathbb{C}$.)

4.50. Limits and co-limits of $\mathcal{MVL}^\sharp$ models
The category of $\mathcal{MVL}^\sharp$-models for a fixed signature has small limits and co-limits. (*Hint:* In the case of the co-limits apply Thm. 4.14.)

4.51. Diagrams for theories
(a) For each institution I with diagrams the institution I^{th} of its theories has diagrams such that the

(original) diagrams of I are 'borrowed' from those of I^{th} along the canonical embedding comorphism $I \to I^{\text{th}}$.
(b) As an application to (a), for any institution with diagrams and initial models for theories, the categories of models of theories have all (co-)limits of the category of signatures.
(c) Does (b) apply to the problem of co-products of groups (see Ex. 4.42)?

4.52. [64] **Weak limits of models**
Consider an institution with diagrams and initial models of theories. Then, for each signature Σ, the category of Σ-models has weak J-limits whenever the category of signatures Sig has J-limits.

4.53. Preserving carriers
A signature morphism $\varphi : \Sigma \to \Sigma'$ *preserves carriers* when

$$\begin{array}{ccc} \Sigma & \xrightarrow{\iota_\Sigma M} & \Sigma_M \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \iota_\varphi 1_M} \\ \Sigma' & \xrightarrow[\iota_{\Sigma'} M']{} & \Sigma'_{M'} \end{array}$$

is a pushout of signature morphisms for all Σ'-models M' and Σ-models M for which $M' \upharpoonright_\varphi = M$.
Then signature morphisms preserving carriers are closed under composition.
In $\mathcal{FOL}$ all signature morphisms which are bijective on sorts preserve the carriers.

4.54. Study the model amalgamation properties of $E(\mathcal{FOL})$, i.e., the sub-institution of $\mathcal{FOL}$ with elementary embeddings as model homomorphisms.

4.5 Inclusion systems

In this section, we introduce a general category-theoretic device that provides support for abstract institution-independent concepts of 'sub-model' and 'quotient model'. Furthermore, we will also discuss applications of this concept to categories of signatures and theories.

The standard inclusion system of $\mathbb{S}et$. Each function $f : A \to B$ can be factored as a composition between a surjection and an inclusion, i.e., $f = e_f \ ; \ i_f$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle e_f}\searrow \quad \nearrow{\scriptstyle i_f} & \\ & fA & \end{array}$$

as follows:

- $fA = \{fa \mid a \in A\}$,
- $e_f a = fa$ for each $a \in A$, and
- $i_f b = b$ for each $b \in fA$.

It is easy to see that this factorization is *unique*, that is for any other factorization $f = e'_f;i'_f$ with e'_f surjection and i'_f inclusion we necessarily have $e'_f = e_f$ and $i'_f = i_f$. The existence and the uniqueness of such a factorization are a consequence of the nature of the surjective functions and of the inclusions. For the uniqueness, it is especially important that inclusions are unique in the sense that there exists at most one inclusion between any two given sets.

This factorization phenomenon may be found in various forms in many other categories, including categories of models. In a salient way, it constitutes an important conceptual device in model theory, especially in its areas that are closer to universal algebra.

Categorical inclusion systems. The factorization property of functions presented above can be expressed at the level of abstract categories. In this book, this property will be used in two different ways: on the one hand, for categories of models in institutions, and on the other hand, for the categories of signatures / theories of institutions.

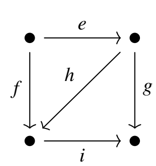
$\langle I, \mathcal{E}\rangle$ is an *inclusion system* for a category $\mathbb{C}$ if I and $\mathcal{E}$ are two broad sub-categories of $\mathbb{C}$, i.e., $|I| = |\mathcal{E}| = |\mathbb{C}|$, such that

1. I is a partial order, and
2. every arrow f in $\mathbb{C}$ can be factored uniquely as $f = e_f \ ; \ i_f$ with $e_f \in \mathcal{E}$ and $i_f \in I$.

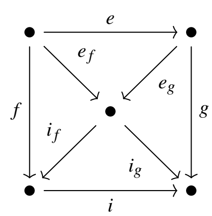
The arrows of I are called *abstract inclusions*, and the arrows of $\mathcal{E}$ are called *abstract surjections*. The domain of the inclusion i_f in the factorization of f is called the *image of f* and is denoted as $\mathrm{Im}(f)$ or fA when $dom(f) = A$. That I is partial order means that between any objects A and B there exists *at most* one arrow in I. Hence it is appropriate to denote abstract inclusions $A \to B$ simply by $A \subseteq B$, which yields a partial order on the objects of $\mathbb{C}$. While the *reflexivity* and *transitivity* of $\subseteq$ follow from the fact that I is a broad sub-category of $\mathbb{C}$, the *anti-symmetry* of $\subseteq$ means that $A \subseteq B$ and $B \subseteq A$ implies $A = B$.

The following property is a useful technical device in many proofs.

Lemma 4.16 (Diagonal-fill)**.** *Given an inclusion system $\langle I, \mathcal{E}\rangle$ in a category $\mathbb{C}$, if $f, g \in \mathbb{C}$, $e \in \mathcal{E}$, $i \in I$, and $f;i = e;g$ then there exists an unique $h \in \mathbb{C}$ such that $e;h = f$ and $h;i = g$.*



Proof. Let us factor $f = e_f;i_f$ and $g = e_g;i_g$. Then $e;e_g;i_g = e;g = f;i = e_f;i_f;i$. By the uniqueness of the factorization of $e;g = f;i$ it follows that $e_f = e;e_g$ and $i_g = i_f;i$ and also that $dom(i_f) = cod(e_g)$. Then $h = e_g; i_f$.



The uniqueness of h follows by noticing that each inclusion is mono and because $h; i = g$. □

Epic inclusion systems. The abstract surjections of some inclusion systems need not necessarily be surjective in the ordinary set-theoretic sense. Consider for example the trivial inclusion system for $\mathbb{S}et$ where each function is an abstract surjection and the abstract inclusions are just the identities. An inclusion system $\langle I, \mathcal{E}\rangle$ is *epic* when all abstract surjections are epis. Therefore the standard inclusion system of $\mathbb{S}et$ presented above is epic, while the trivial one is not.

Unions. An inclusion system $\langle I, \mathcal{E}\rangle$ *has unions* when I has finite least upper bounds (denoted $\cup$). Note that the standard inclusion system of $\mathbb{S}et$ has unions which are exactly the usual unions of sets, while the trivial inclusion system of $\mathbb{S}et$ does not have unions.

Inclusive functors. A functor $\mathcal{U}\colon \langle I, \mathcal{E}\rangle \to \langle I', \mathcal{E}'\rangle$ (between the underlying categories of the inclusion systems) is *inclusive* when it preserves the inclusions, i.e., $\mathcal{U}I \subseteq I'$. Inclusion systems and inclusive functors form a category denoted $\mathbb{IS}$.

Submodels and quotients in $\mathcal{FOL}$

Now we will see how inclusion systems capture submodels and quotient of models from conventional first-order model theory.

Closed and strong model homomorphisms. The category of models for a $\mathcal{FOL}$ signature (S,F,P) admits two meaningful epic inclusion systems which inherit the standard inclusion system of the category of sets and functions. Before discussing them, we have to define some special classes of model homomorphisms.

A model homomorphism $h\colon M \to N$

- is *closed* when $M_\pi = h^{-1}N_\pi$ for each relation symbol $\pi \in P$, and
- is *strong* when $hM_\pi = N_\pi$ for each relation symbol $\pi \in P$.

For each model homomorphism $M \to N$ that is a set inclusion for each sort $s \in S$, let us say that M is a *submodel* of N.

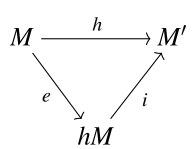
Inclusion systems for $\mathcal{FOL}$ models.

Fact 4.17. *For any $\mathcal{FOL}$ signature (S,F,P), the category of (S,F,P)-models admits the following two inclusion systems:*

inclusion system	abstract surjections	abstract inclusions
closed	surjective homomorphisms	closed sub-models
strong	strong surjective homomorphisms	sub-models

Moreover, for each signature morphism $\varphi: (S,F,P) \to (S',F',P')$, the model reduct functor $Mod\varphi$ is inclusive between both the closed and the strong inclusion systems of $Mod(S',F',P')$ and $Mod(S,F,P)$

The difference between these two inclusion systems can easily be understood when we try to factor a model homomorphism $h: M \to M'$:



Then in both inclusion systems $e\,;\,i$ is the unique factorization of h as (many-sorted) function and $(hM)_s = h_s M_s$ for each sort s. Also, in both inclusion systems the interpretation of the operation symbols is canonically defined by $(hM)_\sigma m = M'_\sigma m$ for each operation symbol $\sigma \in F_{w\to s}$ and each $m \in (hM)_w$. It is easy to see that for the carriers and the operations there is no other possibility. However, the difference between the two inclusion systems occurs at the level of the interpretations of the relation symbols for hM. Given $\pi \in P$, we should have $e(M_\pi) \subseteq (hM)_\pi$ and $i((hM)_\pi) \subseteq M'_\pi$. This means

$$e(M_\pi) \subseteq (hM)_\pi \subseteq i^{-1}(M'_\pi).$$

For the closed inclusion system the interpretation of the relations is defined 'maximally' for i, while in the second situation they are defined 'minimally' for e.

Congruences. Several types of abstract surjections for model homomorphisms correspond to several types of *congruences*. Given a model M for a $\mathcal{FOL}$ signature (S,F,P), an *S-sorted equivalence* relation $\sim$ on M consists of an equivalence relation $\sim_s$ on M_s for each sort s. As a matter of notation for any list of sorts $w = s_1 \dots s_n$, for any $m,m' \in M_w$, $m \sim_w m'$ when $m_1 \sim_{s_1} m'_1, \dots, m_n \sim_{s_n} m'_n$. Then $\sim$ is

- an (S,F)-*congruence* when for each operation symbol $\sigma \in F_{w\to s}$, $M_\sigma m \sim_s M_\sigma m'$ for all $m,m' \in M_w$ with $m \sim_w m'$,
- a (S,P)-*congruence* when for each relation symbol $\pi \in P_w$, $m \sim_w m'$ and $m \in M_\pi$ implies $m' \in M_\pi$ for each $m,m' \in M_w$, and
- an (S,F,P)-*congruence* when it is both an (S,F)-congruence and an (S,P)-congruence.

Quotient models. Given an (S,F)-congruence $\sim$ on M, the *quotient* $M/_{\sim}$ (of the model M by the congruence $\sim$) is defined by

- $(M/_{\sim})_s = \{m/_{\sim} \mid m \in M_s\}$ is the set of equivalence classes for $\sim_s$ for each sort $s \in S$,
- $(M/_{\sim})_{\sigma}(m/_{\sim}) = (M_{\sigma}m)/_{\sim}$ for each operation $\sigma \in F_{w\to s}$ and each $m \in M_w$, and
- $(M/_{\sim})_{\pi} = \{m/_{\sim} \mid m \in M_{\pi}\}$ for each relation symbol $\pi \in P$.

The homomorphism $M \to M/_{\sim}$ mapping each m to its congruence class $m/_{\sim}$ is called a *quotient homomorphism*.

Fact 4.18. *Any quotient homomorphism $M \to M/_{\sim}$ is strong surjective. Moreover when $\sim$ is an (S,F,P)-congruence it is also closed.*

It is also easy to see that each closed abstract surjection is strong too.

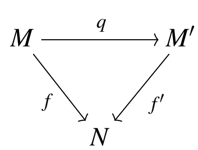
Kernels. Given a model homomorphism $f : M \to N$, its *kernel* is defined by

$$=_f \;=\; \{(a,a') \mid fa = fa'\}$$

Fact 4.19. *The kernel of any homomorphism f is an (S,F)-congruence. Moreover, it is an (S,F,P)-congruence when f is closed.*

The universal property of quotients. Model quotients admit the following universal property:

Proposition 4.20. *Let $q : M \to M'$ be a surjective (S,F,P)-model homomorphism for a signature (S,F,P). Then for each model homomorphism $f : M \to N$, if $=_q \subseteq =_f$, then there exists a unique model homomorphism $f' : M' \to N$ such that $q;f' = f$.*



Moreover, f' is strong when f is strong and it is closed when f is closed.

Proof. f' is defined by $f'(m/_{=_q}) = fm$ for each $m \in M$. This definition is correct since $=_q \subseteq =_f$. The fact that f is an (S,F)-algebra homomorphism implies that f' is an (S,F)-algebra homomorphism. Also, the fact that f is a (S,P)-model homomorphism implies that f' is a (S,P)-model homomorphism. The uniqueness of f' follows from the fact that q is surjective.

Simple calculations show that f being strong / closed, respectively, implies f' is strong / closed, respectively. □

Corollary 4.21. *For each strong surjective model homomorphism $f : M \to N$, $M/_{=_f} \cong N$, i.e., $M/_{=_f}$ and N are isomorphic.*

Proof. In Prop. 4.20 above assume that f is surjective and that q is the quotient $M \to M/_{=_f}$. Then $(=_q) = (=_f)$ implies that f' is injective, while f surjective implies that f' is surjective. Therefore f' is a bijection, which makes it immediately an (S,F)-algebra isomorphism. When f is strong, f' is also strong, which means that for each relation symbol $\pi \in P$, $f'((M/_{=_f})_\pi) = N_\pi$. This implies that the inverse f'^{-1} is also a (S,P)-model homomorphism, hence f' is an (S,F,P)-model isomorphism. □

Signature inclusions in $\mathcal{FOL}$

Now we study inclusion systems at the level of the syntax of $\mathcal{FOL}$.

Fact 4.22. *The category of $\mathcal{FOL}$ signatures admits the inclusion systems given by the table below:*

inclusion system	abstract surjections $\varphi : (S,F,P) \to (S',F',P')$	abstract inclusions $(S,F,P) \hookrightarrow (S',F',P')$
closed	$\varphi^{\mathrm{st}} : S \to S'$ surjective	$S \subseteq S'$ $F_{w\to s} = F'_{w\to s}$ for $w \in S^*$ $P_w = P'_w$ for $s \in S$
strong	$\varphi^{\mathrm{st}} : S \to S'$ surjective $F'_{w'\to s'} = \bigcup_{\varphi^{\mathrm{st}}(ws)=w's'} \varphi^{\mathrm{op}}(F_{w\to s})$ $P'_{w'} = \bigcup_{\varphi^{\mathrm{st}}(w)=w'} \varphi^{\mathrm{rl}}(P_w)$	$S \subseteq S'$ $F_{w\to s} \subseteq F'_{w\to s}$ for $w \in S^*$ $P_w \subseteq P'_w$ for $s \in S$

Other non-trivial inclusion systems for $Sig^{\mathcal{FOL}}$ can be obtained by considering the closed property at the level of the operation symbols and the strong property at the level of the relation symbols, or vice versa.

We can also note that the closed inclusion system does not have unions but the strong one has them:

Fact 4.23. *The strong inclusion system has unions. The union of signatures $(S,F,P) = (S_1,F_1,P_1) \cup (S_2,F_2,P_2)$ is given by*

- $S = S_1 \cup S_2$,
- *for each $w \in S^*$ and $s \in S$, $F_{w\to s} = (F'_1)_{w\to s} \cup (F'_2)_{w\to s}$ where $(F'_k)_{w\to s} = (F_k)_{w\to s}$ when $w \in S_k^*$, $s \in S_k$ and $(F'_k)_{w\to s} = \emptyset$ otherwise, and*
- *for each $w \in S^*$, $P_w = (P'_1)_w \cup (P'_2)_w$ where $(P'_k)_w = (P_k)_w$ when $w \in S_k^*$ and $(P'_k)_w = \emptyset$ otherwise.*

Inclusive institutions. An institution $(Sig, Sen, Mod, \models)$ is called *inclusive* when Sen is an inclusive functor, i.e., the category of signatures comes equipped with an inclusion system such that $Sen\Sigma \subseteq Sen\Sigma'$ whenever $\Sigma \subseteq \Sigma'$ is an inclusion of signatures. For example $\mathcal{PL}$ (propositional logic) is inclusive by considering the inclusion system for its signatures to be the standard inclusion system of $\mathbb{S}et$, but $\mathcal{FOL}$ falls short from being inclusive because of the translations of the quantifiers that update the signature part in the qualification of the variables (i.e. for any inclusion of signatures $\Sigma \subseteq \Sigma'$, a variable (x,s,Σ)

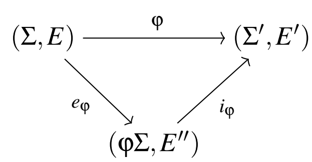
gets translated to (x, s, Σ')). At this moment an adjustment of the definition of $\mathcal{FOL}$ such that it becomes an inclusive institution is an open research problem. An alternative to this is to by-pass this problem by relaxing the inclusive functor condition by allowing each signature inclusion $\Sigma \subseteq \Sigma'$ to get mapped to a designated injection $Sen\Sigma \to Sen\Sigma'$.

Theory inclusions

Let us consider an arbitrary institution I such that its category of signatures is endowed with an inclusion system. Its category *CTh* of closed theories inherits the inclusion systems from the category of signatures in two different ways similar to the ways model homomorphisms in $\mathcal{FOL}$ inherit the conventional inclusion system of $\mathbb{S}et$. A theory morphism $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ in *CTh*

- is *closed* when $E = \varphi^{-1}E'$, and
- is *strong* when $E' = (\varphi E)^{**}$.

Then in *CTh*, we may factor each theory morphism $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ through the inclusion system of the signatures as $\varphi = e_\varphi\, ;\, i_\varphi$



In order to get a factorization of φ in *CTh*, it remains to fix the theory E''. Since E, E', E'' are all closed theories, we can establish easily that

$$(e_\varphi E)^{**} \subseteq E'' \subseteq i_\varphi^{-1}E'.$$

This means that at the general level, we have two choices for E'',

- a 'maximal' one, when i_φ is closed, or
- a 'minimal' one, when e_φ is strong.

Hence we can formulate the following result (and leave it straightforward proof as an exercise to the reader).

Proposition 4.24. *In any institution, an inclusion system of signatures lifts to closed theories in two different ways as shown in the following table:*

inclusion system	abstract surjections $\varphi\colon (\Sigma, E) \to (\Sigma', E')$	abstract inclusions $(\Sigma, E) \subseteq (\Sigma', E')$
closed	$\varphi\colon \Sigma \to \Sigma'$ abstract surjection	$\Sigma \subseteq \Sigma'$ abstract inclusion (i) $E = i^{-1}E'$
strong	$\varphi\colon \Sigma \to \Sigma'$ abstract surjection $E' = (\varphi E)^{**}$	$\Sigma \subseteq \Sigma'$ abstract inclusion

At this point perhaps many of us wonder why *CTh* and not *Th* for establishing meaningful inclusions systems for categories of theory morphisms? If we tried with *Th* then E'' in (Σ'', E'') would be essentially undetermined in any of the 'closed' and 'strong' choices, which are in fact about E''^{**}. The problem here is that there can be many E'' with the same closure. At the general level, the only way out of this problem is to impose that E'' is closed, but such a choice would immediately lead to *CTh* (just by looking at what happens with the identities, which are abstract inclusions and abstract surjections simultaneously).

The following result shows that unions are inherited by the strong inclusion system of theories (its proof is left as an exercise to the reader).

Proposition 4.25. *When the inclusion system of the signatures has unions, the strong inclusion system of theories has unions too by letting*

$$(\Sigma, E) \cup (\Sigma', E') = (\Sigma \cup \Sigma', (iE \cup i'E')^{**})$$

where i / i' denote the inclusions $\Sigma \subseteq \Sigma \cup \Sigma'$ / $\Sigma' \subseteq \Sigma \cup \Sigma'$, respectively.

Exercises

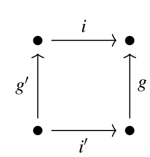
4.55. In any inclusion system the class of the abstract inclusions determines the class of abstract surjections, in the sense that if $\langle I, \mathcal{E}\rangle$ and $\langle I', \mathcal{E}'\rangle$ are two inclusion systems for the same category and if $I \subseteq I'$, then $\mathcal{E}' \subseteq \mathcal{E}$.

4.56. In any inclusion system

- each abstract inclusion is a mono,
- each co-equalizer is an abstract surjection,
- an arrow is both an abstract inclusion and an abstract surjection if and only if it is an identity,
- if $f;g$ is an abstract surjection, then g is an abstract surjection.

4.57. In any category with an inclusion system, the abstract surjections are stable under pushouts.

4.58. In any category with an inclusion system, for each sink $\langle i, g\rangle$ that has a pullback, if i is an abstract inclusion then there exists a unique pullback $\langle g', i'\rangle$ such that i' is an abstract inclusion



Consequently, each sink of abstract inclusions $\bullet \xrightarrow{\subseteq} \bullet \xleftarrow{\supseteq} \bullet$ that has a pullback, has a unique pullback consisting of abstract inclusions. (*Comment:* These allow for the definition of the 'intersection' $A \cap B$ of any two objects in an inclusion system as the pullback of their union $A \cup B$, provided that the latter exists. Moreover, $A \cap B$ is the infimum of A and B in the partial order of the abstract inclusions.)

4.59. In any inclusion system for a category with small limits, each small co-cone $(i_k : N_k \to M)_{k \in I}$ of inclusions has a limit $(i'_k : N \to N_k)_{k \in I}$ consisting of inclusions.

4.60. In any epic inclusion system, the abstract inclusions are *weakly stable under* direct products, i.e. if $(p_i : M \to M_i)_{i \in I}$ is a direct product and $(h_i : N_i \to M_i)_{i \in I}$ are abstract inclusions then there exists a direct product $(q_i : N \to N_i)_{i \in I}$ such that the unique homomorphism $h : N \to M$ such that $h; p_i = q_i; h_i$, $i \in I$, is abstract inclusion.

$$\begin{array}{ccc} N & \xrightarrow{q_i} & N_i \\ h \big\downarrow \subseteq & & \subseteq \big\downarrow h_i \\ M & \xrightarrow[p_i]{} & M_i \end{array}$$

4.61. Properties of $\mathbb{IS}$

- The forgetful functor $\mathbb{IS} \to \mathbb{C}at$ mapping inclusion systems to their underlying categories has a left adjoint and creates small products.
- The category $\mathbb{IS}$ of inclusion systems is cartesian closed.

4.62. Generated closed sub-models
Given an (S,F,P)-model M for a $\mathcal{FOL}$ signature (S,F,P), an arbitrary intersection of [closed] submodels of M is a submodel of M. This allows for the following definition: for any S-sorted set $(X_s)_{s \in S} \subseteq (M_s)_{s \in S}$ we say that N *is the [closed] submodel of* M *generated by* X when N is the least [closed] submodel containing X.

4.63. Intersection of congruences
For any model of any $\mathcal{FOL}$ signature (S,F,P), an arbitrary intersection of (S,F)-congruences is a congruence but only the intersection of a *non-empty* family of (S,F,P)-congruences is an (S,F,P)-congruence.

4.64. For any family $\{f_i : M_i \to N_i \mid i \in I\}$ of model homomorphisms for a fixed $\mathcal{FOL}$ signature, the cartesian product $\prod_{i \in I} f_i : \prod_{i \in I} M_i \to \prod_{i \in I} N_i$ is closed / strong, when the f_i's are closed / strong, respectively.

4.65. Amalgamation of homomorphisms
The sub-institutions of $\mathcal{FOL}$ determined by the closed / strong model homomorphisms, respectively, are exact.

4.66. Inclusion system for preordered algebras
For any algebraic signature (S,F), the category of preordered (S,F)-algebras admits an inclusion system in which the abstract inclusions are *closed* preordered subalgebras, i.e., preordered subalgebras $M \hookrightarrow N$ such that $m \leq_M m'$ if and only if $m \leq_N m'$, and the abstract surjections are just preordered algebra homomorphisms which are (component-wise) surjective functions.

4.67. Preorder algebra congruences
A $\mathcal{POA}$*-congruence* (preorder algebra congruence) on a preorder algebra for a signature (S,F) is a pair $(\sim, \sqsubseteq)$ such that

- $\sim$ is an (S,F)-congruence on M,
- $\sqsubseteq$ is a(n S-sorted) preorder on M compatible with the operations and which contains $M_{\leq}$, i.e., $M_{\leq} \subseteq \sqsubseteq$, and
- $a' \sim a,\ a \sqsubseteq b,\ b \sim b'$ implies $a' \sim b'$ for all elements a, a', b, b' of M.

The $\mathcal{POA}$-*kernel* $ker(h)$ of a preorder algebra homomorphism $h\colon M \to N$ is $(=_h, \leq_h)$ where $a =_h b$ is defined by $ha = hb$ and $a \leq_h b$ by $ha \leq hb$.

Define the quotient of preordered algebras by $\mathcal{POA}$-congruences. Extend Prop. 4.20 to a universal property for preordered algebra quotients.

4.68. Inclusion systems for partial algebra
Let (S, TF, PF) be a $\mathcal{PA}$ signature. A homomorphism $h\colon A \to B$ of partial algebras is

- *full* if whenever $B_\sigma(ha) \in hA$, then there exists $a' \in A_w$ such that $A_\sigma a'$ is defined and $ha' = ha$,
- *closed* when $A_\sigma a$ is defined if $B_\sigma(ha)$ is defined

for each $\sigma \in PF_{w\to s}$.
The category of (S, TF, PF)-partial algebras admits the following inclusion systems:

abstract surjections	*abstract inclusions*
epi homomorphisms	closed inclusions (S_c)
surjective homomorphisms	full inclusions (S_f)
full surjective homomorphisms	(plain) inclusions (S_w)

4.69. Full $\mathcal{FOL}$ model homomorphisms
Let (S, F, P) be a $\mathcal{FOL}$ signature. An (S, F, P)-model homomorphism $h\colon M \to N$ is *full* when $hM_\pi = N_\pi \cap hM_w$ for each relation symbol $\pi \in P$. Then

- any full surjective model homomorphism is strong, and
- any closed model homomorphism is full.

4.70. Which of the institutions $\mathcal{MBA}$, $\mathcal{MA}$, $\mathcal{CA}$, $\mathcal{IPL}$, $\mathcal{HOL}$, and $\mathcal{HNK}$, admit non-trivial inclusion systems for their categories of models?

4.71. Inclusion systems for $\mathcal{MVL}$ models
For any $\mathcal{MVL}$ signature (S, C, P) its category of models admits a 'closed' and a 'strong' inclusion system as follows:

closed: the abstract surjections are the surjective homomorphisms and abstract inclusions are the inclusive homomorphisms $M \to N$ (i.e. $M_s \subseteq N_s$ for each $s \in S$) such that $M_\pi m = N_\pi m$ for all $\pi \in P_w$ and $m \in M_w$.

strong: the abstract surjections are the surjective homomorphisms $h\colon M \to N$ such that $N_\pi y = \bigvee\{M_\pi x \mid h_w x = y\}$ for all $\pi \in P_w$ and $y \in N_w$ and the abstract inclusions are the inclusive homomorphisms.

4.72. Let (Σ, E), (Σ', E'), and (Σ'', E'') be closed theories in an arbitrary institution and $\varphi\colon \Sigma \to \Sigma'$ and $\phi\colon \Sigma' \to \Sigma''$ be signature morphisms such that $\varphi;\phi$ is a theory morphism $(\Sigma, E) \to (\Sigma'', E'')$. Then

- if φ is a strong theory morphism $(\Sigma, E) \to (\Sigma', E')$, then ϕ is a theory morphism $(\Sigma', E') \to (\Sigma'', E'')$, and
- if ϕ is a closed theory morphism $(\Sigma', E') \to (\Sigma'', E'')$, then φ is a theory morphism $(\Sigma, E) \to (\Sigma', E')$.

4.73. The strong inclusion system of $\mathcal{FOL}$ signatures is epic.

4.74. When the inclusion system of signatures is epic, both the closed and the strong inclusion systems of theories are epic too.

4.75. Prove Prop. 4.24.

4.76. Prove Prop. 4.25. Why the closed inclusion system of the theories does not inherit the unions from the inclusion system of the signatures?

4.6 Free models

In this section, we study the existence of free models along theory morphisms, which means the existence of left-adjoint functors to the reducts $Mod(\Sigma',E') \to Mod(\Sigma,E)$ corresponding to theory morphisms $(\Sigma,E) \to (\Sigma',E')$. This general problem comes from specification theory as it supports initial semantics in the context of structured specifications. It can be solved in two steps as follows. First, we establish the existence of initial models for certain theories, and then on this basis we develop the full result. At this point in the book we only have means to prove the existence of initial models for Horn theories in first-order logic. Later on, in Chap. 8, empowered by quasi-varieties, we will be able to develop a general institution-independent approach to initial models of theories. For the second step, we can already develop it in an the institution-independent manner by assuming initial models for theories.

Initial models of Horn theories in $\mathcal{FOL}$

Recall that a *Horn clause* for a signature (S,F,P) is a sentence of the form $(\forall X)H \Rightarrow C$, where H is a finite conjunction of (relational or equational) atoms, C is a (relational or equational) atom, and $H \Rightarrow C$ is the implication of C by H.

For each (S,F,P)-model M and for each set Γ of Horn (S,F,P)-clauses, we define the model M_Γ by

- Let

$$=_\Gamma \; = \; \bigcap\{=_h \mid h\colon\; M \to N \text{ model homomorphism and } N \models \Gamma\}.$$

  Since any intersection of (S,F)-congruences is an (S,F)-congruence, $=_\Gamma$ is an (S,F)-congruence too.

- As (S,F)-algebra, let M_Γ be the quotient $M/_{=_\Gamma}$.
- For each relation symbol $\pi \in P$ let

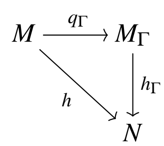
$$(M_\Gamma)_\pi = \{m/_{=_\Gamma} \mid hm \in N_\pi \text{ for each } h\colon\; M \to N \text{ such that } N \models \Gamma\}.$$

We notice easily that the quotient mapping $q_\Gamma\colon\; M \to M_\Gamma$ defined by $q_\Gamma m \;=\; m/_{=_\Gamma}$ is a model homomorphism.

However note also that M_Γ is *not* the quotient $M/_{=_\Gamma}$ of the (S,F,P)-model M by $=_\Gamma$ (as defined in Sect. 4.5). The reason is that they differ on the interpretations of the relation symbols; we have that $(M/_{=_\Gamma})_\pi \subseteq (M_\Gamma)_\pi$ but in general, this is a strict inclusion.

Proposition 4.26. *Let Γ be any set of Horn (S,F,P)-clauses.*

1. *For each (S,F,P)-model homomorphism $h: M \to N$ such that $N \models \Gamma$ there exists a unique model homomorphism $h_\Gamma: M_\Gamma \to N$ such that $q_\Gamma; h_\Gamma = h$.*



2. $M_\Gamma \models \Gamma$.

Proof. 1. This follows from Prop. 4.20 because $(=_{q_\Gamma}) = (=_\Gamma) \subseteq (=_h)$. Note that $h_\Gamma(m/_{=_\Gamma}) = hm$ for each $m \in M$.

2. Let $(\forall X)H \Rightarrow C$ be any Horn clause in Γ. Consider any expansion M'_Γ of M_Γ to $(S,F+X,P)$ such that $M'_\Gamma \models H$. We have to prove that $M'_\Gamma \models C$.

Let M' be any expansion of M to $(S,F+X,P)$ such that $q_\Gamma: M' \to M'_\Gamma$ is an $(S,F+X,P)$-model homomorphism (which means that for each $x \in X$ we choose an element $M'_x \in (M'_\Gamma)_x$).

For any model homomorphism $h: M \to N$ such that $N \models \Gamma$ let N' be the expansion of N to $(S,F+X,P)$ such that $h: M' \to N'$ is an $(S,F+X,P)$-model homomorphism (thus defined by $N'_x = hM'_x$). Then $h_\Gamma: M'_\Gamma \to N'$ becomes an $(S,F+X,P)$-model homomorphism too. Because $M' \models H$ and $h: M' \to N'$, as model homomorphism, preserves the satisfaction of the atoms (the reader is requested to check this) we have that $N' \models H$, which implies $N' \models C$ (because $N \models (\forall X)H \Rightarrow C$).

- When C is an equational atom $t = t'$, $N' \models t = t'$ means that $hM'_t = hM'_{t'}$ which, written differently, means $(M'_t, M'_{t'}) \in =_h$. Since h is arbitrarily chosen, this implies $M'_t =_\Gamma M'_{t'}$. Thus $M'_\Gamma \models t = t'$.
- When C is a relational atom πt (for t an appropriate list of terms), $N' \models \pi t$ means that $hM'_t = N'_t \in N_\pi$ which, since h is arbitrarily chosen, implies $M'_t/_{=_\Gamma} \in (M_\Gamma)_\pi$. But this means that $(M'_\Gamma)_t \in (M_\Gamma)_\pi$ which is the same with $M'_\Gamma \models \pi t$.

Thus $M'_\Gamma \models C$. □

The result of Prop. 4.26 says that M_Γ is the free Γ-model over M. In the language of adjoint functors, this is the same as saying that the forgetful functor $Mod((S,F,P),\Gamma) \to Mod(S,F,P)$ has a left-adjoint.

Initial models of $\mathcal{FOL}$ signatures. To obtain that each Horn theory in $\mathcal{FOL}$ has initial models, we should apply Prop. 4.26 for M being the initial (S,F,P)-model. So we have to establish the existence of initial models in $Mod(S,F,P)$.

Proposition 4.27. *For any $\mathcal{FOL}$-signature (S,F,P) there exists an initial (S,F,P)-model $0_{(S,F,P)}$ defined by*

– *for each sort $s \in S$, let $(0_{(S,F,P)})_s = (T_{(S,F)})_s$ be the set of all (S,F)-terms of sort s;*
– *for each operation symbol $\sigma \in F_{w \to s}$ and for each list of terms $(t_1,\dots,t_n) \in (T_{(S,F)})_w$,*

$$(0_{(S,F,P)})_\sigma(t_1,\dots,t_n) = \sigma(t_1,\dots,t_n);$$

– *for each relation symbol $\pi \in P_w$, $(0_{(S,F,P)})_\pi = \emptyset$.*

Proof. For each (S,F,P)-model N, there exists a unique model homomorphism $h\colon\ 0_{(S,F,P)} \to N$ defined by

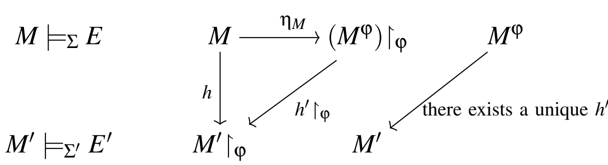
$$h_s(\sigma t) = N_\sigma(h_w t)$$

(for each operation symbol $\sigma \in F_{w \to s}$ and each list of terms $t \in (T_{(S,F)})_w$). □

Corollary 4.28. *For any set Γ of Horn (S,F,P)-clauses, the model $0_\Gamma = (0_{(S,F,P)})_\Gamma$ is the initial Γ-model, i.e., the initial model in $Mod((S,F,P),\Gamma)$.*

Liberal theory morphisms

In any institution, a theory morphism $\varphi\colon\ (\Sigma,E) \to (\Sigma',E')$ is *liberal* if and only if the reduct functor $Mod\varphi\colon\ Mod(\Sigma',E') \to Mod(\Sigma,E)$ has a left-adjoint $(\_)^\varphi$. In other words, for each (Σ,E)-model M there exists a (Σ',E')-model M^φ and a Σ-model homomorphism $\eta_M\colon\ M \to (M^\varphi)\restriction_\varphi$



such that for each (Σ',E')-model M' and for each Σ-model homomorphism $h\colon\ M \to M'\restriction_\varphi$, there exists a unique Σ'-model homomorphism $h'\colon\ M^\varphi \to M'$ such that $\eta_M \mathbin{;} h'\restriction_\varphi = h$. The pair (η_M, M^φ) is called the *free extension of M along* φ, or the *free (Σ',E')-model over M*. We already have an example of this in $\mathcal{FOL}$: (q_Γ, M_Γ) is the free extension of M along the theory morphism $1_{(S,F,P)}\colon\ ((S,F,P),\emptyset) \to ((S,F,P),\Gamma)$.

Note that by the composition of adjunctions (see Sect. 2.3), the composition of liberal theory morphisms is liberal. An institution is *liberal* if and only if each theory morphism is liberal.

In any institution with initial signatures (let Z be one of them) which are mapped by the model functor to the terminal category (a property which holds in any exact institution), it is easy to see that the existence of initial models for a theory (Σ,E) is the same as the liberality of the unique theory morphism $(Z,\emptyset) \to (\Sigma,E)$. This shows that under very mild conditions the existence of initial models of theories is a special case of liberality. In what follows we will work towards the converse of this, to establish how general liberality may follow from the existence of free models of theories. The crucial move in this direction is to decompose the general problem of free extensions along theory morphisms $\varphi\colon\ (\Sigma,E) \to (\Sigma',E')$ into two of its special cases:

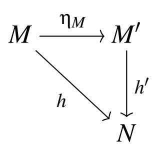
1. when φ is a theory inclusion of the form $1_\Sigma : (\Sigma, \emptyset) \to (\Sigma, E)$, and
2. when φ is a signature morphism $\Sigma \to \Sigma'$.

Proposition 4.29. *Consider an institution with diagrams ι such that each theory has an initial model. Then*

1. *for each theory (Σ, E), the forgetful functor $Mod(\Sigma, E) \to Mod\Sigma$ has a left-adjoint, and*
2. *if in addition the institution has pushouts of signatures and is semi-exact, then for each signature morphism φ the reduct functor $Mod\varphi$ has a left adjoint.*

Proof. For each theory (Σ, E), we denote its initial model by $0_{\Sigma,E}$.

1. Consider a theory (Σ, E) and let M be a Σ-model. Let $E' = (\iota_\Sigma M)E$. We show that $M' = (0_{\Sigma_M, E_M \cup E'})\upharpoonright_{\iota_\Sigma M}$ is the free (Σ, E)-model over M with the universal arrow $\eta_M = (M_M \to 0_{\Sigma_M, E_M \cup E'})\upharpoonright_{\iota_\Sigma M} : M \to M'$. Thus we have to prove that for each model homomorphism $h : M \to N$ such that $N \models_\Sigma E$, there exists a unique $h' : M' \to N$ such that $\eta_M ; h' = h$.

   Let $N_h = i^{-1}_{\Sigma,M} h$. Then

1	$N_h \models E'$	$N \models E$, $N_h\upharpoonright_{\iota_\Sigma M} = N$, Satisfaction Condition
2	$N_h \models E_M$	definition of N_h.

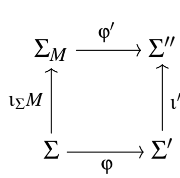
   Hence $N_h \models E_M \cup E'$. Let h'' be the unique model homomorphism $h'' : 0_{\Sigma_M, E_M \cup E'} \to N_h$. Let $h' = h''\upharpoonright_{\iota_\Sigma M}$. Then

   $$\eta_M ; h' = (M_M \to 0_{\Sigma_M, E_M \cup E'})\upharpoonright_{\iota_\Sigma M} ; h''\upharpoonright_{\iota_\Sigma M} = (M_M \to N_h)\upharpoonright_{\iota_\Sigma M} = h.$$

   The uniqueness of h' follows by the hom-sets bijection

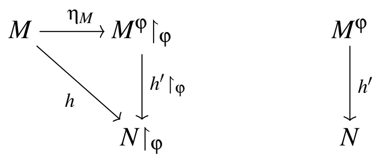
   $$(M/Mod\Sigma)(\eta_M, h) \;\cong\; Mod(\Sigma_M, E_M)(0_{\Sigma_M, E_M \cup E'}, N_h).$$

2. Let $\varphi : \Sigma \to \Sigma'$ be a signature morphism and let M be a Σ-model. Consider the following pushout square of signature morphisms:

   (4.2)

We define $M^{\varphi} = (0_{\Sigma'',\varphi' E_M})\upharpoonright_{\iota'}$ and the candidate universal arrow $\eta_M : M \to (M^{\varphi})\upharpoonright_{\varphi}$ to be $(M_M \to (0_{\Sigma'',\varphi' E_M})\upharpoonright_{\varphi'})\upharpoonright_{\iota_\Sigma M}$. For proving the universal property of η_M, consider $h : M \to N\upharpoonright_{\varphi}$ with N any Σ'-model. We have to prove the existence of a unique Σ'-model homomorphism $h' : M^{\varphi} \to N$ such that the following diagram commutes:



Let $M_h = i^{-1}_{\Sigma,M} h$. Then $M_h\upharpoonright_{\iota_\Sigma M} = N\upharpoonright_{\varphi}$ which by the semi-exactness hypothesis applied to the pushout square (4.2), allows for the existence of a unique amalgamation $N \otimes M_h$. We have that:

1	$(N \otimes M_h)\upharpoonright_{\varphi'} = M_h$	definition of $N \otimes M_h$
2	$M_h \models E_M$	definition of M_h
3	$N \otimes M_h \models \varphi' E_M$	1, 2, Satisfaction Condition.

Therefore there exists a unique model homomorphism $h'' : 0_{\Sigma'',\varphi' E_M} \to N \otimes M_h$. We define $h' = h''\upharpoonright_{\iota'}$. Note that $h' : M^{\varphi} \to N$. It follows that:

1	$\eta_M \,;\, h'\upharpoonright_{\varphi} = \eta_M \,;\, h''\upharpoonright_{\iota'}\upharpoonright_{\varphi}$	definition of h'
2	$h''\upharpoonright_{\iota'}\upharpoonright_{\varphi} = h''\upharpoonright_{\varphi'}\upharpoonright_{\iota_\Sigma M}$	commutativity of (4.2), functoriality of *Mod*
3	$\eta_M = (M_M \to (0_{\Sigma'',\varphi' E_M})\upharpoonright_{\varphi'})\upharpoonright_{\iota_\Sigma M}$	by definition
4	$(M_M \to M_h)\upharpoonright_{\iota_\Sigma M} = h$	definition of M_h
5	$\eta_M \,;\, h'\upharpoonright_{\varphi} = h$	1, 2, 3, 4.

It remains to justify the uniqueness of h'. This follows from the uniqueness of h'' (from the initiality property of $0_{\Sigma'',\varphi' E_M}$) and from the uniqueness side of the amalgamation property for model homomorphisms (by using the semi-exactness hypothesis since h'' is the amalgamation of h' and $h''\upharpoonright_{\varphi'}$).

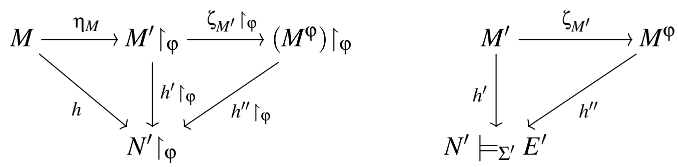
□

Corollary 4.30. *A semi-exact institution with diagrams and pushouts of signatures is liberal when each theory has an initial model.*

Conversely, if the institution has initial signatures and is finitely exact, each theory has an initial model whenever the institution is liberal.

Proof. The second part of this corollary has been already discussed above. For the first part let us consider a theory morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$. Let M be a (Σ, E)-model.

By Prop. 4.29 let (η_M, M') be its free extension along φ as a signature morphism $\Sigma \to \Sigma'$ and let $(\zeta_{M'}, M^\varphi)$ be the free (Σ', E')-model over the Σ'-model M'.



Then it is easy to see that $(\eta_M\ ;\ \zeta_{M'}\upharpoonright_\varphi,\ M^\varphi)$ is the free extension of M along the theory morphism $\varphi\colon\ (\Sigma, E) \to (\Sigma', E')$. □

A concrete application of Cor. 4.30 is the following:

Corollary 4.31. *The institution $\mathcal{HCL}$ is liberal.*

Liberal institution mappings

It is also useful to consider free models also across the institution morphisms or comorphisms. An institution morphism $(\Phi, \alpha, \beta)\colon\ I' \to I$ is *liberal* when the model translations $\beta_{\Sigma'}\colon\ Mod'\Sigma' \to Mod(\Phi\Sigma')$ have left adjoints for all I'-signatures Σ'. Similarly, an institution comorphism (Φ, α, β) is *liberal* when all β_Σ's have left adjoints.

Persistently liberal institution (co-)morphisms. Especially useful for the transfer of institutional properties across institution mappings is the case when these adjunctions corresponding to the model translations β_Σ are persistent, which means that the left-adjoint to β_Σ is also a left-inverse (up to isomorphism) to β_Σ. In many actual situations, persistently liberal institution comorphisms determine useful 'representations' of a more complex institution into a simpler one. The following is an example.

Encoding relations as operations. Recall the comorphism $\mathcal{FOL} \to \mathcal{FOEQL}$ discussed in Sect. 3.3. For each $\mathcal{FOL}$-signature (S, F, P), the adjunction between $Mod^{\mathcal{FOL}}(S, F, P)$ and $Mod^{\mathcal{FOL}}(S + \{\mathbf{b}\}, F + \overline{P} + \{\mathbf{true}, \emptyset\})$ is persistently liberal, with the free $(S + \{\mathbf{b}\}, F + \overline{P} + \{\mathbf{true}\})$-algebra M' over a model M interpreting 'freely' the non-true values by $M'_{\mathbf{b}} = \{M'_{\mathbf{true}}\} \uplus \{\pi m \mid \pi \in P, m \notin M_\pi\}$. Hence

Proposition 4.32. *The encoding of relations as operations $\mathcal{FOL} \to \mathcal{FOEQL}$ is a persistently liberal comorphism.*

Exercises

4.77. Give a counterexample showing that $\mathcal{FOL}$ is not liberal.

4.78. From Prop. 4.29 derive that each $\mathcal{FOL}$ signature morphism is liberal.

4.79. Γ-congruences
Let Γ be a set of Horn clauses for an algebraic signature (S,F). A congruence $\equiv$ on an algebra A is a Γ-*congruence* if and only if for any sentence $(\forall X)H \Rightarrow (t=t')$ in Γ and for each expansion A' of A to $(S,F+X)$, $A'_t \equiv A'_{t'}$ if $A'_{t_1} \equiv A'_{t_2}$ for all $t_1 = t_2$ in H. Then $=_\Gamma$ is the least Γ-congruence.

4.80. Liberality in $\mathcal{PA}$
In $\mathcal{PA}$ each morphism between presentations of universally quantified (possibly conditional) existence equations is liberal.

4.81. Liberality in $\mathcal{POA}$
The institution $\mathcal{HPOA}$ (of Horn $\mathcal{POA}$-sentences) is liberal. (*Hint:* Extend the concept of Γ-congruence of Ex. 4.79 to $\mathcal{POA}$-congruences of Ex. 4.67 and show that for each $\mathcal{POA}$-algebra M, the quotient $q_\Gamma : M \to M/_{(=_\Gamma,\leq_\Gamma)}$ is the free preordered algebra satisfying Γ, where $(=_\Gamma,\leq_\Gamma)$ is the least Γ-$\mathcal{POA}$-congruence.)

4.82. Give a counterexample showing that in the institution $\mathcal{MA}$ of multialgebras not all sets of atoms have initial models.

4.83. (a) Give a counterexample showing in general, in $\mathcal{HOL}$, signatures do not admit initial models. (b) On the other hand, all $\mathcal{HNK}$-signatures which have at least one constant for each type have initial models. (*Hint:* Consider the comorphism $(\Phi,\alpha,\beta) : \mathcal{HNK} \to \mathcal{FOEQL}^{\text{th}}$ of Ex. 4.12. Then for each $\mathcal{HNK}$-signature Σ, the $\mathcal{FOEQL}$-theory $\Phi\Sigma$ has initial models, one of them being just the term model.)

4.84. Each $\mathcal{LA}$-signature morphism is liberal.

4.85. Each $\mathcal{CA}$-signature has initial algebras. (S,F,q) in $\mathcal{CA}$ has an initial algebra. (*Hint:* For any $\mathcal{CA}$-signature (S,F,q) the S-sorted set T_F^ω of (possibly) infinite terms can be organized as a contraction (S,F,q)-algebra with the distance between two terms t and t' being $q^{\alpha(t,t')}$, where $\alpha(t,t')$ is the minimum depth at which t and t' differ.)

4.86. Liberality of comorphism $\mathcal{FOL} \to (\mathcal{FOL}^1)^{\text{th}}$
The encoding of many-sorted logic into single-sorted logic described in Sect. 4.1 is a liberal comorphism. (*Hint:* For each $\mathcal{FOL}$-signature (S,F,P) and any (S,F,P)-model M, we first take the disjoint union $\biguplus_{s\in S} M_s$. Then we take the free $\overline{F}$-algebra over $\biguplus_{s\in S} M_s$ where $\overline{F}$ is the single-sorted variant of F. Then we take take its quotient under the congruence generated by the pairs $\langle \sigma m, m' \rangle$ for which $M_\sigma m = m'$ for all $\sigma \in F$. The final step is to organize this quotient $\overline{F}$-algebra as an $(\overline{F}, \overline{P} \cup \{(-: s) \mid s \in S\})$-model; this is done in a canonical way.)

4.87. Institution representations
An *institution representation* $I \to I'$ is just a persistently liberal institution comorphism $I \to I'^{\text{th}}$ from I to the theories of I'. Institution representations compose and form a category. (*Hint:* For any institution representation $I \to I'$, the induced institution comorphism $I^{\text{th}} \to I'^{\text{th}}$ is persistently liberal.)

4.88. [186] **Creating liberality along institution comorphisms**
Persistently liberal institution comorphisms $(\Phi,\alpha,\beta) : I \to I'$ create liberality in the sense that any theory morphism $\varphi : (\Sigma_1,E_1) \to (\Sigma_2,E_2)$ is liberal if $\Phi\varphi : (\Phi\Sigma_1,\alpha E_1) \to (\Phi\Sigma_2,\alpha E_2)$ is liberal. Apply this to the following comorphisms:

- $\mathcal{PA} \to \mathcal{FOL}^{\text{th}}$ (the operational encoding introduced in Sect. 4.1),
- $\mathcal{PA} \to \mathcal{FOEQL}^{\text{th}}$ of Ex. 4.10,

- $\mathcal{POA} \to \mathcal{FOL}^{\text{th}}$ of Ex. 4.9,
- $\mathcal{MBA} \to \mathcal{FOL}$ of Ex. 3.22,
- $\mathcal{AUT} \to \mathcal{FOL}^1$ of Ex. 3.23,
- $\mathcal{IPL} \to (\mathcal{FOEQL}^{\mathbf{1}})^{\text{th}}$ of Ex. 4.11, and
- $\mathcal{LA} \to (\mathcal{FOEQL}^{\mathbf{1}})^{\text{th}}$ of Ex. 4.14,

and from all these deduce corresponding liberality results for $\mathcal{PA}$, $\mathcal{POA}$, $\mathcal{MBA}$, $\mathcal{AUT}$, $\mathcal{IPL}$, and $\mathcal{LA}$.

4.89. Comorphism $\mathcal{EQL} \to Cat\mathcal{EQL}$
Construct a canonical institution comorphism $\mathcal{EQL} \to Cat\mathcal{EQL}$ (see Ex. 3.7) by mapping

- each algebraic signature (S,F) to the category $Alg(S,F)$ of (S,F)-algebras, and
- each (S,F)-equation $(\forall X)t = t'$ to the $Alg(S,F)$-equation $(\forall T_{(S,F)}(X))t^\sharp = t'^\sharp$ where $t^\sharp$, $t'^\sharp$ are the unique extensions of t, t' to (S,F)-algebra homomorphisms $T_{(S,F)}(\{*\}) \to T_{(S,F)}(\{X\})$ from the (S,F)-algebra free over the singleton set $\{*\}$ to the (S,F)-algebra free over the set X.

4.90. [64] **Model pushouts**
In any liberal institution with diagrams the category of models of any theory has pushouts. Moreover, if the institution is also exact and has initial signatures, then the category of models of any theory has finite co-limits. (*Hint:* the pushout of model homomorphisms is the same with the universal arrow to a canonical functor between comma categories of models.)

Notes. In many institution theory texts, especially those from formal / algebraic specification, our 'theories' are called 'presentations'. Our choice of terminology here is aligned with the mainstream terminology in logic.

Both the 'operational' and the 'relational' encoding comorphisms $\mathcal{PA} \to \mathcal{FOL}^{\text{th}}$ appear in [186]. Encoding modalities in relational logic is known in modal logic literature under the name of 'standard translation'. The ideas behind the comorphism $\mathcal{HNK} \to \mathcal{FOEQL}^{\text{th}}$ appear in [178].

Co-limits of theories have been playing a very important role in algebraic specification [124, 96, 219]; one could say that the search for an institution-independent approach to compositionality of specification theories was one of the origins of institutions. By contrast, theory limits seem to be much less important in applications.

Institution theory is the only model theory that first properly identified [218] and then gradually realized the importance [96] of the model amalgamation (exactness) properties of logics. Since then semi-exactness has been intensively used as a basic institutional property by various works in algebraic specification. In practice very often the weak version of exactness suffices. This has been already considered in several works [60, 231] and is especially important for the case of the multi-paradigm or heterogeneous institutions obtained by a Grothendieck construction on institutions [62]. Model amalgamation has been extended to arbitrary co-cones in works such as [221].

The model amalgamation proof for $\mathcal{FOL}$ is similar in flavour to the functorial semantics of [160], and appears in the form we have presented here in [221].

The method of diagrams pervades much of conventional model theory [42]. The institution-independent method of diagrams used here was developed in [64] and has been used in [64, 140, 139] etc. A precursor of the method of diagrams has been used for developing quasi-variety theorems and the existence of free models within the context of the so-called 'abstract algebraic institutions' [227, 228]. Elementary homomorphisms have been introduced in [139]. The existence of limits and co-limits of models via diagrams has been obtained in [64].

Inclusion systems and inclusive institutions were introduced in [96] for the institution-independent study of structuring specifications, however there they were defined in a stronger version corresponding to our epic inclusion systems with unions. In [96, 130] they provide the underlying mathematical concept for module imports, which are the most fundamental structuring constructs. Inclusions of models are used in [212, 64, 81] for an institution-independent approach to quasi-varieties of models. Mathematically, inclusion systems capture categorically the concept of set-theoretic 'inclusion' in a way reminiscent of the well-known factorization systems [31]; however in many applications the former are more convenient than the latter. In [53] the original definition of [96] has been weakened to what they called 'weak inclusion systems', which are just our inclusion systems.

Our (S,F,P)-congruences are elsewhere called 'closed' congruences.

Liberality has played a central role in institution theory from its beginning [124]. This was due to the traditionally important role played in algebraic specification by initial algebra semantics. Free models along theory morphisms provide semantics for initial denotation modules in structured algebraic specifications [124]. Our institution comorphisms $I \to (I')^{\mathrm{th}}$ have been studied in [183, 130] under the name of 'simple theoroidal comorphisms'.

Chapter 5

Internal Logic

The definition of the satisfaction relation between models and sentences in $\mathcal{FOL}$ was a two-layered process. At the base level, we have defined the satisfaction of atomic sentences. Then we performed an induction step on the structure of the sentences. This Tarskian process of determining the actual satisfaction between models and sentences is a common pattern for a multitude of concrete institutions. This is the case whenever the sentences are defined inductively. In this chapter, we develop an abstract institution-independent approach to this process by providing a uniform general treatment to Boolean connectives, to quantifiers, and to some extent even to atomic sentences.

Our approach to atomic sentences is based on a simplified form of categorical injectivity. When considering Horn sentences at an institution-independent level, their satisfaction is equivalent to (full) categorical injectivity. Later on in the book this will prove very useful for the development of Birkhoff-style axiomatizability results.

Many important results in model theory rely upon quantification being first order. First-order quantifiers are handled at the institution-independent level by the concept of '(quasi-)representable' signature morphisms. This is a rather semantic property as its formulation involves the models of the institution.

Other topics of this chapter include substitutions (in continuation to our approach to quantifiers) and a deepening of the study of elementary homomorphisms.

5.1 Boolean connectives

Given a signature Σ in an institution and a class $\mathbb{M}$ of Σ-models by $\overline{\mathbb{M}}$ we denote $Mod\Sigma \setminus \mathbb{M}$. Then a Σ-sentence ρ' is a *semantic*

- *negation* of ρ when $\rho'^* = \overline{\rho^*}$;
- *conjunction* of the Σ-sentences ρ_1 and ρ_2 when $\rho'^* = \rho_1^* \cap \rho_2^*$;
- *disjunction* of the Σ-sentences ρ_1 and ρ_2 when $\rho'^* = \rho_1^* \cup \rho_2^*$;

– *implication* of the Σ-sentences ρ_1 and ρ_2 when $\rho'^* = \overline{\rho_1^*} \cup \rho_2^*$;

– *equivalence*[1] of the Σ-sentences ρ_1 and ρ_2 when $\rho'^* = (\rho_1^* \cap \rho_2^*) \cup (\overline{\rho_1^*} \cap \overline{\rho_2^*})$;

– *true* when $\rho'^* = Mod\Sigma$; and

– *false* when $\rho'^* = \emptyset$.

A more informal way to express these connectives is by relying on a meta-level. For example ρ' is the negation of ρ when for each Σ-model M, $M \models \rho'$ if and only if $M \not\models \rho$. Or ρ' is the conjunction of ρ_1 and ρ_2 when for each Σ-model M, $M \models \rho'$ if and only if $M \models \rho_1$ *and* $M \models \rho_2$, etc.

Fact 5.1. *Negations, conjunctions, disjunctions, implications, and equivalences of sentences are unique up to semantical equivalence.*

An institution *has (semantic) negation* when each sentence of the institution has a negation. It has *(semantic) conjunctions* when every two sentences (of the same signature) have a conjunction. Similar definitions can be formulated for disjunctions, implications, and equivalences. Designated Boolean connectives are denoted in the familiar way, negations by $\neg\rho$, conjunctions by $\rho_1 \wedge \rho_2$, disjunctions by $\rho_1 \vee \rho_2$, implications by $\rho_1 \Rightarrow \rho_2$, equivalences by $\rho_1 \Leftrightarrow \rho_2$, true by true, and false by false.

When they exist, the semantic Boolean connectives are inter-definable as shown by the following result which is familiar to us from propositional logic. In this sense the institution-independent semantics of the Boolean connectives represent an ‘internalisation’ of propositional logic in abstract institutions.

Fact 5.2. *In any institution having the corresponding Boolean connectives we have that*

- disjunction: $\rho_1 \vee \rho_2 \models\!\mid \neg(\neg\rho_1 \wedge \neg\rho_2)$*;*
- implication: $\rho_1 \Rightarrow \rho_2 \models\!\mid \neg\rho_1 \vee \rho_2$*;*
- equivalence: $\rho_1 \Leftrightarrow \rho_2 \models\!\mid (\rho_1 \Rightarrow \rho_2) \wedge (\rho_2 \Rightarrow \rho_1)$*;*
- false: $\mathsf{false} \models\!\mid \rho \wedge \neg\rho$*;*
- *etc.*

In this book, we will sometimes use a notation like $\wedge E$ to denote the conjunction of a finite set of sentences E. While the meaning of this is straightforward when the institution has conjunctions and E is non-empty, $\wedge\emptyset$ is just true. This means that the use of the notation $\wedge E$ for any finite set E of sentences requires that the institution has true. Sometimes it is not necessary to assume this explicitly since it may hold as a consequence of other conditions, for example, when the institution has conjunctions and negations.

An institution which has all semantic Boolean connectives is called a *Boolean complete institution*.

The following gives the situation of the semantic Boolean connectives in some institutions (the reader is invited to check this table by himself):

[1]Not to be confused with the semantical equivalence *relation* $\models\!\mid$ between sentences.

institution	$\wedge$	$\neg$	$\vee$	$\Rightarrow$	$\Leftrightarrow$
$\mathcal{FOL}$, $\mathcal{PL}$, $\mathcal{HOL}$, $\mathcal{HNK}$, $\mathcal{MFOL}^\sharp$, $\mathcal{MPL}^\sharp$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\mathcal{FOL}^+$	$\surd$		$\surd$		
$\mathcal{EQL}$, $\mathcal{HCL}$, $\mathcal{MVL}^\sharp$					
$\mathcal{EQLN}$		$\surd$			
$\mathcal{MFOL}^*$, $\mathcal{MPL}^*$, $\mathcal{IPL}$	$\surd$				

Note that while designated connectives are semantic connectives, the presence of a semantic connective does not necessarily mean it is designated. For instance if we define $\mathcal{PL}$ only using the connectives $\neg$ and $\wedge$, we would still have the other semantic Boolean connectives but those are not designated. For instance $\neg(\neg\rho_1 \wedge \neg\rho_2)$ is a semantic disjunction of ρ_1 and ρ_2 but it is not a designated one.

Exercises

5.1. Preservation of Boolean connectives along signature morphisms
Let ρ' be a semantic conjunction of the Σ-sentences ρ_1 and ρ_2. Let $\varphi\colon\ \Sigma \to \Sigma'$ be a signature morphism. Then $\varphi\rho$ is a semantic conjunction of $\varphi\rho_1$ and $\varphi\rho_2$. Prove similar preservation properties for the other Boolean connectives.

5.2. In any institution, for any sets of Σ-sentences Γ and any Σ-sentences ρ, ρ_1, ρ_2 we have

1. $\Gamma \models \rho_1$ and $\Gamma \models \rho_2$ if and only if $\Gamma \models \rho_1 \wedge \rho_2$,
2. $\rho_1 \models \Gamma$ and $\rho_2 \models \Gamma$ if and only if $\rho_1 \vee \rho_2 \models \Gamma$,
3. $\mathsf{false} \models \Gamma$,
4. $\Gamma \cup \{\rho\} \models \mathsf{false}$ if and only if $\Gamma \models \{\neg\rho\}$,
5. $\rho \models\!\!\mid \neg\neg\rho$,
6. $\neg\rho \models\!\!\mid \rho \Rightarrow \mathsf{false}$, and
7. $\Gamma \cup \{\rho_1\} \models \rho_2$ if and only if $\Gamma \models \rho_1 \Rightarrow \rho_2$.

5.3. Weak propositional logic ($\mathcal{WPL}$, see Ex. 3.27) does have all semantic Boolean connectives apart from negation.

5.4. [70] **Finitary sentences (Ex. 4.22 continued)**
(a) In any institution the negation of a finitary sentence is finitary.
(b) If the category of signatures has binary co-products, then any binary Boolean connection of finitary sentences is finitary too.

5.2 Quantifiers

Let us first recall the semantics of quantifiers in a concrete institution such as $\mathcal{FOL}$. Given a $\mathcal{FOL}$-signature (S,F,P) and a set X of variables for (S,F,P), let ρ' be an $(S,F+X,P)$-sentence and M be an (S,F,P)-model. Then

$$M \models (\exists X)\rho' \text{ if and only if } M' \models \rho' \text{ for some } (S,F+X,P)\text{-expansion } M' \text{ of } M.$$

General institution-independent quantifiers are defined similarly to the above by abstracting from $\mathcal{FOL}$ signature inclusions $(S,F,P) \hookrightarrow (S,F+X,P)$ to any signature morphisms $\chi\colon\ \Sigma \to \Sigma'$ in any arbitrary institution.

– A Σ-sentence ρ is a *(semantic) existential χ-quantification* of a Σ'-sentence ρ' when $\rho^* = (Mod\chi)\rho'^*$; designated existential quantification may be written as $(\exists\chi)\rho'$,

– A Σ-sentence ρ is a *(semantic) universal χ-quantification* of a Σ'-sentence ρ' when $\rho^* = \overline{(Mod\chi)\overline{\rho'^*}}$; designated universal quantification may be written as $(\forall\chi)\rho'$.

A more informal way to express semantic existential / universal quantifiers, which uses meta-level 'all' and 'some', is as follows:

– $M \models_\Sigma (\exists\chi)\rho'$ when there *exists* a χ-expansion M' of M such that $M' \models_{\Sigma'} \rho'$, and

– $M \models_\Sigma (\forall\chi)\rho'$ when $M' \models_{\Sigma'} \rho'$ for *all* χ-expansions M' of M.

When they exist, in the presence of negation, the universal and the existential quantifiers are inter-definable:

Fact 5.3. *In any institution with negation*

$$(\exists\chi)\rho \models\!\!\!\!\;\!\!\;=\!\!\!| \ \neg(\forall\chi)\neg\rho.$$

Usually, quantification is considered only for a restricted class of signature morphisms. For example, quantification in $\mathcal{FOL}$ considers only the finitary signature extensions with constants. For a class $\mathcal{D} \subseteq Sig$ of signature morphisms, we say that the institution has semantic universal / existential *$\mathcal{D}$-quantification* when for each $\chi\colon\ \Sigma \to \Sigma'$ in $\mathcal{D}$, each Σ'-sentence has a universal / existential χ-quantification, respectively. The table below shows some internal quantifications in some institutions.[2]

institution	$\mathcal{D}$	$\forall$	$\exists$
$\mathcal{FOL}$	finitary injective sign. extensions with constants	$\surd$	$\surd$
$\mathcal{SOL}$, $\mathcal{HOL}$, $\mathcal{HNK}$	finitary injective sign. extensions	$\surd$	$\surd$
$\mathcal{PA}$	finitary injective sign. extensions with total constants	$\surd$	$\surd$
$\mathcal{EQL}$, $\mathcal{HCL}$, $\mathcal{MVL}^\sharp$	finitary injective sign. extensions with constants	$\surd$	
$\mathcal{MFOL}^*$	finitary injective sign. extensions with rigid constants	$\surd$	
$\mathcal{MFOL}^\sharp$	finitary injective sign. extensions with rigid constants	$\surd$	$\surd$

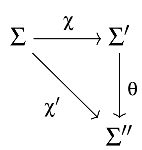
The situations listed in the table above can be traced back to the definitions of the quantifiers in the respective institutions. However, this is not immediately obvious as they go beyond the respective designated quantifications, a situation that shows that the semantic quantifications may be broader than the designated quantifications in a rather different

[2] Where 'injective signature extension' means a signature morphism with all components injective, 'finitary' means that there is only a finite number of symbols outside the image of the signature morphism, and 'with constants' means that all symbols outside the image of the signature morphism are constants.

way than the cases when for instance the universal quantifications can be defined in terms of existential quantifications and negations. For example, the definition of quantifications in $\mathcal{FOL}$ considers only *inclusive* signature extensions with finite *blocks of variables*, while the $\mathcal{FOL}$ entry in the table above allows for a the more general situation, when the extensions may be only injective and when the new additional constants are not restricted to be variables in the precise sense defined when $\mathcal{FOL}$ has been introduced as an institution. The following result clarifies this situation and its main idea can be replicated to other cases, such as those listed in the table above.

Proposition 5.4. *$\mathcal{FOL}$ has semantic universal / existential χ-quantifications for any χ finitary injective signature extension with constants.*

Proof. It is enough to do this for existential quantification. Let $\chi : \Sigma \to \Sigma'$ and let ρ' be any Σ'-sentence. There exists a signature extension $\chi' : \Sigma \to \Sigma''$ of Σ with a finite block of variables X such that there exists an isomorphism of signatures (i.e. bijective componentwise) $\theta : \Sigma' \to \Sigma''$ with $\chi ; \theta = \chi'$.



We define ρ as $(\exists X)\theta\rho'$ and prove that it is an existential χ-quantification of ρ'. Let M be any Σ-model. Then:

1 $M \models_\Sigma \rho$ if and only if $M \models_\Sigma (\exists X)\theta\rho'$ — definition of ρ

2 if and only if there exists M'', $M''\restriction_{\chi'} = M$, $M'' \models_{\Sigma''} \theta\rho'$ — 1.

We define $M' = M''\restriction_\theta$. Then:

3 $M'\restriction_\chi = M''\restriction_\theta\restriction_\chi = M''\restriction_{\chi'} = M$

4 $M' = M''\restriction_\theta \models \rho'$ — 2, Satisfaction Condition.

5 there exists M', $M'\restriction_\chi = M$, $M' \models \rho'$ — 3, 4.

Because θ is isomorphism, M' and M'' determine each other ($M'' = M'\restriction_{\theta^{-1}}$), hence from 2 and 5 we get the equivalence that shows that ρ is an existential χ-quantification of ρ'. □

In general, in institutions, one may consider quantification only up to what the respective concept of signature supports. For example, $\mathcal{FOL}$ signatures support quantifications only up to second order (by considering signature extensions also with operation / relation symbols, and even with sorts). Quantifications higher than the second order require thus another concept of signature involving higher-order types, such an example is given by $\mathcal{HOL}$ or $\mathcal{HNK}$.

Quantification systems. The following concept provides a formal general approach to designated quantifications. A *quantification system* consists of a designated class $\mathcal{D} \subseteq Sig^I$ of signature morphisms and a designated class of pushout squares

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma_1 \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi(\varphi)} \\ \Sigma' & \xrightarrow[\varphi[\chi]]{} & \Sigma'_1 \end{array}$$

for any $(\chi : \ \Sigma \to \Sigma') \in \mathcal{D}$ and $\varphi : \ \Sigma \to \Sigma_1$, with $\chi(\varphi) \in \mathcal{D}$ and such that

(*HCOMP)* the 'horizontal' composition of such designated pushout squares is again a designated pushout square, i.e., for the pushout squares in the following diagram

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\varphi} & \Sigma_1 & \xrightarrow{\theta} & \Sigma_2 \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi(\varphi)} & & \downarrow{\scriptstyle\chi(\varphi)(\theta)} \\ \Sigma' & \xrightarrow[\varphi[\chi]]{} & \Sigma'_1 & \xrightarrow[\theta[\chi(\varphi)]]{} & \Sigma'_2 \end{array}$$

we have that $\varphi[\chi]\ ;\ \theta[\chi(\varphi)] = (\varphi;\theta)[\chi]$ and $\chi(\varphi)(\theta) = \chi(\varphi;\theta)$,

(*UNIT)* we also have $\chi(1_\Sigma) = \chi$ and $1_\Sigma[\chi] = 1_{\Sigma'}$, and

(*QAMG)* each designated pushout is a model amalgamation square.

Given any quantification system $\mathcal{D}$ for an institution I we may extend I to another institution $I^{\mathcal{D}}$ which adds sentences of the form $(\forall\chi)\rho$ and $(\exists\chi)\rho$, for $\chi : \ \Sigma \to \Sigma' \in \mathcal{D}$ and $\rho \in Sen^I\Sigma'$, and which extends the satisfaction relation by using the definitions of semantic universal and existential quantifications.

Proposition 5.5. *$I^{\mathcal{D}}$ is an institution.*

Proof. We have only to establish the functoriality of the $I^{\mathcal{D}}$-sentence functor $Sen^{I^{\mathcal{D}}}$ and the Satisfaction Condition for the extended satisfaction relation of $I^{\mathcal{D}}$. The former follows immediately from (*HCOMP*) and (*UNIT*), while the latter makes essential use of (*QAMG*) in the style of the induction step corresponding to quantifications in the proof the $\mathcal{FOL}$ Satisfaction Condition in Sect. 3.1. □

Now we look back to the emblematic $\mathcal{FOL}$ case. The quantification system $\mathcal{D}$ involved by the designated quantifiers in $\mathcal{FOL}$ consists of the signature extensions $\chi : \ (S,F,P) \to (S,F+X,P)$ with finite blocks of variables X and with the designated pushout squares defined as follows. Given any signature morphism $\varphi : \ (S,F,P) \to (S',F',P')$, we set

- $\chi(\varphi)$ to be the signature extension $(S',F',P') \to (S',F'+X^{\varphi},P')$, and

- $\varphi[\chi]: (S, F+X, P) \to (S', F'+X^{\varphi}, P')$ to be the extension of φ with the canonical bijection $X \to X^{\varphi}$ that maps each variable $(x, s, (S, F, P))$ to $(x, \varphi^{\mathrm{st}} s, (S', F', P'))$.

$$\begin{array}{ccc} (S,F,P) & \xrightarrow{\ \varphi\ } & (S',F',P') \\ \chi\big\downarrow & & \big\downarrow\chi(\varphi) \\ (S,F+X,P) & \xrightarrow[\varphi[\chi]]{} & (S',F'+X^{\varphi},P') \end{array}$$

Some actual institutions of interest arise as institutions of the form $I^{\mathcal{D}}$, or as parts of institutions $I^{\mathcal{D}}$. For example, under the $\mathcal{FOL}$ quantification system $\mathcal{D}$, the institution $\mathcal{EQL}$ of equational logic is the universally quantified part of $\mathcal{AEQL}^{\mathcal{D}}$ (i.e., where $\mathcal{AEQL}$ is the sub-institution of $\mathcal{FOL}$ that has only equations $t = t'$ as sentences).

Conservative quantifications

In any institution with false, $M \models (\forall\chi)\mathsf{false}$ for all models M which do not admit a χ-expansion. This indicates that quantification behaves 'well' only for the signature morphisms $\chi: \Sigma \to \Sigma'$ for which each Σ-model admits at least one χ-expansion. Then we say that χ *has the model expansion property*.

Model expansion property in $\mathcal{FOL}$. Characterisations of the signature morphisms with the model expansion property similar to the $\mathcal{FOL}$ characterisation below are common to many concrete institutions.

Fact 5.6. *Let $\varphi: \Sigma \to \Sigma'$ be a $\mathcal{FOL}$ signature morphism such that Σ has non-empty sorts, i.e., there exists at least one term for each of its sorts. Then φ has the model expansion property if and only if it is injective, i.e., φ^{st}, φ^{op}, and φ^{rl} are injective.*

The role of the non-empty sorts condition is to guarantee that a new operation introduced by Σ' such that its sort is in Σ can always get an interpretation. In the absence of this situation the non-empty sorts condition is not necessary, such as in $\mathcal{REL}$.

Finitary quantifications

The finiteness of quantifications is necessary for many important model theory results. This concept can be defined at the level of abstract institutions as follows.

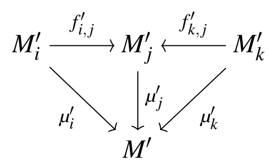
Finitary signature morphisms. A signature morphism $\chi: \Sigma \to \Sigma'$ is *finitary* when for each co-limit $(\mu_i)_{i\in I}$ of a directed diagram $(f_{i,j})_{(i<j)\in(I,\leq)}$ of Σ-models

$$\begin{array}{ccc} M_i & \xrightarrow{\ f_{i,j}\ } & M_j \\ & {\scriptstyle\mu_i}\searrow \quad \swarrow{\scriptstyle\mu_j} & \\ & M & \end{array}$$

and for each χ-expansion M' of M

– there exists an index $i \in I$ and a χ-expansion $\mu_i' : M_i' \to M'$ of μ_i, and

– any two different expansions as above can be 'unified' in the sense that for any χ-expansions μ_i' and μ_k' as above there exists an index $j \in I$ with $i,k \leq j$, a χ-expansion μ_j' as above and $f_{i,j}', f_{k,j}'$ χ-expansions of $f_{i,j}, f_{k,j}$ such that the following diagram commutes



The following is a standard example.

Proposition 5.7. *In $\mathcal{FOL}$ each injective signature extension with a finite number of constants is finitary.*

Proof. This is based on the remark that directed co-limits of $\mathcal{FOL}$ models are lifted from the corresponding directed co-limits of the underlying carrier sets (see Prop. 6.8 below for a proof of this fact). Then we have just to note that expansions of models M along signature extensions with constants $\Sigma \hookrightarrow \Sigma + X$ are just functions $X \to M$ and use the fact that X being finite is a finitely presented object in the category $\mathbb{S}et$. □

Note that extending by an infinite number of constants, or by non-constant operations yields a non-finitary signature morphism. On the other hand extending by relation symbols, in any number, yields a finitary signature morphism. So in concrete situations this concept of finitary is meaningful mainly for first-order quantifications.

Accessibility

For each class of sentences E and each set O of connectives (Boolean connectives, quantifications), let $O(E)$ be the least set of 'internal' sentences closed under O and containing E. In general, the actual institution does not necessarily have all sentences of $O(E)$. A sentence ρ of the institution is *(semantically) accessible from E by O* when ρ is semantically equivalent to a sentence from $O(E)$. The following table illustrates the accessibility situation in some concrete institutions, in all listed examples all sentences being accessible from $O(E)$.

institution	E	O
$\mathcal{FOL}$	atomic equations and relations	negation, conjunctions, $\mathcal{FOL}$ universal quantifications
$\mathcal{EQL}$	atomic equations	$\mathcal{FOL}$ universal quantifications
$\mathcal{HCL}$	atomic equations and relations	conjunctions, implications $\mathcal{FOL}$ universal quantifications
$\mathcal{PA}$	atomic existence equations	negation, conjunctions, universal quantifications with finite blocks of total variables

In the list above only $\mathcal{HCL}$ does not have all sentences from $O(E)$.

Exercises

5.5. In each institution

$$(\forall\chi)\rho \Rightarrow (\rho_1 \wedge \rho_2) \;\models\!\!\!\dashv\; ((\forall\chi)\rho \Rightarrow \rho_1) \wedge ((\forall\chi)\rho \Rightarrow \rho_2).$$

5.6. In any institution with weak model amalgamation let $\mathcal{D}$ be a class of signature morphism with the model expansion property which is stable under pushouts. Then

$$(Q\chi_1)\rho_1 \ (c)\ (Q\chi_2)\rho_2 \;\models\!\!\!\dashv\; (Q\chi_1;\chi_2)\theta_1\rho_1 \ (c)\ \theta_2\rho_2$$

where $Q \in \{\forall, \exists\}$, $(c) \in \{\wedge, \vee\}$ and the following is a pushout square of signature morphisms of $\mathcal{D}$:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\chi_1} & \Sigma_1 \\ {\scriptstyle \chi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

5.7. $\mathcal{FOL}$ admits infinitary quantifiers
Let $\chi_2 = \chi_1;\chi$ be signature morphisms such that χ has the model expansion property. Then $(\forall\chi_2)\chi\rho \models\!\!\!\dashv (\forall\chi_1)\rho$. Apply this for showing that $\mathcal{FOL}$ admits semantic infinitary quantifications.

5.8. Generalization Rule
For each signature morphism $\chi : \Sigma \to \Sigma'$ and each set E of Σ-sentences

$$E \models_\Sigma (\forall\chi)e \text{ if and only if } \chi E \models_{\Sigma'} e.$$

5.9. Stability under pushouts of finitary signature morphisms
In any semi-exact institution, the finitary signature morphisms are stable under pushouts along those signature morphisms for which their model reducts preserve directed co-limits of models.

5.10. Finite models
(a) In any institution with diagrams ι a model is ι-*finite* when its diagram E_M is finite. If the institution has finite conjunctions and existential quantification over elementary extensions along finite models, any two elementary equivalent finite models are homomorphically related. (*Hint:* For a model M consider the sentence $(\exists\iota_\Sigma M) \wedge E_M$.)
(b) In any finite $\mathcal{FOL}$-signature, any two elementary equivalent models with finite carriers are isomorphic. (*Hint:* The sub-institution of $\mathcal{FOL}$ determined by the closed and injective model homomorphisms admits a system of diagrams ι such that a model is ι-finite whenever its signature is finite and it has finite carrier sets.)

5.3 Substitutions

Substitutions are an important logical device, especially when some proof theory is involved. The most notorious concept of substitution comes from the first order logic. Here we recall this in a form that emphasises its institution theoretic properties. By taking these properties as axioms we define an abstract institution-independent concept of substitution.

First order substitutions in $\mathcal{FOL}$. Given a $\mathcal{FOL}$ signature $\Sigma = (S, F, P)$ and two blocks X and Y of variables for Σ a *first order Σ-substitution from X to Y* consists of a sort preserving mapping $\psi\colon X \to T_\Sigma Y$ of the variables X with Σ-terms over Y.

On the semantics side, each first order Σ-substitution $\psi\colon X \to T_\Sigma Y$ determines a functor

$$Mod\psi\colon Mod^{\mathcal{FOL}}(S, F+Y, P) \to Mod^{\mathcal{FOL}}(S, F+X, P)$$

defined by

- $((Mod\psi)M)_x = M_x$ for each sort $x \in S$, or operation symbol $x \in F$, or relation symbol $x \in P$, and
- $((Mod\psi)M)_x = M_{\psi x}$, i.e., the evaluation of the term ψx in M, for each $x \in X$.

On the syntax side, ψ determines a sentence translation function

$$Sen\psi\colon Sen^{\mathcal{FOL}}(S, F+X, P) \to Sen^{\mathcal{FOL}}(S, F+Y, P)$$

which in essence replaces all symbols from X with the corresponding $(S, F+Y)$-terms according to ψ. This can be formally defined as follows:

- $(Sen\psi)(t = t')$ is defined as $\psi^{\mathrm{tm}}t = \psi^{\mathrm{tm}}t'$ for each $(S, F+X, P)$-equation $t = t'$, where $\psi^{\mathrm{tm}}\colon T_\Sigma X \to T_\Sigma Y$ is the unique extension of ψ to an (S, F, P)-homomorphism (ψ^{tm} replaces the variables $x \in X$ with ψx in each $(S, F+X, P)$-term t).
- $(Sen\psi)\pi(t_1, \ldots, t_n)$ is defined as $\pi(\psi^{\mathrm{tm}}t_1, \ldots, \psi^{\mathrm{tm}}t_n)$ for each $(S, F+X, P)$-relational atom $\pi(t_1, \ldots, t_n)$.
- $(Sen\psi)(\rho_1 \wedge \rho_2)$ is defined as $(Sen\psi)\rho_1 \wedge (Sen\psi)\rho_2$ for each conjunction $\rho_1 \wedge \rho_2$ of $(S, F+X, P)$-sentences, and similarly for the case of any other Boolean connectives.
- $(Sen\psi)(\forall Z)\rho = (\forall Z)(Sen\psi_Z)\rho$ for each $(S, F+X+Z, P)$-sentence ρ, where ψ_Z is the trivial extension of ψ to an $(S, F+Z, P)$-substitution.

$$\begin{array}{ccccc}
\Sigma + X & & Sen(\Sigma + X) \xleftarrow{(\forall Z)_} Sen(\Sigma + X + Z) & & \Sigma + X + Z \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle Sen\psi} \qquad\qquad\qquad \downarrow{\scriptstyle Sen\psi_Z} & & \downarrow{\scriptstyle \psi_Z} \\
\Sigma + Y & & Sen(\Sigma + Y) \xleftarrow[(\forall Z)_]{} Sen(\Sigma + Y + Z) & & \Sigma + Y + Z
\end{array}$$

Note that we have extended the notation used for the models functor *Mod* and for the sentence functor *Sen* from the signatures to the first-order substitutions. This notational extension is justified by the Satisfaction Condition developed below:

Proposition 5.8. *For each $\mathcal{FOL}$-signature (S,F,P), each (S,F,P)-substitution $\psi: X \to T_\Sigma Y$, each $(S,F+Y,P)$-model M and each $(S,F+X,P)$-sentence ρ:*

$$(Mod\psi)M \models \rho \ \textit{ if and only if } \ M \models (Sen\psi)\rho.$$

Proof. The proof follows similar steps as in the proof of the $\mathcal{FOL}$ Satisfaction Condition of Prop. 3.2 We can establish by induction on the structure of terms that for each $(S,F+X)$-term t we have that $((Mod\psi)M)_t = M_{\psi^{\mathrm{tm}}t}$. Then, on this basis, we establish the Satisfaction Condition of the proposition by induction on the structure of the sentences. □

General substitutions. The Satisfaction Condition property expressed in Prop. 5.8 allows for the definition of a general concept of substitution by abstracting

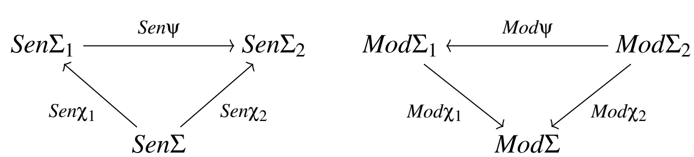
- $\mathcal{FOL}$ signatures (S,F,P) to signatures Σ in arbitrary institutions, and
- sets of first-order variables X for (S,F,P) to signature morphisms $\Sigma \to \Sigma_1$.

For any signature Σ of an institution, and any signature morphisms $\chi_1: \Sigma \to \Sigma_1$ and $\chi_2: \Sigma \to \Sigma_2$, a Σ-*substitution* $\psi: \chi_1 \to \chi_2$ consists of a pair $(Sen\psi, Mod\psi)$, where

– $Sen\psi: Sen\Sigma_1 \to Sen\Sigma_2$ is a function, and

– $Mod\psi: Mod\Sigma_2 \to Mod\Sigma_1$ is a functor

such that both of them preserve Σ, i.e., the following diagrams commute:



and such that the following Satisfaction Condition holds for each Σ_2-model M_2 and each Σ_1-sentence ρ_1:

$$(Mod\psi)M_2 \models \rho_1 \ \text{ if and only if } \ M_2 \models (Sen\psi)\rho_1 \tag{5.1}$$

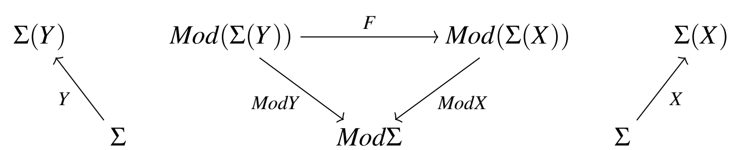
Note that we have again extended the notations *Mod* and *Sen* from the model and the sentence functors of the institution to the model and the sentence components of substitutions.

Fact 5.9. *The Σ-substitutions come equipped with a natural composition satisfying the category axioms by inheriting the composition of the function and functor components. Let this* category of Σ-substitutions *be denoted by $Subst\Sigma$.*

Substitution systems. In actual situations, one often considers only substitutions between signature morphisms which are used in quantifications. The main motivation for this practice is proof-theoretic. Therefore, for any class $\mathcal{D}$ of signature morphisms in an institution, let us say that a *$\mathcal{D}$-substitution* is just a substitution between signature morphisms in $\mathcal{D}$.

When $\mathcal{D}$ is a quantification system, a *$\mathcal{D}$-substitution system* consists of a $|Sig|$-indexed family $\mathcal{S} = \{\mathcal{S}_\Sigma \mid \Sigma \in |Sig|\}$ such that

1. for each $\Sigma \in |Sig|$, $\mathcal{S}_\Sigma$ is a sub-category of the category $Subst(\Sigma)$ of all Σ-substitutions (cf. Fact 5.9);
2. $|\mathcal{S}_\Sigma| = \{X \in \mathcal{D} \mid dom(X) = \Sigma\}$;
3. for any $X, Y \in |\mathcal{S}_\Sigma|$ and any functor F making the triangle below commute



   there exists a unique $\psi \in \mathcal{S}_\Sigma$ such that $F = Mod\psi$.

The Σ-substitutions that belong to $\mathcal{S}_\Sigma$ are called *$\mathcal{S}_\Sigma$-substitutions*.

For example, it is easy to see that the first order substitutions in $\mathcal{FOL}$ constitute an example of a system of $\mathcal{D}$-substitutions when $\mathcal{D}$ is the standard $\mathcal{FOL}$ quantification system. The key to this is to analyse the latter condition above. In $\mathcal{FOL}$, $\Sigma(X) = \Sigma + X$ and $\Sigma(Y) = \Sigma + Y$. Then we consider the result of applying the functor F to $0_{\Sigma+Y}$, the initial mode in $Mod^{\mathcal{FOL}}(\Sigma + Y)$. The substitution $\psi\colon X \to T_\Sigma Y$ that corresponds to F is defined by $\psi x = (F\ 0_{\Sigma+Y})_x$.

Equivalent substitutions. Since general substitutions are a semantic concept, semantical equivalence on substitutions is more meaningful than the strict equality. In other words, what matters about substitution is their semantic effect. Two substitutions $\psi, \psi'\colon \chi_1 \to \chi_2$ are *equivalent* when $Mod\psi = Mod\psi'$. For instance in $\mathcal{PL}$ if χ_i, $i = 1,2$, are extensions of signatures $\Sigma \subseteq \Sigma \uplus \{\pi_i\}$, $i = 1,2$, and $\psi, \psi'\colon \chi_1 \to \chi_2$ are the Σ-substitutions defined by $\psi\pi_1 = \pi_2$ and $\psi\pi_1 = \neg\neg\pi_2$, respectively, then $Mod\psi = Mod\psi'$ while $Sen\psi \neq Sen\psi'$. However in general the translation of models determines the translation on sentences *up to semantical equivalence*:

Fact 5.10. *If ψ and ψ' are equivalent substitutions, then $(Sen\psi)\rho_1 \models\!\!\mid (Sen\psi')\rho_1$ for each Σ_1-sentence ρ_1.*

Exercises

5.11. Substituting relations by sentences

Let (S, F, P) be a $\mathcal{FOL}$ signature and P_1 a set of new relation symbols for S. Each mapping ψ

of relation symbols $\pi \in (P_1)_w$ to sentences $\psi\pi \in Sen(S, F+X, P)$ where $X = \{x_1, \ldots, x_n\}$ such x_i is a variable of sort s_i where $w = s_1 \ldots s_n$, can be extended to a mapping $Sen(S, F, P+P_1) \to Sen(S, F, P)$ by replacing each relational atom $\pi(t_1, \ldots, t_n)$ with $(\psi\pi)(t_1, \ldots, t_n)$. This determines a general substitution in $\mathcal{FOL}$ between $(S, F, P) \hookrightarrow (S, F, P+P_1)$ and $1_{(S,F,P)}$.

5.12. Substitution rule
In any institution, for any substitution $\psi : \chi \to \chi'$ and any sentence ρ,

$$(\forall\chi)\rho \models (\forall\chi')(Sen\psi)\rho.$$

5.13. Institution of substitutions
For each signature Σ of an institution $(Sig, Sen, Mod, \models)$, let $(Subst(\Sigma), Sen, Mod, \models)$ denote the *institution of* Σ*-substitutions*. Its signatures are the signature morphisms with Σ as their domain of the original institution and its signature morphisms are the Σ-substitutions. Then each signature morphism $\varphi : \Sigma \to \Sigma'$ determines canonically a functor $Subst(\varphi) : Subst(\Sigma) \to Subst(\Sigma')$. This construction further determines a functor $Sig^{\mathrm{op}} \to \mathbb{I}ns$.

5.14. For any signatures $(S+S_1, F+F_1)$ and $(S+S_2, F+F_2)$ in $\mathcal{HOL}$ / $\mathcal{HNK}$ each pair consisting of a mapping $\psi^{\mathrm{st}} : S_1 \to \overrightarrow{(S+S_2)}$ and of a family of mappings $\{\psi_s^{\mathrm{op}} : (F_1)_s \to (T_{F+F_2})_{\psi^{\mathrm{type}}s} \mid s \in \overrightarrow{(S+S_1)}\}$ determines a general (S,F)-substitution in $\mathcal{HOL}$ / $\mathcal{HNK}$ between $(S,F) \hookrightarrow (S+S_1, F+F_1)$ and $(S,F) \hookrightarrow (S+S_2, F+F_2)$.

5.4 Representable signature morphisms

$\mathcal{FOL}$ quantifications, called *first order quantifications*, provide a good balance between expressivity and good model-theoretic properties. A great deal of mathematics is first order, and this part can receive great logical / model-theoretic support. On the other hand, a lot of arguments in mathematics have a second or even higher-order nature. This part of mathematics is more problematic concerning logical support. The special strength of first-order model theory owes to quantifications being first order. In this section, we develop an abstract institution-independent approach to first-order quantifications. This will happen on two slightly different levels. The weaker level allows for more concrete examples, while the stronger level allows for more properties.

Quasi-representable signature morphisms

In any institution, a signature morphism $\chi : \Sigma \to \Sigma'$ is *quasi-representable* when for each Σ'-model M', the canonical functor determined by the reduct functor $Mod\chi$ is an isomorphism (of comma categories)

$$M'/Mod\Sigma' \;\cong\; (M'{\restriction_\chi})/Mod\Sigma.$$

This means that each Σ-model homomorphism $h : M'{\restriction_\chi} \to N$ admits a unique χ-expansion $h' : M' \to N'$.

Proposition 5.11. *In $\mathcal{FOL}$ any injective signature extension with constants is quasi-representable.*

Proof. Let $\chi: \Sigma \to \Sigma'$ be any injective $\mathcal{FOL}$ signature extension with constants. Each Σ-homomorphism $h: M'\upharpoonright_\chi \to N$ determines uniquely a χ-expansion $h': M' \to N'$ of h defined, for each constant $x \in \Sigma'$ which does not occur in Σ, by $N'_x = hM'_x$. $\square$

Note that in $\mathcal{FOL}$ quasi-representability fails when we extend signature morphisms with some relation or some non-constant operation symbols.

Quasi-representability of signature extensions with constants holds in various institutions in ways similar to Prop. 5.11. For example, it also works in the institution $E(\mathcal{FOL})$ of the $\mathcal{FOL}$ elementary embeddings. However, in some cases, quasi-representability goes beyond extensions with constants. An example is given by the restriction of $\mathcal{FOL}$ to strong model homomorphisms (recall that $h: M \to N$ is strong when $hM_\pi = N_\pi$ for each relation symbol π). In this institution any signature extension with constants or relation symbols is quasi-representable.

Structural properties of quasi-representability. The following result provides a list of basic structural properties of quasi-representable signature morphisms.

Proposition 5.12. *In any institution*

1. *The quasi-representable signature morphisms are closed under composition.*
2. *If the institution is semi-exact, then quasi-representable signature morphisms are stable under pushouts.*
3. *If the institution is directed-exact, then any directed co-limit of quasi-representable signature morphisms consists of quasi-representable signature morphisms.*
4. *If φ and $\varphi;\chi$ are quasi-representable, then χ is quasi-representable.*

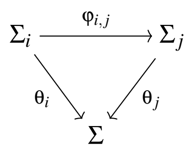
Proof. 1. That composition of quasi-representable morphisms is quasi-representable follows immediately from the definition.

2. Consider a pushout of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\chi} & \Sigma' \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ \Sigma_1 & \xrightarrow[\chi_1]{} & \Sigma'_1 \end{array}$$

such that χ is quasi-representable. We have to show that χ_1 is quasi-representable. Consider a Σ_1-model homomorphism $h_1: M'_1\upharpoonright_{\chi_1} \to N_1$. Let $h: M \to N$ be its θ-reduct. Then $M = M'_1\upharpoonright_{\chi_1}\upharpoonright_\theta = M'_1\upharpoonright_{\theta'}\upharpoonright_\chi$. Because χ is quasi-representable, let $h': M'_1\upharpoonright_{\theta'} \to N'$ be the unique χ-expansion of h. By the semi-exactness of the institution, the unique amalgamation h'_1 of h_1 and h' is the unique χ_1-expansion of h_1 as a homomorphism $M'_1 \to N'_1$.

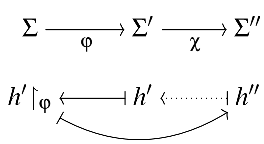
3. Let $(\varphi_{i,j})_{(i<j)\in(I,\leq)}$ be a directed diagram of quasi-representable signature morphisms and let $(\theta_i)_{i\in I}$ be its co-limit.



For each $i \in I$ we show that θ_i is quasi-representable. Let $h_i : M\restriction_{\theta_i} \to N_i$ be a Σ_i-homomorphism for some Σ-model M. We have to show that h_i has an unique θ_i-expansion to a Σ-model homomorphism $h : M \to N$.

- For each $j \in I$, let $M_j = M\restriction_{\theta_j}$. Notice that $M_j\restriction_{\varphi_{i,j}} = M_i$ when $j > i$. For each $j > i$, because $\varphi_{i,j}$ is quasi-representable, let $h_j : M_j \to N_j$ be the unique $\varphi_{i,j}$-expansion of h_i. By the uniqueness of expansion for quasi-representable signature morphisms, we can show that $h_{j'}\restriction_{\varphi_{j,j'}} = h_j$ for each $i \leq j < j'$.
- Now let $(J,\leq)$ be the sub-poset of $(I,\leq)$ determined by the elements $\{j \mid i \leq j\}$. Because $(J,\leq)$ is a final sub-poset of $(I,\leq)$, by Thm. 2.4 we have that $(\theta_i)_{i\in J}$ is a co-limit of $(\varphi_{j,j'})_{(j<j')\in(J,\leq)}$. Because the institution is directed-exact, let $h : M \to N$ be the unique Σ-homomorphism such that $h\restriction_{\theta_j} = h_j$ for each $j \in J$, i.e., h is the unique amalgamation of $(h_j)_{j\in J}$. Then h is the unique θ_i-expansion of h_i to a Σ-homomorphism $M \to N$.

4. Let $\varphi : \Sigma \to \Sigma'$ and $\chi : \Sigma' \to \Sigma''$ be signature morphisms. Consider any Σ'-model homomorphism $h' : M''\restriction_\chi \to N'$. We show that the unique $(\varphi;\chi)$-expansion of $h'\restriction_\varphi$ to a Σ''-model homomorphism $h'' : M'' \to N''$ constitutes the unique χ-expansion of h' to a Σ''-model homomorphism $M'' \to N''$.



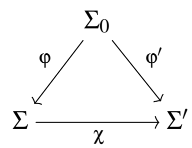
- That $h' = h''\restriction_\chi$ follows by the uniqueness property of the quasi-representability of φ since $h'\restriction_\varphi = (h''\restriction_\chi)\restriction_\varphi$ and h' and $h''\restriction_\chi$ both have $M''\restriction_\chi$ as their domain.
- The uniqueness of h'' as a χ-expansion of h' follows by the uniqueness of h'' as a $(\varphi;\chi)$-expansion of $h'\restriction_\varphi$.

□

Quasi-representable signature morphisms in $\mathcal{FOL}$. We know that the $\mathcal{FOL}$ signature extensions with constants are quasi-representable. The question is: are these all the quasi-representable signature morphisms? Below we give an answer to this question in the form of a complete description of the quasi-representability in $\mathcal{FOL}$.

Proposition 5.13. *A $\mathcal{FOL}$ signature morphism is quasi-representable if and only if it is bijective on sort symbols, relation symbols, and non-constant operation symbols.*

Proof. Consider such a $\mathcal{FOL}$-signature morphism $\chi\colon \Sigma \to \Sigma'$. Then there exists a signature Σ_0 and injective extensions with constants $\varphi\colon \Sigma_0 \to \Sigma$ and $\varphi'\colon \Sigma_0 \to \Sigma'$ such that the following triangle commutes:



Because both φ and φ' are (injective) extensions with constants, they are quasi-representable (Prop. 5.11), hence by χ is quasi-representable too (Prop. 5.12 4.).

Conversely, let us assume that χ is quasi-representable. If one of χ^{st}, χ^{op} restricted to non-constant operation symbols, or χ^{rl}, is not surjective, respectively not injective, then we can find a Σ-homomorphism $h\colon M \to N$ and a χ-expansion M' of M such that h has more than one, respectively does not have any, χ-expansion $h'\colon M' \to N'$. We leave the details of this argument to the reader. □

Corollary 5.14. *A $\mathcal{FOL}$ signature morphism has the model expansion property and is quasi-representable if and only if it is an injective extension with constants.*

Finitary quasi-representable signature morphisms. Any quasi-representable signature morphism $\chi\colon \Sigma \to \Sigma'$ determines a canonical functor $(Mod\chi)^{-1}\colon Mod\Sigma \to \mathbb{C}lass$

- that maps each Σ-model M to $\{M' \in |Mod\Sigma'| \mid M'\restriction_\chi = M\}$, and
- that maps each Σ-model homomorphism $h\colon M \to N$ to the class function $(Mod\chi)^{-1}h\colon (Mod\chi)^{-1}M \to (Mod\chi)^{-1}N$ such that for each χ-expansion M' of M, $((Mod\chi)^{-1}h)M' = N'$ where $h'\colon M' \to N'$ is the unique χ-expansion of h from M'.

These considerations allow us to express at the abstract institution-independent level the signature morphisms that are both finitary and quasi-representable utilizing a preservation property.

Fact 5.15. *A quasi-representable signature morphism χ is* finitary *if and only if $(Mod\chi)^{-1}$ preserves the directed co-limits.*

Below, in Cor. 5.17 we will establish in concrete terms when the $\mathcal{FOL}$ quasi-representable signature morphisms χ are finitary.

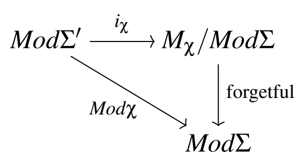
Representable signature morphisms

Consider a quasi-representable signature morphism $\chi\colon \Sigma \to \Sigma'$ and assume that $Mod(\Sigma')$ has an initial model $0_{\Sigma'}$. We have the following canonical isomorphisms:

$$Mod\Sigma' \;\cong\; 0_{\Sigma'}/Mod\Sigma' \;\cong\; (0_{\Sigma'}\restriction_\chi)/Mod\Sigma.$$

This situation shows that the Σ'-models M' can be 'represented' isomorphically by Σ-model homomorphisms $0_{\Sigma'}\upharpoonright_{\chi} \to M'\upharpoonright_{\chi}$.

A signature morphism $\chi\colon \Sigma \to \Sigma'$ is *representable* if and only if there exists a Σ-model M_χ (called the *representation of* χ) and an isomorphism i_χ of categories such that the following diagram commutes:



Since 1_{M_χ} is initial in $M_\chi/Mod\Sigma$, by the isomorphism i_χ it follows that $Mod\Sigma'$ necessarily has an initial model whose χ-reduct is precisely M_χ.

Fact 5.16. *A signature morphism* $\chi\colon \Sigma \to \Sigma'$ *is representable if and only if it is quasi-representable and* $Mod\Sigma'$ *has an initial model.*

For example, since $\mathcal{FOL}$ has initial models of signatures, in $\mathcal{FOL}$ representable and quasi-representable signature morphisms are the same concept. Given a set X of first-order variables for a $\mathcal{FOL}$ signature (S,F,P), the representation of the signature inclusion $(S,F,P) \hookrightarrow (S,F+X,P)$ is given by the model of the $(S,F+X,P)$-terms $T_{(S,F,P)}X$, which is the free (S,F,P)-model over X. This is due to the fact that $(S,F+X,P)$-models M are in canonical bijection with valuations of variables from X to the carrier sets of M. By the freeness property of $T_{(S,F,P)}X$, these valuations are in canonical bijection with (S,F,P)-model homomorphisms $T_{(S,F,P)}X \to M$.

By Fact 5.16, examples that fall between representability and quasi-representability can be found only in institutions which do not have initial models of signatures. Examples include the local and global $\mathcal{MFOL}$ institutions and $\mathcal{HOL}$. A special class of institutions without initial models for signatures arises by narrowing the class of model homomorphisms in institutions; examples include the sub-institution $E(\mathcal{FOL})$ of $\mathcal{FOL}$ elementary embeddings, and the sub-institution of strong $\mathcal{FOL}$-model homomorphisms. In all examples mentioned above the signature extensions with constants are quasi-representable but in general, they are not representable.

Finitary representable signature morphisms. Since for any χ representable signature morphism we have that $(Mod\chi)^{-1}$ is isomorphic to $(Mod\Sigma)(M_\chi, -)$ it follows immediately that:

Corollary 5.17. *A representable signature morphism* $\chi\colon \Sigma \to \Sigma'$ *is finitary if and only if its representation* M_χ *is finitely presented.*

For instance, Cor. 5.17 together with the existence of initial models for $\mathcal{FOL}$ signatures (Prop. 4.27) and the characterisation of quasi-representability in $\mathcal{FOL}$ (Prop. 5.13) enable us to establish quite easily the following (we leave the details of the argument as an exercise for the reader).

Corollary 5.18. *The finitary representable $\mathcal{FOL}$ signature morphisms χ are precisely those that extend with a finite number of constants and such that the set of pairs of constants $\{(c_1, c_2) \mid \chi c_1 = \chi c_2, c_1 \neq c_2\}$ is finite too. In particular, the finitary representable $\mathcal{FOL}$ signature morphisms that have the model expansion property are just the injective extensions with a finite number of constants.*

Representable substitutions

The $\mathcal{FOL}$ situation that each first order (S,F,P)-substitution $\psi\colon X \to T_{(S,F,P)}Y$ (of variables X with (S,F,P)-terms over Y) can be extended uniquely to a model homomorphism $h_\psi\colon T_{(S,F,P)}X \to T_{(S,F,P)}Y$ is a mere reflection of the more general fact that substitutions between representable signature morphisms can be 'represented' as model homomorphisms.

Proposition 5.19. *Any substitution $\psi\colon \chi_1 \to \chi_2$ between representable signature morphisms $\chi_1\colon \Sigma \to \Sigma_1$ and $\chi_2\colon \Sigma \to \Sigma_2$ determines canonically a Σ-model homomorphism $M_\psi\colon M_{\chi_1} \to M_{\chi_2}$ between the representations of the signature morphisms χ_1 and χ_2 such that the diagram below commutes:*

$$\begin{array}{ccc}
Mod\Sigma_2 & \xrightarrow[\sim]{i_{\chi_2}} & M_{\chi_2}/Mod\Sigma \\
{\scriptstyle Mod\psi}\downarrow & & \downarrow{\scriptstyle M_\psi;_} \\
Mod\Sigma_1 & \xrightarrow[i_{\chi_1}]{\sim} & M_{\chi_1}/Mod\Sigma
\end{array} \tag{5.2}$$

Moreover, the mapping of the substitutions ψ to the model homomorphisms M_ψ is functorial and faithful modulo substitution equivalence.

Proof. We define $M_\psi = (i_{\chi_2}^{-1}\,; Mod\psi\,;\, i_{\chi_1})1_{M_{\chi_2}}$.

- Let us show that $M_\psi\colon M_{\chi_1} \to M_{\chi_2}$. Since $M_\psi \in |M_{\chi_1}/Mod\Sigma|$ it is obvious that $dom(M_\psi) = M_{\chi_1}$. That $cod(M_\psi) = M_{\chi_2}$ follows by analysing the results of applying in succession the functors in the diagram below to $1_{M_{\chi_2}} \in |M_{\chi_2}/Mod\Sigma|$.

$$\begin{array}{ccccc}
 & Mod\Sigma_2 & \xrightarrow{Mod\psi} & Mod\Sigma_1 & \\
{\scriptstyle i_{\chi_2}^{-1}}\nearrow & & {\scriptstyle Mod\chi_2}\searrow \quad \swarrow{\scriptstyle Mod\chi_1} & & \searrow{\scriptstyle i_{\chi_1}} \\
M_{\chi_2}/Mod\Sigma & \xrightarrow[\text{forgetful}]{} & Mod\Sigma & \xleftarrow[\text{forgetful}]{} & M_{\chi_1}/Mod\Sigma
\end{array} \tag{5.3}$$

- For proving the commutativity of the diagram (5.2) we consider any arrow $f\colon 1_{M_{\chi_2}} \to f$ in $M_{\chi_2}/Mod\Sigma$. From the commutativity of the diagram (5.3) we get that $f\colon M_\psi \to (i_{\chi_2}^{-1}\,; Mod\psi\,;\, i_{\chi_1})f$ which implies $(i_{\chi_2}^{-1}\,; Mod\psi\,;\, i_{\chi_1})f = M_\psi; f$.
- Let ψ and ψ' be substitutions that can be composed. By putting together the commutative squares (5.2) that correspond to ψ and to ψ' and by applying the functoriality

of *Mod* (the model functor for substitutions) we get that $M_\psi ; M_{\psi'} = M_{\psi;\psi'}$. That $M_{1_\chi} = 1_{M_\chi}$ follows by considering the diagram (5.2) for the identity substitution.

- If ψ and ψ' are equivalent substitutions then $M_\psi = M_{\psi'}$ since by its definition M_ψ are uniquely determined by the model translations $Mod\psi$.

□

The converse of the representation result of Prop. 5.19 is that each model homomorphism $h\colon M_{\chi_1} \to M_{\chi_2}$ determines a unique equivalence class of substitutions $\psi_h\colon \chi_1 \to \chi_2$ such that $h = M_{\psi_h}$. Thus we say that an institution *has representable $\mathcal{D}$-substitutions* for a class $\mathcal{D}$ of signature morphisms when for each signature Σ the canonical functor (of Prop. 5.19) from the category of the Σ-$\mathcal{D}$-substitutions between representable signature morphisms to the category of Σ-models is full. Although a general criterion for an institution to have representable substitutions is not to be expected, this property can be established rather easily for some particular institutions. $\mathcal{FOL}$ is a rather typical example.

Proposition 5.20. *$\mathcal{FOL}$ has all representable substitutions.*

Proof. Let $\chi_1\colon \Sigma \to \Sigma_1$ and $\chi_2\colon \Sigma \to \Sigma_2$ be representable signature morphisms in $\mathcal{FOL}$ and let $h\colon M_{\chi_1} \to M_{\chi_2}$ be a Σ-model homomorphism.

By Prop. 5.13, without any loss of generality, we may assume that $\Sigma = (S, F + X, P)$, $\Sigma_1 = (S, F + X_1, P)$ and $\Sigma_2 = (S, F + X_2, P)$ where X, X_1, X_2 are sets of constants and χ_1 and χ_2 keep (S, F, P) invariant but on X manifest as functions $f_1\colon X \to X_1$ and $f_2\colon X \to X_2$. Then $M_{\chi_1} = (T_{(S,F,P)}X_1)\upharpoonright_{\chi_1}$ and $M_{\chi_2} = (T_{(S,F,P)}X_2)\upharpoonright_{\chi_2}$. Note that because h is a $(S, F + X, P)$-homomorphism we have that $h(f_1 x) = f_2 x$ for each $x \in X$.

The desired substitution ψ is defined as the first order substitution given by the restriction of h to a function $X_1 \to T_{(S,F,P)}X_2$. Although ψ appears as a substitution between $(S, F, P) \hookrightarrow (S, F + X_1, P)$ and $(S, F, P) \hookrightarrow (S, F + X_2, P)$, the equality $h(f_1 x) = f_2 x$ guarantees that ψ is a substitution $\chi_1 \to \chi_2$. Finally, we notice that $M_\psi = h$ indeed. □

Exercises

5.15. Does $\mathcal{PL}$ have non-bijective representable signature morphisms?

5.16. For any quasi-representable signature morphism χ, the model reduct functor $Mod\chi$ is faithful.

5.17. Representable signature morphisms in $\mathcal{HNK}$
In $\mathcal{HNK}$ the signature extensions with constants $\chi\colon \Sigma \to \Sigma'$, although in general are not representable, they are however quasi-representable. Moreover, χ is representable whenever Σ' has at least a constant operation symbol for each type.

5.18. In any institution the quasi-representable signature morphisms preserve the epi model homomorphisms, i.e., the model homomorphism reduct $h\upharpoonright_\chi$ is epi when h is epi and χ is quasi-representable.

5.19. Quasi-representable theory morphisms
In any institution I for each theory (Σ, E) and each quasi-representable signature morphism $\chi\colon \Sigma \to$

Σ', the theory morphism $\chi\colon (\Sigma,E) \to (\Sigma',\chi E)$ is quasi-representable (as signature morphism in I^{th}). For non-liberal institutions this constitutes another source of examples of quasi-representable signature morphisms which are not representable.

5.20. Quasi-representability along institution comorphisms
Any exact institution comorphism (Φ,α,β) preserves quasi-representable signature morphisms in the sense that $\Phi\chi$ is quasi-representable when χ is quasi-representable.

5.21. In $\mathcal{IPL}$ each morphism of signatures is representable. (*Hint:* Consider the comorphism $\mathcal{IPL} \to (\mathcal{FOEQL}^1)^{\text{th}}$ of Ex. 4.11 and use the combined conclusions of Exercises 5.19 and 5.20.)

5.22. Liberal representable signature morphisms
In any institution with binary co-products of models, for each signature each representable signature morphism is liberal.

5.23. In any institution the finitary quasi-representable signature morphisms are closed under composition.

5.24. Co-products of substitutions
In an institution with representable substitutions which has pushouts of signature morphisms, which is semi-exact and its categories of models have finite co-products, for each signature Σ, the category of the Σ-substitutions modulo substitution equivalence between representable signature morphisms has finite co-products.

5.25. Representable substitutions for theories
Consider a liberal institution I and a class $\mathcal{D}$ of representable signature morphisms such that

- for all theories (Σ,E) the units of the adjunctions determined by the forgetful functors $Mod^{\text{th}}(\Sigma,E) \to Mod\Sigma$ are epi, and
- the representations M_χ of the signature morphisms $\chi \in \mathcal{D}$ are projective.

Let $\mathcal{D}^{\text{th}}$ be the class of strong theory morphisms $\chi\colon (\Sigma,E) \to (\Sigma',E')$ for which $(\chi\colon \Sigma \to \Sigma') \in \mathcal{D}$. Then the institution I^{th} has representable $\mathcal{D}^{\text{th}}$-substitutions. Apply this general result for establishing that $\mathcal{AFOL}^{\text{th}}$ (where $\mathcal{AFOL}$ is the atomic sub-institution of $\mathcal{FOL}$) has representable $\mathcal{D}^{\text{th}}$-substitutions for $\mathcal{D}$ the class of $\mathcal{FOL}$ signature extensions with a finite number of constants.

5.5 Satisfaction by injectivity

In this chapter we have already introduced the semantics of Boolean connectives and quantifiers at a general institution-independent level. To complete the institution-independent expression of the fundamental built of first-order model theory it remains to develop an abstract approach to atomic sentences. Unlike with the Boolean connectives and with the quantifiers, at the institution-independent level the semantics of atoms can only be approximated by relying on categorical injectivity. The satisfaction of the atoms can be explained at the abstract level as a restricted form of categorical injectivity. On this basis, we define a general abstract concept of Horn sentences whose satisfaction can be expressed as full categorical injectivity. Moreover, we show that satisfaction by categorical injectivity cannot go beyond such sentences.

Basic sentences

In any $\mathcal{FOL}$-signature $\Sigma = (S, F, P)$ let E be a set of atoms. Recall from Sect. 4.6 that E has an initial model 0_E constructed as follows: on the quotient $(0_\Sigma)_{=_E}$ of the term model 0_Σ by the congruence generated by the equational atoms of E, we interpret each relation symbol $\pi \in P$ by $(0_E)_\pi = \{(t_1/_{=_E}, \ldots, t_n/_{=_E}) \mid \pi(t_1, \ldots, t_n) \in E\}$.

Fact 5.21. *For each set E of $\mathcal{FOL}$-atoms and for each model M, $M \models E$ if and only if there exists a model homomorphism $0_E \to M$.*

The categorical characterization of atomic satisfaction above can serve as a first institution-independent approximation for the concept of atom: in any institution, a set E of Σ-sentences is *basic* if there exists a Σ-model M_E, called a *basic model of* E, such that for each Σ-model M,

$$M \models_\Sigma E \text{ if and only if there exists a model homomorphism } M_E \to M.$$

Given a basic set E of sentences, in general the basic models M_E are not necessarily unique, not even up to isomorphisms. Often M_E is the initial model of E; we have already seen this in Fact 5.21. One may think that the existence of an initial model for a set of sentences implies that the respective set of sentences is basic. This is not true, and a simple counterexample in $\mathcal{FOL}$ is given by the negation $t_1 \neq t_2$ of an equation $t_1 = t_2$ (where t_1 and t_2 are different terms). The negation has the term model 0_Σ as its initial model but is not basic.

On the other hand, being basic covers significantly more than atomic sentences. For instance in $\mathcal{FOL}$ the existentially quantified atoms are basic too. This even works at the general level as follows.

Fact 5.22. *Basic sentences are closed under quasi-representable existential quantification. Moreover if χ is quasi-representable and $M_{\rho'}$ is a basic model for ρ' then $M_{\rho'}\restriction_\chi$ is a basic model for $(\exists\chi)\rho'$.*

Epi basic sentences. The concept of 'epi basic' sentences constitutes a more accurate institution-independent capture of the actual atoms than the concept of basic sentences. An *epi basic* set of Σ-sentences E is a basic set of sentences that admits a basic model M_E – called the *epi basic model of* E – which is initial in $Mod(\Sigma, E)$.

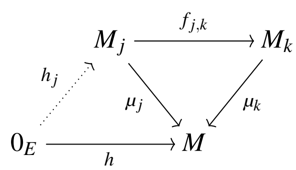
Since the initial models are unique up to isomorphisms, for each epi basic set E of sentences we may refer to any of its epi basic models as its epi basic *model*. Notice also that if E is an epi basic set of sentences it does not necessarily mean that any of its basic models is epi basic. For instance, let us consider a $\mathcal{FOL}$ signature with one sort s and two constants a and b. Then the model N defined by $N_s = \{a, c\}$ and $N_a = N_b = a$ is a basic model for the equation $a = b$ but it is not its epi basic model.

Fact 5.23. *All sets E of $\mathcal{FOL}$ atoms are epi basic. However in general the existential quantifications of $\mathcal{FOL}$ atoms are* not *epi basic.*

Finitary basic sentences. A basic set of Σ-sentences E is *finitary* if it has a basic model M_E that is finitely presented in the category $Mod\Sigma$.

Proposition 5.24. *All finite sets of $\mathcal{FOL}$ atoms are finitary basic.*

Proof. We prove that for each set E of Σ-atoms in $\mathcal{FOL}$, their initial model 0_E is finitely presented. Consider a model homomorphism $h: 0_E \to M$ where the $(\mu_i : M_i \to M)_{\in I}$ is a co-limit of a directed diagram $(f_{i,j})_{(i<j)\in(I,\leq)}$ of Σ-model homomorphisms. Since $\mathcal{FOL}$ model homomorphisms preserve the satisfaction of the atomic sentences, it follows that $M \models E$. It is enough to find $j \in I$ and a homomorphism $h_j : 0_E \to M_j$ because in that case both $h_j ; \mu_j = h$ and the second condition defining finitely presented objects hold trivially by the uniqueness side of the initiality property of 0_E. Moreover h_j is always guaranteed if $M_j \models E$, so the problem reduces to finding j such that $M_j \models E$.



We do this first for the case when E consists of a single equational atom, then for the case when E consists of single relational atoms, and finally for the general case. At some moment in our proof, we will have to rely on the fact that forgetful functors from $\mathcal{FOL}$ models to their carrier sets preserve directed co-limits; this will be proved only in Chap. 6 (Prop. 6.8).

- Let $E = \{t = t'\}$. Then

1	$M_t = M_{t'}$	$M \models E$.

  Let us consider any $i \in I$. Then

2	$\mu_i\ (M_i)_t = M_t,\ \mu_i\ (M_i)_{t'} = M_{t'}$	μ_i homomorphism
3	$\mu_i\ (M_i)_t = \mu_i\ (M_i)_{t'}$	1, 2.

  Since the forgetful functors from $\mathcal{FOL}$ models to their carrier sets preserve directed co-limits, from 3 it follows that there exists $j \geq i$ such that

4	$f_{i,j}\ (M_i)_t = f_{i,j}\ (M_i)_{t'}$	directed co-limits in $\mathbb{S}et$.

  Since

5	$f_{i,j}\ (M_i)_t = (M_j)_t,\ f_{i,j}\ (M_i)_{t'} = (M_j)_{t'}$	$f_{i,j}$ homomorphism

  from 4 it follows that $(M_j)_t = (M_j)_{t'}$ which means $M_j \models E$.

- Let $E = \{\pi t\}$ such that πt is a relational atom (t represents a string of terms of appropriate sorts). Since μ is a co-limit, $M_\pi = \bigcup_{i \in I} \mu_i \ (M_i)_\pi$. Thus there exists $i \in I$ such that

1	$M_t \in \mu_i \ (M_i)_\pi$.	

  It follows that

2	there exists $m_i \in (M_i)_\pi,\ M_t = \mu_i m_i$	1
3	$M_t = \mu_i \ (M_i)_t$	μ_i homomorphism
4	there exists $j \geq i$, $f_{i,j} m_i = f_{i,j} \ (M_i)_t$	2, 3, m_i, t finite strings, directed co-limits in $\mathbb{S}et$
5	$f_{i,j} m_i \in (M_j)_\pi$	$m_i \in (M_i)_\pi$, $f_{i,j}$ homomorphism
6	$f_{i,j} \ (M_i)_t = (M_j)_t$	$f_{i,j}$ homomorphism
7	$(M_j)_t \in (M_j)_\pi$	4, 5, 6.

  Hence there exists $j \in I$ such that $M_j \models \pi t$.

- For the general case, we perform a proof by induction on the size of E. The base case has been solved above. For the induction step, we consider any partition $E = E_1 \uplus E_2$. Then by the induction hypothesis let j_i, $i = 1,2$, such that $M_{j_i} \models E_i$, $i = 1,2$. For any $k \geq j_1, j_2$, since $f_{j_i,k}$, $i = 1,2$, as homomorphisms do preserve the satisfaction of the atoms, we have that $M_k \models E_i$, $i = 1,2$. Hence $M_k \models E$.

□

Basic diagrams. In concrete situations, it is quite common that the elementary extensions that come with the respective concept of diagram are first order. The following result develops a general consequence of such a situation.

Proposition 5.25. *In any institution with diagrams ι such that the elementary extensions are quasi-representable, the diagrams are epi basic.*

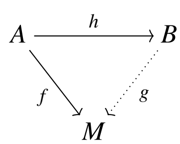
Proof. Let M be a Σ-model and let $\iota_\Sigma M : \ \Sigma \to (\Sigma_M, E_M)$ be its diagram. Let M_M be the initial model of this diagram. We prove that M_M is the epi basic model of E_M.

- Let N' be any Σ_M-model. If $N' \models E_M$, then because M_M is the initial (Σ_M, E_M)-model there exists a unique model homomorphism $M_M \to N'$.
- Conversely, assume that there exists a model homomorphism $h' : \ M_M \to N'$ and let $h = h' \upharpoonright_{\iota_\Sigma M} : \ M \to N' \upharpoonright_{\iota_\Sigma M}$. Let $N_h = i^{-1}_{\Sigma,M} h$. Then $N_h \models E_M$. Because $\iota_\Sigma M$ is quasi-representable and $M_M \upharpoonright_{\iota_\Sigma M} = M$, there exists a unique $\iota_\Sigma M$-expansion of h to a Σ_M-model homomorphism from M_M, which is necessarily h'. Therefore $N_h = N'$, hence $N' \models E_M$.

□

The result of Prop. 5.25 is quite telling about the concept of epic basic sets of sentences. On the one hand, it is quite common for diagrams to be sets of atomic sentences. On the other hand, there are many relevant situations when this is not so, the table of Section 4.4 showing examples of diagrams in some sub-institutions of $\mathcal{FOL}$ determined by various concepts of model homomorphisms that illustrate well this. Perhaps all these tell us that from a semantic perspective the concept of the atom as the most primitive constituent of sentences should be considered according to the actual concept of model homomorphism.

Satisfaction by injectivity. The semantics of the basic sentences constitutes a simple example of satisfaction by injectivity. Recall that a model M is injective with respect to a model homomorphism $h\colon\ A \to B$ when for each homomorphism $f\colon\ A \to M$ there exists a homomorphism $g\colon\ B \to M$ such that $h\,;\,g = f$.



Let us denote this by $M \models^{\mathrm{inj}} h$. For each homomorphism h let $Inj(h)$ be the class of models injective with respect to h, and for each class of homomorphisms H let $Inj(H) = \bigcap_{h\in H} Inj(h)$.

Fact 5.26. *Let Σ be a signature with initial model 0_Σ. For any basic set E of Σ-sentences, any basic model M_E for E, and for any Σ-model M,*

$$M \models e \quad \text{if and only if} \quad M \models^{\mathrm{inj}} (0_\Sigma \to M_E),$$

(where $0_\Sigma \to M_E$ represents the unique model homomorphism given by the initiality of 0_Σ).

General Horn sentences. The abstract concepts introduced so far allow us to define a general institution-independent concept of Horn sentence as follows. In any institution, for a designated class $\mathcal{D}$ of signature morphisms, a *$\mathcal{D}$-universal Horn sentence* is any sentence that is semantically equivalent to $(\forall\chi)(E \Rightarrow E')$ where

– $\chi\colon\ \Sigma \to \Sigma'$ is a representable signature morphism in $\mathcal{D}$,

– E is a set of epic basic Σ'-sentences, and

– E' is a basic set of Σ'-sentences.

A universal Horn sentence $(\forall\chi)(E \Rightarrow E')$ is *finitary* when χ is finitary and E is finite.

While using the notation $(\forall\chi)(E \Rightarrow E')$ we do not assume that designated universal quantifications or implications exist, instead, as usual, we just assume that they exist in an implicit form.

Note that the general concept of finitary Horn sentence defined above covers more sentences than some of the actual concepts of Horn sentence in concrete institutions. This is because the basic, and even the epi basic, sentences are usually more than the actual atoms in institutions. For example, $(\forall X)(\exists Y)t = t'$ is a general finitary Horn sentence in $\mathcal{FOL}$ but it is not a $\mathcal{HCL}$-sentence.

What remains from this section is devoted to establishing the equivalence between the satisfaction of Horn sentences and the satisfaction by categorical injectivity.

Satisfaction of Horn sentences is injectivity

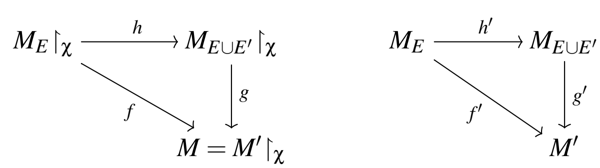
For the rest of this section let us assume that in our institutions the basic sets of sentences are closed under finite unions. This is a mild condition in the applications as this happens whenever there exist co-products of models by letting a basic model be $M_{E \cup E'} = M_E + M_{E'}$. The following extends the encoding of satisfaction of sentences as injectivity from basic sentences to Horn sentences.

Proposition 5.27. *In any institution, for any universal Horn sentence $(\forall \chi)(E \Rightarrow E')$ there exists a model homomorphism h such that for each Σ-model M,*

$$M \models^{\text{inj}} h \quad \textit{if and only if} \quad M \models (\forall \chi)(E \Rightarrow E').$$

Proof. Let M_E / $M_{E'}$ / $M_{E \cup E'}$ be basic models for E / E' / $E \cup E'$, respectively. Since E is epi basic let us assume that $M_E = 0_E$, the initial model satisfying E. Because $E \cup E' \models E$, $M_{E \cup E'} \models E$. Therefore there exists a homomorphism $h' : M_E \to M_{E \cup E'}$. We let $h = h' \restriction_\chi$.

- First, let us assume that a model M is injective for h. Consider any χ-expansion M' of M such that $M' \models E$. Hence there exists a model homomorphism $f' : M_E \to M'$. Because M is injective for h, there exists a model homomorphism $g : M_{E \cup E'} \restriction_\chi \to M$ such that $h \,;\, g = f' \restriction_\chi$. Because χ is quasi-representable, we get $g' : M_{E \cup E'} \to M'$ such that $g' \restriction_\chi = g$. This means that $M' \models E \cup E'$, which implies $M' \models E'$.



- Conversely, assume that $M \models (\forall \chi)E \Rightarrow E'$. Because χ is quasi-representable, each Σ-model homomorphism $f : M_E \restriction_\chi \to M$ admits an expansion to an Σ'-model homomorphism $f' : M_E \to M'$. This implies $M' \models E$, therefore $M' \models E'$. Hence $M' \models E \cup E'$ which guarantees the existence of a model homomorphism $g' : M_{E \cup E'} \to M'$. Because M_E is initial it follows that $h' \,;\, g' = f'$, which implies $h \,;\, g' \restriction_\chi = f$.

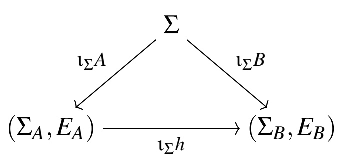
□

The model homomorphism $h : M_E \to M_{E \cup E'}$ of Prop. 5.27 has the flavour of a 'quotient' because $E \subseteq E \cup E'$. In some common cases, this can indeed be a proper quotient, for instance when E and E' consist of equations.

Injectivity is satisfaction of Horn sentences

We have seen that the satisfaction of Horn sentences can be expressed as categorical injectivity for a model homomorphism. Now we show that the other way around is also true, that the injectivity with a model homomorphism that resembles a 'quotient' can be expressed as the satisfaction of a Horn sentence. But this result requires significantly more conceptual infrastructure than the result of Prop. 5.27 because we have to 'produce' a Horn sentence from a model homomorphism. For this we use diagrams. The 'quotient' property of the model homomorphism involved is axiomatised in relation to the respective diagrams as follows.

ι-conservative model homomorphisms. Given an institution with diagrams ι, a Σ-model homomorphism $h\colon A \to B$ is ι-*conservative* when the theory morphism $\iota_\Sigma h\colon (\Sigma_A, (\iota_\Sigma h)^{-1}E_B^{**}) \to (\Sigma_B, E_B)$ has the model expansion property.



The informal meaning of this property is that the target of h does not contain anything that cannot be expressed by the entities of the source of h. The following easy-to-check fact gives an example of such a concrete situation.

Fact 5.28. *Consider $\mathcal{FOL}$ with its standard system of diagrams. Then each surjective model homomorphism h is ι-conservative.*

In the absence of the surjectivity of h the ι-conservativity may fail as suggested by the following simple example. Let Σ consist of one sort and one unary relation symbol π. Let the carrier set of A consist of only one element such that $A_\pi = \{a\}$ and the carrier set of B consist of two elements a and b such that $B_\pi = \{a,b\}$. Let $ha = a$. Let M be the Σ_A-model consisting of two elements a and b and such that $M_a = a$, $M_\pi = \{a\}$. Then $M \models (\iota_\Sigma h)^{-1}E_B^{**}$ but M does not admits a $\iota_\Sigma h$-expansion (to a (Σ_B, E_B)-model) because $b \notin M_\pi$.

Proposition 5.29. *In any institution with diagrams ι, let h be a Σ-model homomorphism $h\colon A \to B$. Let M be any Σ-model. Then*

1. *$M \models^{\mathrm{inj}} h$ implies $M \models_\Sigma (\forall \iota_\Sigma A)(E_A \Rightarrow (\iota_\Sigma h)^{-1}E_B^{**})$.*
2. *If h is ι-conservative, then $M \models_\Sigma (\forall \iota_\Sigma A)(E_A \Rightarrow (\iota_\Sigma h)^{-1}E_B^{**})$ implies $M \models^{\mathrm{inj}} h$.*

Consequently, if the elementary extensions of ι are representable then the satisfaction by injectivity for any ι-conservative h is the same as the satisfaction of a general Horn sentence.

Proof. 1. Let M' be any $\iota_\Sigma A$-expansion of M such that $M' \models E_A$. Let $f = i_{\Sigma,A}\, M'$. By the injectivity of M, let g be such that $h\,;\, g = f$ and let $M_g = i^{-1}_{\Sigma,B}\, g$.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\iota_\Sigma A} & \Sigma_A \\ & \searrow^{\iota_\Sigma B} & \downarrow \iota_\Sigma h \\ & & \Sigma_B \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow^{f} & \downarrow g \\ & & M \end{array}$$

By the naturality of i as shown in the following diagram

$$\begin{array}{ccc} Mod(\Sigma_A, E_A) & \xrightarrow{i_{\Sigma,A}} & A/Mod\Sigma \\ {\scriptstyle Mod(\iota_\Sigma h)}\uparrow & & \uparrow{\scriptstyle h;\_} \\ Mod(\Sigma_B, E_B) & \xrightarrow[i_{\Sigma,B}]{} & B/Mod\Sigma \end{array} \tag{5.4}$$

we get that $M_g\restriction_{\iota_\Sigma h} = M'$. Then

1	$M_g \models E_B^{**}$	$M_g = i^{-1}_{\Sigma,B}\, g \in Mod(\Sigma_B, E_B)$
2	$(\iota_\Sigma h)((\iota_\Sigma h)^{-1} E_B^{**}) \subseteq E_B^{**}$	
3	$M_g \models (\iota_\Sigma h)((\iota_\Sigma h)^{-1} E_B^{**})$	1, 2
4	$M_g\restriction_{\iota_\Sigma h} = M'$	above
5	$M' \models (\iota_\Sigma h)^{-1} E_B^{**}$	3, 4, Satisfaction Condition

2. Consider a model M such that $M \models_\Sigma (\forall \iota_\Sigma A)(E_A \Rightarrow (\iota_\Sigma h)^{-1} E_B^{**})$ and a model homomorphism $f : A \to M$. Let $M_f = i^{-1}_{\Sigma,A}\, f$.

1	$M_f\restriction_{\iota_\Sigma A} = M$	definition of M_f, naturality of i
2	$M_f \models E_A$	definition of M_f
3	$M_f \models (\iota_\Sigma h)^{-1} E_B^{**}$	1, 2, $M \models (\forall \iota_\Sigma A)(E_A \Rightarrow (\iota_\Sigma h)^{-1} E_B^{**})$
4	there exists $M' \in Mod(\Sigma_B, E_B)$, $M'\restriction_{\iota_\Sigma h} = M_f$	3, h is ι-conservative.

Let $g = i_{\Sigma,B}\, M'$. By the naturality of i, as shown in the diagram (5.4), applied to M', we have $h\,;\, g = f$.

The final conclusion holds because E_A is epi basic (cf. Prop. 5.25) and because of the following claim:

> If all elementary extensions are quasi-representable and h is ι-conservative, then $(\iota_\Sigma h)^{-1} E_B^{**}$ is basic with the $(B_B)\restriction_{\iota_\Sigma h}$ being one of its basic models.

To prove this claim let us consider any Σ-model M.

- On the one hand, if $M \models (\iota_\Sigma h)^{-1} E_B^{**}$, then because h is ι-conservative, there exists a $\iota_\Sigma h$-expansion M' of M to a (Σ_B, E_B)-model. Let f be the unique (Σ_B, E_B)-model homomorphism $B_B \to M'$. Then $f\restriction_{\iota_\Sigma h} : (B_B)\restriction_{\iota_\Sigma h} \to M$.
- On the other hand, let us consider a Σ_A-model homomorphism $(B_B)\restriction_{\iota_\Sigma(h)} \to M$. Then

1	$\iota_\Sigma h$ quasi-representable	$\iota_\Sigma A$, $\iota_\Sigma B$ quasi-representable, $\iota_\Sigma A \,;\, \iota_\Sigma h = \iota_\Sigma B$, Prop. 5.12 4.
2	there exists homomorphism $B_B \to M'$ expansion of $(B_B)\restriction_{\iota_\Sigma(h)} \to M$	1
3	B_B epic basic model for E_B	Prop. 5.25
4	$M' \models E_B$	2, 3
5	$(\iota_\Sigma h)((\iota_\Sigma h)^{-1} E_B^{**}) \subseteq E_B^{**}$	
6	$M \models (\iota_\Sigma h)^{-1} E_B^{**}$	$M = M'\restriction_{\iota_\Sigma h}$, 4, 5, Satisfaction Condition.

□

Exercises

5.26. Unions of finitary basic sets of sentences
Finite co-products of finitely presented objects are still finitely presented. If finite co-products of models exist then the union of finitary basic sets of sentences is still finitary basic.

5.27. Finitary basic sentences are closed under finitary quasi-representable existential quantifications. (*Hint:* The model reducts corresponding to the finitary quasi-representable signature morphisms preserve the finitely presented models.)

5.28. In any institution, any liberal signature morphism $\varphi : \Sigma \to \Sigma'$ preserve the (epi) basic sentences, i.e. if E is a set of (epi) basic sentences then φE is (epi) basic too.

5.29. Representable theory morphisms
Let $\varphi : \Sigma \to \Sigma'$ be a quasi-representable signature morphism in an institution $\mathcal{I}$. If E' is epi basic then each theory morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is representable (as a signature morphism of $\mathcal{I}^{\mathrm{th}}$).

5.30. Preservation of Horn sentences
A sentence ρ is *preserved by a limit* $(M_i \xrightarrow{\mu_i} M)_{i \in |J|}$ of a diagram of models $(M_i \xrightarrow{f_u} M_j)_{u \in J}$ of models when $M_i \models \rho$ for each $i \in |J|$ implies $M \models \rho$.
In any institution:

1. Small products of models preserve Horn sentences.
2. Small limits of models preserve all Horn sentences $(\forall \chi)E \Rightarrow E'$ for which E' is *epi* basic.
3. Directed co-limits of models preserve the finitary Horn sentences.

5.31. Basic sentences modulo theories
Let $\mathcal{I}$ be a liberal institution.

1. Each set of sentences which is (epi) basic in $\mathcal{I}$ is (epi) basic in the institution $\mathcal{I}^{\mathrm{th}}$ too.
2. If each sentence of $\mathcal{I}$ is preserved by directed co-limits, then any finitary basic sets of sentences in $\mathcal{I}$ is finitary basic in $\mathcal{I}^{\mathrm{th}}$ too.

5.32. Borrowing basic sentences along comorphisms
Any persistently liberal institution comorphism (Φ,α,β) 'borrows' the (finitary) epi basic sentences, i.e., E is (finitary) epi basic when αE is so. This can be applied in conjunction with the results of Ex. 5.31 for the comorphisms of Ex. 4.88 for showing that finite sets of existence equations in $\mathcal{PA}$ or of any atoms of several other institutions (such as $\mathcal{POA}$, $\mathcal{AUT}$, $\mathcal{MBA}$) are finitary epi basic.

5.33. [47] $\mathcal{HNK}$ has atoms that are not basic. In the signature (S,F) defined by $S=\{s,s'\}$ and $F_{s\to s}=\{f\}$, $F_{(s\to s)\to s'}=\{\sigma_1,\sigma_2\}$, for other types x, F_x being empty, the atom $\sigma_1 f=\sigma_2 f$ is *not* basic. (*Hint:* Consider the model M defined by M_s empty, and $M_{s'}$, $M_{s\to s}$ and $M_{(s\to s)\to s'}$ containing only one element. Then $M\models\sigma_1 f=\sigma_2 f$. For any other model N satisfying $\sigma_1 f=\sigma_2 f$ but such that $N_{\sigma_1}\neq N_{\sigma_2}$ there exists no model homomorphism $M\to N$. Deduce from here that $\sigma_1 f=\sigma_2 f$ cannot be basic.)

5.34. In $\mathcal{MVL}^\sharp$ all sets of sentences of the form $(\pi(c_1,\dots,c_n),\kappa)$ where π is relation symbol, $c_1,\dots,c_n$ are symbols of constants, and κ is an element of the residuated lattice, are epi basic.

5.35. An institution of injectivity
The following defines an institution $\mathcal{INJ}$:

1. $Sig^{\mathcal{INJ}}$ is the category of the adjunctions, i.e., the signatures are categories and the signature morphisms are adjunctions $(\mathcal{U},\mathcal{F},\eta,\varepsilon)\colon\ \mathbb{A}\to\mathbb{B}$ where $\mathcal{U}\colon\ \mathbb{B}\to\mathbb{A}$ is the right adjoint and $\mathcal{F}$ is the left adjoint.
2. $Sen^{\mathcal{INJ}}(\mathbb{A})=arr(\mathbb{A})$ (the class of all arrows of $\mathbb{A}$), and $Sen^{\mathcal{INJ}}(\mathcal{U},\mathcal{F},\eta,\varepsilon)=\mathcal{F}$ for adjunctions,
3. $Mod^{\mathcal{INJ}}(\mathbb{A})=\mathbb{A}$ for each category $\mathbb{A}$ and $Mod^{\mathcal{INJ}}(\mathcal{U},\mathcal{F},\eta,\varepsilon)=\mathcal{U}$ for adjunctions, and
4. $A\models^{\mathcal{INJ}} f$ if and only if $A\models^{\mathrm{inj}} f$.

There exists an institution comorphism from the sub-institution of $Cat\mathcal{EQL}$ for which the categories have co-equalizers to $\mathcal{INJ}$ mapping each categorical equation $(\forall B)l=r$ to a designated co-equalizer.

5.6 Elementary homomorphisms revisited

In Sect. 4.4 we have introduced the following concept of elementary homomorphism: in any institution with diagrams ι, a Σ-model homomorphism $h\colon\ M\to N$ is ι-elementary when $N_h\models(M_M)^*$ where N_h is the canonical expansion of N to a Σ_M-model, determined by h , i.e., $N_h=i^{-1}_{\Sigma,M}N$. This diagram-based definition of elementary homomorphism does not support some desirable structural properties of elementary homomorphisms, such as closure under composition. In this section, we provide an alternative concept of elementary homomorphism which we show to coincide with the diagram-based one under some general 'normality' condition on the system of diagrams. The most important gain of the convergence between these two different perspectives on elementary homomorphisms are good structural properties which can be summed up by the fact that the elementary homomorphisms of the institution determine themselves an institution with diagrams.

$\mathcal{D}$-elementary homomorphisms. Given a class $\mathcal{D} \subseteq Sig$ of signature morphisms, a Σ-model homomorphism $h : M \to N$ is *$\mathcal{D}$-elementary* when $M'^* \subseteq N'^*$ for each $\mathcal{D}$-expansion $h' : M' \to N'$ of h.

In the actual institutions, $\mathcal{D}$ often consists of the class of all signature extensions with constants, or of the injective signature extensions with constants. Notice that in the case of $\mathcal{FOL}$, and in fact in all institutions with finitary sentences, elementariness for signature extensions with an arbitrary number of constants is equivalent to elementariness for extensions adding *finite* numbers of constants.

The following applies to the cases when $\mathcal{D}$ contains all (injective) signature extensions with constants.

Fact 5.30. *In any institution with diagrams ι such that $\mathcal{D}$ contains all the elementary extensions, any $\mathcal{D}$-elementary homomorphism is ι-elementary.*

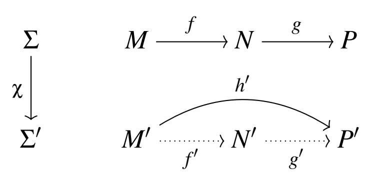
Structural properties of elementary homomorphisms. We now give some general conditions on $\mathcal{D}$ which ensure that the $\mathcal{D}$-elementary homomorphisms determine a subinstitution of the original institution.

Proposition 5.31. *Let $\mathcal{D}$ be a class of signature morphisms.*

1. *If each morphism in $\mathcal{D}$ is quasi-representable, then the $\mathcal{D}$-elementary homomorphisms are closed under composition.*
2. *If the institution is weakly semi-exact and $\mathcal{D}$ is stable under pushouts, then the $\mathcal{D}$-elementary homomorphisms are preserved by any model reduct functor.*
3. *If $\mathcal{D}$ is closed under compositions then the $\mathcal{D}$-elementary homomorphisms are closed under $\mathcal{D}$-expansions.*

Proof. 1. Let $f : M \to N$ and $g : N \to P$ be $\mathcal{D}$-elementary homomorphisms and for any $\chi \in \mathcal{D}$ let $h' : M' \to P'$ be a χ-expansion of $f;g$. Then because χ is quasi-representable

- f and M' determine a unique χ-expansion $f' : M' \to N'$ of f, and
- g and N' determine a unique χ-expansion $g' : N' \to P''$ of g.



Therefore $f';g'$ is the unique χ-expansion of $f;g$, hence $P'' = P'$ and $f' \; ; \; g' = h'$. Since f and g are $\mathcal{D}$-elementary it follows that $M'^* \subseteq N'^*$ and that $N'^* \subseteq P'^*$ hence $M'^* \subseteq P'^*$. This shows that $f;g$ is $\mathcal{D}$-elementary.

2. Let $h_1 : M_1 \to N_1$ be a $\mathcal{D}$-elementary Σ_1-model homomorphism and $\varphi : \Sigma \to \Sigma_1$ be any signature morphism. In order to prove that $h_1\!\upharpoonright_\varphi$ is $\mathcal{D}$-elementary, we consider

$(\chi : \Sigma \to \Sigma') \in \mathcal{D}$. Because $\mathcal{D}$ is stable under pushouts, in the pushout square below we also have that $\chi_1 \in \mathcal{D}$.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\chi} & \Sigma' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ \Sigma_1 & \xrightarrow[\chi_1]{} & \Sigma'_1 \end{array}$$

Let $h' : M' \to N'$ be any χ-expansion of $h_1\restriction_{\varphi}$. Then

1	there exists $h'_1 : M'_1 \to N'_1,\ h'_1\restriction_{\varphi'} = h',\ h'_1\restriction_{\chi_1} = h_1$	weak semi-exactness hyp.
2	$(M'_1)^* \subseteq (N'_1)^*$	1, h_1 $\mathcal{D}$-elementary
3	$M'^* \subseteq N'^*$	2, Satisfaction Condition for φ'.

3. Let $\chi : \Sigma \to \Sigma'$ belong to $\mathcal{D}$, $h : M \to N$ be $\mathcal{D}$-elementary and $h' : M' \to N'$ be a χ-expansion of h. We prove that h' is $\mathcal{D}$-elementary. Let $\chi' : \Sigma' \to \Sigma''$ belong to $\mathcal{D}$ and let $h'' : M'' \to N''$ be a χ'-expansion of h'.

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\chi} & \Sigma' & \xrightarrow{\chi'} & \Sigma'' \\ & & & & \\ M & & M' & & M'' \\ {\scriptstyle h}\downarrow & & {\scriptstyle h'}\downarrow & & {\scriptstyle h''}\downarrow \\ N & & N' & & N'' \end{array}$$

Then $M''^* \subseteq N''^*$ by the $\mathcal{D}$-elementariness of h for $\chi\,;\chi'$ (which belongs to $\mathcal{D}$ because $\mathcal{D}$ is closed under compositions).

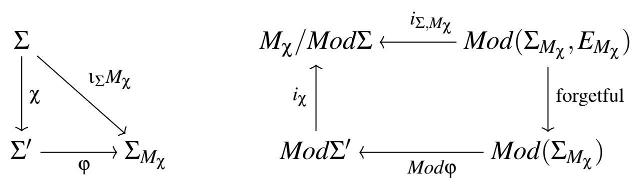
□

Corollary 5.32. *Under the first two conditions of Prop. 5.31, the $\mathcal{D}$-elementary homomorphisms determine a sub-institution of the original institution.*

Normal diagrams

We have already seen that $\mathcal{D}$-elementary homomorphisms are ι-elementary under the natural assumption that the elementary extensions belong to $\mathcal{D}$. This was rather easy. To establish the equivalence between the two concepts of elementary homomorphism we need a stronger connection between $\mathcal{D}$ and ι as follows. For any class $\mathcal{D}$ of *representable* signature morphisms, the diagrams ι of an institution are *$\mathcal{D}$-normal* if for each $(\chi : \Sigma \to$

$\Sigma') \in \mathcal{D}$, represented by M_χ, there exists a signature morphism $\varphi : \Sigma' \to \Sigma_{M_\chi}$ such that the diagrams below commute:



(5.5)

We can say that the $\mathcal{D}$-normal signature morphisms are representable in such a way that they can be 'realised' by the diagram of their representation. Each homomorphism $M_\chi : N$ corresponds to a Σ'-model on the one hand, and to a model of the diagram of M_χ on the other hand. These two representations are isomorphically related through the canonical isomorphisms from the definitions of representability and of the diagrams and this isomorphism can also be realised based on the model reduct corresponding to φ. The following concrete example shows how this works in $\mathcal{FOL}$. It illustrates a rather typical actual situation in the sense that its main idea can be replicated in many other concrete contexts.

Proposition 5.33. *$\mathcal{FOL}$ has $\mathcal{D}$-normal diagrams for $\mathcal{D}$ consisting of all (injective) signature extensions with constants.*

Proof. Without any loss of generality, we treat only the case of the strict signature extensions with constants. Let $\chi : \Sigma \to \Sigma'$ extend the $\mathcal{FOL}$ signature Σ with a set of constant symbols X. Then M_χ is the free (term) model $T_\Sigma X$, while φ is defined on each $x \in X$ by x (but regarded as a term in $T_\Sigma X$). The commutativity of the left-hand side triangle of (5.5) is trivial.

To see that the corresponding condition on the model categories holds, let N be any $(\Sigma_{M_\chi}, E_{M_\chi})$-model. Let $h_N = i_{\Sigma,M_\chi} N$ and $g = i_\chi(N\upharpoonright_\varphi)$. The commutativity of the right-hand side square of (5.5) is shown if we proved that $h_N = g$:

1	for each $t \in M_\chi$, $h_N t = N_t$	definition of h_N

For each $x \in X$:

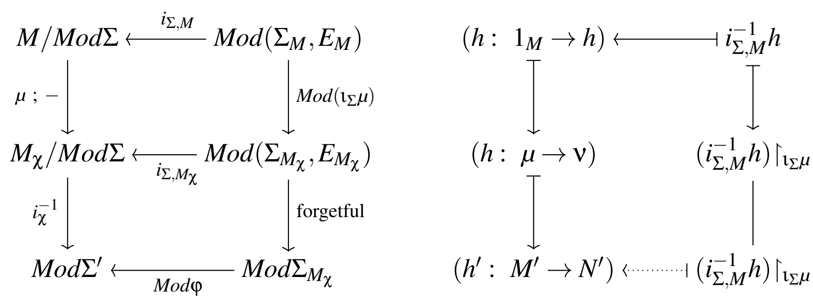
2	$gx = (N\upharpoonright_\varphi)_x$	definition of i_χ
3	$\varphi x = x$	definition of φ
4	$(N\upharpoonright_\varphi)_x = N_{\varphi x}$	definition of φ-reducts
5	$gx = N_x$	2, 3, 4
6	$h_N x = gx$	1, 5.

From 6, since M_χ is free over X, it follows that $h_N = g$. □

The following result shows that the general concept of $\mathcal{D}$-normal diagram serves its purpose.

Proposition 5.34. *In any institution with $\mathcal{D}$-normal diagrams ι, any ι-elementary homomorphism is $\mathcal{D}$-elementary.*

Proof. Consider $h : M \to N$ a ι-elementary Σ-homomorphism and let $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ and $h' : M' \to N'$ be a χ-expansion of h. We have to show that $M'^* \subseteq N'^*$. Because the diagrams are $\mathcal{D}$-normal, χ is representable (represented by M_χ) and there exists $\varphi : \Sigma' \to \Sigma_{M_\chi}$ such that the diagrams of (5.5) commute. Let $\mu = i_\chi M : M_\chi \to M$ and $\nu = i_\chi N : M_\chi \to N$. We consider the diagrams below:



The left-hand side diagram above is a categorical diagram that commutes because both its constituent squares commute, the upper square because of the naturality of i and the lower square by (5.5). Then we consider h as an arrow $1_M \to h$ in $M/Mod\Sigma$. The right-hand side diagram above shows how $i^{-1}_{\Sigma,M}h$ (as a (Σ_M, E_M)-model homomorphism) gets mapped through the functors of the left-hand side diagram. On the right hand side the vertical and the upper horizontal mappings hold by the respective definitions, while the lower horizontal mapping is proved by the commutativity of the left-hand side categorical diagram. Hence

1	$h' = (i^{-1}_{\Sigma,M}h)\upharpoonright_{\iota_\Sigma \mu}$	
2	$i^{-1}_{\Sigma,M}h : M_M \to N_h$	$h : M \to N$
3	$(M_M)^* \subseteq (N_h)^*$	h ι-elementary
4	$M' = M_M\upharpoonright_{\varphi;\iota_\Sigma \mu}$, $N' = N_h\upharpoonright_{\varphi;\iota_\Sigma \mu}$	1, 2
5	$M'^* \subseteq N'^*$	3, 4, Satisfaction Condition.

□

Corollary 5.35. *In any institution with $\mathcal{D}$-normal diagrams ι such that $\mathcal{D}$ contains all elementary extensions, a homomorphism is $\mathcal{D}$-elementary if and only if it is ι-elementary.*

The following sums up the developments of this section.

Corollary 5.36. *Let I be an institution with diagrams ι and a class $\mathcal{D}$ of signature morphisms such that*

- *I is semi-exact,*
- *all elementary extensions of ι belong to $\mathcal{D}$,*

- *$\mathcal{D}$ is stable under pushouts,*
- *all signature morphisms in $\mathcal{D}$ are quasi-representable.*

Then the elementary homomorphisms form a sub-institution $E(I)$ of I called the elementary sub-institution of I. Moreover $E(I)$ has diagrams such that its elementary extensions are those of ι and for each Σ-model M its diagram in $E(I)$ is $(M_M)^*$.

Corollary 5.36 can be instantiated to $\mathcal{FOL}$ as follows:

Corollary 5.37. *In $\mathcal{FOL}$ the elementary embeddings form an institution.*

Exercises

5.36. Taking the elementary sub-institution is an idempotent operation on institutions, i.e., $E(E(I)) = E(I)$.

Notes. The institution-independent semantics of Boolean connectives is rather folklore of institution theory, perhaps this was introduced first time in [226]. The institution-independent semantics of quantifiers had been introduced first time by [228] and has been used intensively in [63]. Although quantification by signature extensions is well-known in conventional mathematical logic [224, 150] it is quite rare in the usual presentations of conventional logic or model theory. Quantification systems have been introduced in [72] under a slightly different terminology.

The institution-independent concept of substitution has been introduced in [65]. A rather different approach has been developed within the framework of the 'context institutions' of [201].

The institution-independent approach to first order quantifiers via representable signature morphisms have been developed in [63] which also introduced the concept of finitary representable signature morphisms as an abstract categorical treatment for finitary first order quantification. Quasi-representable signature morphisms have been introduced in [139]. In the jargon of more advanced category theory that χ is quasi-representable is equivalent to saying that $Mod\chi$ is a *discrete opfibration*. Finitary quasi-representable signature morphisms are introduced here. Prop. 5.13 has been proved in [49].

Satisfaction by injectivity is a well-known concept in categorical universal algebra and it has been intensively used in the general study of Birkhoff axiomatizability in arbitrary categories [9]. According to [197] injectivity was first used to represent satisfaction in [15]. In [9] the injectivity is extended to arbitrary cones which cover the satisfaction of all first-order formulæ, however, this leads to enormous conceptual and proof complexity without going beyond the boundaries of first-order satisfaction. The same satisfaction power, and even much more, can be achieved only by basic sentences, internal quantification and logical connectives, but in a much simpler framework. This is due to the advantage of using the multi-signature framework based on institutions as opposed to the other more rigid single-signature categorical abstract model-theoretic frameworks.

The institution-independent concept of elementary homomorphism, due to [139], unifies various concepts of model embeddings from the literature, such as elementary embeddings from conventional model theory [42] for $\mathcal{FOL}$, elementary embeddings of partial algebras [36], $L_{\infty,\omega}$- and $L_{\alpha,\omega}$-elementary embeddings from infinitary model theory [155, 168, 149], the existentially closed embeddings of [149] for $(\Pi\cup\Sigma)^0_1$, the Σ^0_n-extensions [42] for $(\Pi\cup\Sigma)^0_n$. $\mathcal{D}$-elementary homomorphisms have been introduced by [139], which also proved their equivalence to (ordinary) elementary homomorphisms under the normality condition on the diagrams.

Part II

Advanced topics

Chapter 6

Model Ultraproducts

The method of ultraproducts represents a true powerhouse of model theory as it has significant applications in almost all branches of mathematics. For instance, it is the best method to establish semantic compactness, compactness being one of the central themes in logic. Some voices even consider the study of semantic compactness as the main purpose of model theory. In other mathematical areas – such as algebra, especially fields theory, algebraic geometry, combinatorics, etc. – through the method of ultraproducts a multitude of new more elegant and simpler proofs of known results have been developed. Moreover, in some cases, new important results have been obtained. Perhaps the most famous application of ultraproducts is the theory of 'hyperreals' and its associated 'non-standard' analysis of Abraham Robinson. This represents an elegant and fully rigorous recovery of the original approach to analysis by Newton, Leibniz, and Euler based on the calculus of infinitesimals. Whilst the 'standard' analysis of Cauchy, Weistrass, etc., is based on the ε-δ calculus, which uses inequalities (for handling limits), the 'non-standard' analysis rather uses equalities (and no need for limits). In the face of the conceptual and aesthetic superiority of the latter, one may be left wondering why 'non-standard' analysis is not standard and vice versa. The answer to this is deceptively simple: because of intellectual and educational inertia. Deep-rooted habits are difficult to uproot especially when the (mathematical) education system worldwide does not encourage unconventionality.

The main purpose of this chapter is to develop the fundamentals of the method of ultraproducts in an institution-independent manner, which liberates it from its traditional classical first-order logic context, thus making it easily available to a multitude of non-conventional logic contexts. At the core of this enterprise lies the abstract category-theoretic definition of ultraproducts.

Most applications of the method of ultraproducts rely on a famous theorem of the Polish mathematician, logician, economist and philosopher Jerzy Łoś. Here we develop a general institution-independent version of this crucial result. For this, we make explicit at the level of abstract institutions a series of properties whose role is implicit in traditional first-order logic.

The last parts of this chapter are devoted to several applications of the institution-independent version of Łoś Theorem. The most notable such application is a general institution-independent semantic compactness result which can be applied easily to a multitude of concrete logical systems.

The developments in this chapter require mostly familiarity with material from the first four sections of Chap. 5. Only Sec. 6.5 involves material from the latter two sections of Chap. 5.

6.1 Filtered products

Ultraproducts of models are a special kind of 'filtered' products of models. Results in model theory may sometimes refer to various classes of filtered products that are more general than ultraproducts, but in terms of applications, especially when negation is involved, the ultraproducts are chief. In this section, we first illustrate the filtered products construction for the particular example of the $\mathcal{FOL}$ models. Then we introduce the general concept of filtered products in arbitrary categories.

Filters. For each non-empty set I, a *filter F over I* is defined to be a set $F \subseteq \mathcal{P}I$ (the set of all subsets of I) such that

- $I \in F$,
- $X \cap Y \in F$ if $X \in F$ and $Y \in F$, and
- $Y \in F$ if $X \subseteq Y$ and $X \in F$.

A filter F is *proper* when F is not $\mathcal{P}I$, is *principal* if there exists $Z \subseteq I$ such that $F = \{X \subseteq I \mid Z \subseteq X\}$, and it is an *ultrafilter* when $X \in F$ if and only if $(I \setminus X) \notin F$ for each $X \in \mathcal{P}I$. Notice that ultrafilters are proper filters. We will always assume that all our filters are proper.

Filtered products in $\mathcal{FOL}$

Given a $\mathcal{FOL}$ signature Σ, let $(M_i)_{i\in I}$ be a family of Σ-models and let F be a filter over I. Let M be the following Σ-model.

For each sort s of Σ, for each element $m \in M_s$ let $m = (m_i)_{i\in I}$ with $m_i \in (M_i)_s$ for each $i \in I$. We let also

- For each sort s of Σ let M_s be the cartesian product of the sets $(M_i)_s$.
- For each σ operation symbol and $(m^1,\dots,m^k)$ list of appropriate arguments for M_σ, $M_\sigma(m^1,\dots,m^k) = (M_\sigma(m_i^1,\dots,m_i^k))_{i\in I}$.
- For each π relation symbol in Σ and $(m^1,\dots,m^k)$ list of appropriate arguments for M_π, $(m^1,\dots,m^k) \in M_\pi$ if and only if $(m_i^1,\dots,m_i^k) \in (M_i)_\pi$ for each $i \in I$.

Then we can establish easily that M together with the projections $p_i : M \to M_i$ (where $p_i m = m_i$) constitute a categorical product of the family $(M_i)_{i\in I}$.

By defining the equivalence $\sim_F$ on M by

$$m \sim_F m' \text{ if and only if } \{i \mid m_i = m'_i\} \in F$$

(which is correctly defined because F is a filter), we construct an *F-product M_F of $(M_i)_{i\in I}$* by

- $(M_F)_s = M_s/_{\sim_F}$ for each sort s of Σ,
- $(M_F)_\sigma(m/_{\sim_F}) = M_\sigma(m)/_{\sim_F}$ for each operation σ of Σ and each list m of arguments for M_σ, and
- $(M_F)_\pi = \{m/_{\sim_F} \mid \{i \in I \mid m_i \in (M_i)_\pi\} \in F\}$ for each relation π of Σ and each list m of arguments for M_π.

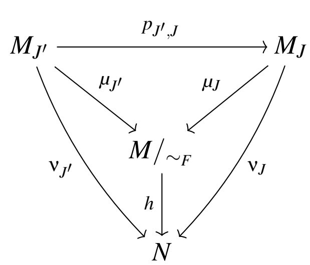
Routine calculations based on the filter property of F give the correctness of the definition of M_F.

Hyperreals. An emblematic example of the F-product construction is that of the ordered field of the hyperreals, ${}^*\mathbb{R}$. We take Σ to be the signature of the theory of ordered fields, $I = \omega$ (the set of the natural numbers), each $M_i = \mathbb{R}$ (the ordered field of the real numbers), and F *any* non-principal ultrafilter on ω. Then $M_F = {}^*\mathbb{R}$. It has been shown that the choice of F is immaterial if we assumed the *Continuum Hypothesis*, so, in this case, the system of the hyperreal numbers is unique (up to isomorphism).

The categorical representation of $\mathcal{FOL}$ F-products. For each $J \in F$, let M_J be the cartesian product of the models $(M_i)_{i\in J}$ and let $\mu_J : M_J \to M_F$ be the model homomorphism such that $\mu_J m = m'/_{\sim_F}$ for each m' such that $m = (m'_i)_{i\in J}$. Because F is a filter, μ_J is well defined and is a model homomorphism. The reader is invited to check this.

Proposition 6.1. *$(\mu_J)_{J\in F}$ is a co-limit of the directed diagrams of (canonical) projections $\{p_{J\supseteq J'} : M_J \to M_{J'} \mid J' \subseteq J \in F\}$.*

Proof. That μ is a co-cone for the respective diagram is obvious from its definition. Let $(\nu_J : M_J \to N)_{J\in F}$ be a co-cone over the same diagram.



There exists a unique many-sorted function $h : M_F \to N$ such that $h_s\ m/_{\sim_F} = \nu_I m$ for each $m \in M_s$ and for each sort s. Notice that the definition of h is correct because for each $m \sim_F m'$,

$$\nu_I m = (p_{I\supseteq J};\nu_J)m = (p_{I\supseteq J};\nu_J)m' = \nu_I m'$$

where $J = \{i \mid m_i = m'_i\}$.

We prove that h is a model homomorphism $M_F \to N$.

- For each operation σ and each list of arguments m for M_σ, we successively have

$$\begin{aligned}
h((M_F)_\sigma\, m/_{\sim_F}) &= h((M_F)_\sigma(\mu_I m)) && \text{definition of } \mu_I \\
&= h(\mu_I(M_\sigma m)) && \mu_I \text{ homomorphism} \\
&= \nu_I(M_\sigma m) && \nu = \mu\,;\,h \\
&= N_\sigma(\nu_I m) && \nu_I \text{ homomorphism} \\
&= N_\sigma(h(\mu_I m)) && \nu = \mu\,;\,h \\
&= N_\sigma(h\, m/_{\sim_F}) && \text{definition of } \mu_I.
\end{aligned}$$

  therefore h commutes with the interpretations of the operations.

- For each relation symbol π with arity w assume that $m/_{\sim_F} \in (M_F)_\pi$ for some $m \in M_w$. Then $J = \{i \in I \mid m_i \in (M_i)_\pi\} \in F$. We have that

$$\begin{aligned}
h\, m/_{\sim_F} &= (\mu_I;h)m && \text{definition of } \mu_I \\
&= (\mu_J;h)(p_{I\supseteq J} m) && \mu_I = p_{I\supseteq J};\mu_J \\
&= \nu_J(p_{I\supseteq J} m) \in N_\pi && \nu = \mu;h,\;\; p_{I\supseteq J} m = (m_i)_{i\in J} \in (M_J)_\pi,\;\; \nu_J \text{ homomorphism.}
\end{aligned}$$

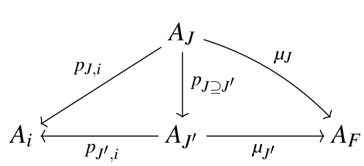
  Therefore h preserves the interpretations of the relation symbols too.

□

Prop. 6.1 has the merit that it gives a purely categorical description of F-products of $\mathcal{FOL}$ models. This indicates that F-products can be defined at the level of abstract categories.

Categorical F-products

Consider a filter F over the set of indices I and a family of objects $(A_i)_{i\in I}$ in any category $\mathbb{C}$ with small products. Then an *F-filtered product of* $(A_i)_{i\in I}$ (or *F-product*, for short) is a co-limit $\{\mu_J : A_J \to A_F \mid J \in F\}$ of the directed diagram of canonical projections $\{p_{J\supseteq J'} : A_J \to A_{J'} \mid J' \subseteq J \in F\}$, where for each $J \in F$, $\{p_{J,i} : A_J \to A_i \mid i \in J\}$ is a direct product of $(A_j)_{j\in J}$.



Occasionally, by abuse of terminology, instead of the whole co-cone we will just refer to the vertex A_F as the F-product. Obviously, as co-limits of diagrams of products, F-products are unique up to isomorphisms.

Note that the co-limits defining F-products are directed. Therefore a sufficient condition for the existence of F-products, which applies to many institutions, is the existence of small products and of directed co-limits of models. Note however that strictly speaking the latter condition is not a necessary condition because only co-limits over diagrams of projections are involved. We will see examples when directed co-limits of models do not exist in general but some F-products do exist.

If F is an ultrafilter then F-products are called *ultraproducts*. When $A_i = A$ for all $i \in I$, then a F-product is called *F-power*. F-powers corresponding to ultrafilters are called *ultrapowers*. The model of the hyperreals $^*\mathbb{R}$ is such an ultrapower. Note that a direct product of $(A_i)_{i\in I}$ is the same as the F-product $A_{\{I\}}$, i.e. where $F = \{I\}$.

Filter reductions. Let F be a filter over I and $I' \subseteq I$. The *reduction of F to I'* is denoted by $F|_{I'}$ and defined as $\{I' \cap X \mid X \in F\}$.

Fact 6.2. *The reduction of any filter is still a filter.*

A class $\mathcal{F}$ of filters is *closed under reductions* if and only if $F|_J \in \mathcal{F}$ for each $F \in \mathcal{F}$ and $J \in F$. Examples of classes of filters closed under reductions include the class of all filters, the class of all ultrafilters, the class of the singleton filters $\{\{I\} \mid I \text{ set}\}$, etc. The following is a useful property of filter reductions which will be used in several situations.

Proposition 6.3. *Let F be a filter over I and $(A_i)_{i\in I}$ a family of objects in a category $\mathbb{C}$. For each $J \in F$, the F-products $A_{F|_J}$ and A_F are isomorphic.*

Proof. Note that the inclusion of posets $(F|_J, \supseteq) \subseteq (F, \supseteq)$ is a final functor since for each $J' \in F$ we have that $J' \cap J \in F|_J$. Then the conclusion follows directly from Thm. 2.4. □

Exercises

6.1. Filtered products in $\mathcal{PL}$
In $\mathcal{PL}$, for any filter F over a set I, and for each family $(M_i)_{i\in I}$ of models, its F-product is $\bigcup_{J\in F} \bigcap_{i\in J} M_i$.

6.2. Let Σ be the $\mathcal{FOL}$ signature having only one sort and only one binary relation symbol R. Let $(M_i)_{i\in I}$ be a family of models and F be a filter over I. Prove that $(M_F)_R$ is reflexive, symmetric, or transitive when $(M_i)_R$ is reflexive, symmetric, respectively transitive for each $i \in I$.

6.3. The class of all ultrafilters is closed under reductions.

6.4. Borrowing F-products along institution comorphisms
For any persistently liberal institution comorphism $I \to I'$ the institution I has the limits and the co-limits of models that I' has. This leads to the existence of F-products of models in several institutions (such as $\mathcal{POA}$, $\mathcal{PA}$, $\mathcal{AUT}$, $\mathcal{MBA}$, $\mathcal{LA}$, etc) via the examples of Ex. 4.88 and to the existence of direct products of models in $\mathcal{HNK}$ via the comorphism of Ex. 4.12.

6.5. [47] In general, $\mathcal{HNK}$ does not have directed co-limits of models. However, it has co-limits of directed 'injective' diagrams, i.e., diagrams consisting of injective model homomorphisms.

6.6. $\mathcal{IPL}$ has direct products of models. Moreover, if we let $\mathcal{IPL}'$ be the sub-institution of $\mathcal{IPL}$ that is determined by considering only complete Heyting algebras then $\mathcal{IPL}'$ has both direct products and directed co-limits.

6.7. The categories $Mod^{\mathcal{WPL}}P$ (of $\mathcal{WPL}$-models and model homomorphisms; see Ex. 4.48) have directed co-limits but in general lack direct products.

6.8. Both $\mathcal{MFOL}^*$ and $\mathcal{MFOL}^\sharp$ have F-products of models.

6.9. The categories of multialgebras ($\mathcal{MA}$) do have direct products and directed co-limits.

6.10. [135] The categories of contraction algebras ($\mathcal{CA}$) do have direct products and directed co-limits.

6.2 Fundamental theorem

In this section, we develop an institution-independent version of Łoś Theorem. First, we define some properties of signature morphisms regarding the interaction between model reducts and F-products; these are required by the parts of the Łoś Theorem that deal with quantified sentences.

Preservation / lifting / creation of filtered products. Consider a functor $G\colon\ \mathbb{C}' \to \mathbb{C}$ and F a filter over a set I. Then G *preserves / lifts / creates F-products* when

- it preserves / lifts / creates direct products, and
- it preserves / lifts / creates the directed co-limits defining the F-products.

The following fact gives the expected hierarchy of these three concepts.

Fact 6.4. *If G creates F-products then it also lifts them. If G lifts F-products and $\mathbb{C}$ has F-products then $\mathbb{C}'$ has F-products which are preserved by G.*

The following fact applies to many institutions when the role of G is played by model reduct functors.

Fact 6.5. *A functor G preserves / creates F-products if it preserves / creates direct products and directed co-limits.*

Inventing filtered products. Let $\mathcal{F}$ be a class of filters closed under reductions. A functor $G\colon\ \mathbb{C}' \to \mathbb{C}$ *invents $\mathcal{F}$-products* when for each $F \in \mathcal{F}$, for each F-product $\{\mu_J\colon\ M_J \to M_F \mid J \in F\}$ of a family $(M_i)_{i\in I}$ in $|\mathbb{C}|$, and for each $N \in |\mathbb{C}'|$ such that $GN = M_F$,

– there exists $J \in F$ and $(M'_i)_{i\in J}$ a family in $|\mathbb{C}'|$ such that $G(M'_i) = M_i$ for each $i \in J$ and such that

– there exists an $F|_J$-product $\{\mu'_{J'} : M'_{J'} \to M'_{F|_J} \mid J' \in F|_J\}$ of $(M'_i)_{i\in J}$ such that $M'_{F|_J} \cong N$.

When $G\mu'_{J'} = \mu_{J'}$ for each $J' \in F|_J$ we say that G *invents strongly* the respective F-product. When $J = I$ we say that G *invents completely* . (Note that in this case, the closure of $\mathcal{F}$ under reductions is redundant.)

In essence, this inventing property means that each $\mathcal{F}$-product construction of GN can be established as the image by G of an $\mathcal{F}$-product construction of B using a filter reduction. This is significantly more demanding than the lifting or preservation of $\mathcal{F}$-products and therefore in the actual institutions inventing of F-products holds for a narrower class of signature morphisms than preservation. However, as will see below the first-order quantifications are covered well by both properties.

The Fundamental Theorem

Sentences preserved by F-factors / products. Let $\mathcal{F}$ be a family of filters. The following notions of preservation by $\mathcal{F}$-factors and by $\mathcal{F}$-products are dual to each other. For a signature Σ in an institution, for each $F \in \mathcal{F}$, and each F-product $\{\mu_J : M_J \to M_F \mid J \in F\}$ of any family $(M_i)_{i\in I}$ of Σ-models, a Σ-sentence e is

- *preserved by $\mathcal{F}$-factors* if $M_F \models_\Sigma e$ implies $\{i \in I \mid M_i \models_\Sigma e\} \in F$, and
- *preserved by $\mathcal{F}$-products* if $\{i \in I \mid M_i \models_\Sigma e\} \in F$ implies $M_F \models_\Sigma e$.

When $\mathcal{F}$ is the class of all ultrafilters, preservation by $\mathcal{F}$-factors, respectively products, are called *preservation by ultrafactors*, respectively *ultraproducts*.

Theorem 6.6 (Fundamental ultraproducts theorem)**.** *Consider any institution that has appropriate filtered products of models. For any filter F:*

1. *The basic sentences are preserved by all F-products.*
2. *The finitary basic sentences are preserved by all F-products and all F-factors.*
3. *The sentences preserved by F-factors and the sentences preserved by $\mathcal{F}$-products are both closed under conjunction.*
4. *The sentences preserved by F-products are closed under infinite conjunctions.*
5. *If F is a proper filter, then for any sentence that is preserved by F-factors its negation is preserved by F-products.*

If F is an ultrafilter then:

6. *If a sentence is preserved by F-products then any of its negations is preserved by F-factors.*
7. *The sentences preserved by both F-products and F-factors are closed under negation.*

For any class $\mathcal{F}$ of filters closed under reductions (in particular the class of all ultrafilters):

8. *The sentences preserved by $\mathcal{F}$-products are closed under existential χ-quantification, when Modχ preserves $\mathcal{F}$-products.*
9. *The sentences preserved by $\mathcal{F}$-factors are closed under existential χ-quantification, when Modχ invents $\mathcal{F}$-products.*

Proof. In this proof we let $(M_i)_{i\in I}$ denote an arbitrary family of Σ-models for a signature Σ and $\{\mu_J : M_J \to M_F \mid J \in F\}$ denote an F-product of this family.

1. Let e be a basic sentence. Let M_e be a basic model for e. Consider $J = \{i \in I \mid M_i \models_\Sigma e\}$. There exists a model homomorphism $M_e \to M_i$ for each $i \in J$, therefore by the universal property of the products, there exists a model homomorphism $M_e \to M_J$. By composing this with $\mu_J : M_J \to M_F$, we get a model homomorphism $M_e \to M_F$, which implies that $M_F \models e$.
2. Consider a finitary basic Σ-sentence e. By 1. we have to prove only that e is preserved by F-factors. If $M_F \models e$, then there exists a model homomorphism $M_e \to M_F$. Since e is finitary basic we may consider that M_e is finitely presented. Hence there exists a model homomorphism $M_e \to M_J$ for some non-empty $J \in F$, which, by the product projections, means that $M_i \models e$ for all $i \in J$. Therefore $\{i \in I \mid M_i \models_\Sigma e\} \in F$ because $J \subseteq \{i \in I \mid M_i \models_\Sigma e\}$.
3. Let e be the conjunction of Σ-sentences e' and e''. Let $J = \{i \in I \mid M_i \models e\}$, $J' = \{i \in I \mid M_i \models e'\}$ and $J'' = \{i \in I \mid M_i \models e''\}$. Then

1	$J = J' \cap J''$	e conjunction of e' and e''
2	$J \in F$ if and only if $J', J'' \in F$	1, F filter.

   - Suppose that e', e'' are preserved by F-products and that $J \in F$. Then $J', J'' \in F$ (cf. 2) which implies $M_F \models e', e''$. Hence $M_F \models e$.
   - Now suppose that e', e'' are preserved by F-factors and that $M_F \models e$. Then $M_F \models e', e''$, hence $J', J'' \in F$. Thus $J \in F$ (cf. 2).

4. Let $(e_\ell)_{\ell\in L}$ be a family of Σ-sentences preserved by F-products and let e be a conjunction of $(e_\ell)_{\ell\in L}$. Suppose that $\{i \in I \mid M_i \models e\} \in F$. Then

1	$\{i \in I \mid M_i \models e_\ell\} \supseteq \{i \in I \mid M_i \models e\}$ for each $\ell \in L$	e conjunction of $(e_\ell)_{\ell\in L}$
2	$\{i \in I \mid M_i \models e_\ell\} \in F$ for each $\ell \in L$	1, F filter
3	$M_F \models e_\ell$ for each $\ell \in L$	e_ℓ preserved by F-products
4	$M_F \models e$	3, e conjunction of $(e_\ell)_{\ell\in L}$.

5. Let e be a negation of a Σ-sentence e' such that e' is preserved by F-factors. Let us assume that $J = \{j \in I \mid M_j \models e\} \in F$. We have to prove that $M_F \models e$. By *Reductio ad Absurdum* we suppose $M_F \not\models e$. Then

1	$M_F \models e'$	e' negation of e, $M_F \not\models e$
2	$J' = \{j \in I \mid M_j \models e'\} \in F$	1, e' preserved by F-factors
3	$J \cap J' \in F$	$J \in F$, 2, F filter
4	there exists $j \in J \cap J'$	3, F proper
5	$M_j \models e, e'$	4.

But 5 represents a contradiction.

6. Let e be a negation of a Σ-sentence e' such that e' is preserved by $\mathcal{F}$-products. Let F be any ultrafilter in $\mathcal{F}$ and assume that $M_F \models e$. By *Reductio ad Absurdum* we suppose $J = \{j \in I \mid M_j \models e\} \notin F$. Then

1	$J' = \{j \in I \mid M_j \models e'\} = I \setminus J$	e' negation of e
2	$J' \in F$	1, F ultrafilter
3	$M_F \models e'$	2, e' preserved by F-products.

But 3 and $M_F \models e$ are contradictory, hence $J \in F$.

7. From 5. and 6.

8. Let $\chi\colon\ \Sigma \to \Sigma'$ be a signature morphism that preserves $\mathcal{F}$-products. Let e' be a Σ'-sentence preserved by $\mathcal{F}$-products, and let e be an existential χ-quantification of e'. Consider a filter $F \in \mathcal{F}$ over a set I, and assume that $J = \{i \in I \mid M_i \models_\Sigma e\} \in F$. We have to prove that $M_F \models_\Sigma e$. For each $i \in J$ let M'_i be a χ-expansion of M_i such that $M'_i \models_{\Sigma'} e'$. Then

1	$M'_{F\|_J} \models_{\Sigma'} e'$	$J \in F$, $F\|_J \in \mathcal{F}$, e' preserved by $\mathcal{F}$-products
2	$M'_{F\|_J}\upharpoonright_\chi \cong M_{F\|_J}$	$Mod\chi$ preserves $\mathcal{F}$-products
3	$M_{F\|_J} \cong M_F$	Prop. 6.3
4	$M'_{F\|_J}\upharpoonright_\chi \models_\Sigma e$	1, e is existential χ-quantification of e'
5	$M_F \models e$	2, 3, 4, satisfaction invariant with respect to model isomorphisms.

9. Under the same framework as in the item above, but assuming instead that χ invents $\mathcal{F}$-products, let us consider that $M_F \models_\Sigma e$. We have to prove that $\{i \in I \mid M_i \models_\Sigma e\} \in F$. Let N be a χ-expansion of M_F such that $N \models_{\Sigma'} e'$. Because $Mod\chi$ invents $\mathcal{F}$-products, there exists $J \in F$ such that for each $i \in J$ there exists a χ-expansion M'_i of M_i such that $M'_{F|_J} \cong N$. Then

1	$M'_{F\|_J} \models e'$	$N \models e'$, $M'_{F\|_J} \cong N$, satisfaction invariant with respect to model isomorphisms
2	$J' = \{i \in J \mid M'_i \models_{\Sigma'} e'\} \in F\|_J$	1, $F\|_J \in \mathcal{F}$, e' preserved by $\mathcal{F}$-factors
3	$F\|_J \subseteq F$	$J \in F$, F filter, filter properties

4	$J' \in F$	2, 3
5	$J' \subseteq \{i \in I \mid M_i \models_\Sigma e\}$	e is existential χ-quantification of e'
6	$\{i \in I \mid M_i \models_\Sigma e\} \in F$	4, 5, F filter.

□

The result of Theorem 6.6 has been developed in a modular manner. This allows for a great deal of flexibility in the applications, as one may invoke only the parts that are required by a respective application. The last two parts of Theorem 6.6 are concerned with quantifiers and these are the only ones that require some specific conditions. In the remaining part of the section, we address these conditions.

Establishing the preservation and invention of filtered products

From these two properties, in the applications preservation is in general easier to establish that invention. Moreover the former can be treated as a combination of two separate 'smaller' problems, namely the preservation of direct products and preservation of directed co-limits.

Preservation of direct products. Since right adjoint functors preserve all limits (see Prop. 2.6), one way to see that model reducts preserve direct products is to invoke liberality of the signature morphisms. According to Prop. 4.29 a sufficient set of conditions for this is the existence of signature pushouts, semi-exactness, the existence of diagrams, and the existence of initial models of presentations. At first glance the latter condition might seem quite strong, however, it is not since we need only the sub-institution of those sentences which are involved in the diagrams. As we know, in the case of the standard concepts of model homomorphisms, these are the atomic sentences of the institution. $\mathcal{FOL}$ is a typical case, since for the standard concept of model homomorphism we thus need that only (sets of) atoms have initial models, a property which is easy to establish (see Cor. 4.28).

Note that restricted concepts of model homomorphisms may break the argument above. Consider for example the injective $\mathcal{FOL}$-model homomorphisms. Recall that the corresponding diagrams consist of (equational and relational) atoms plus negations of equational atoms. Arbitrary sets of atoms and negations of equational atoms do not necessarily have an initial model, even when homomorphisms are all injective. Moreover, in this case even the existence of (direct) products of models is lost. This shows that categorical F-products require appropriate model homomorphisms which guarantee good structural properties for the categories of models.

In most applications of Theorem 6.6 the quantifications are first-order, and in those situations the following general simple result can be applied immediately, thus avoiding the method discussed above.

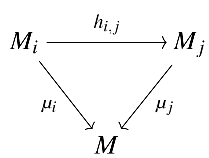
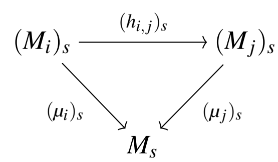
Proposition 6.7. *All model reduct functors corresponding to representable signature morphisms create limits of models.*

Proof. Let $\chi : \Sigma \to \Sigma'$ be a representable signature morphism. The proposition holds by the general categorical argument that the forgetful functor from a comma category to the base category creates all limits (Prop. 2.3) applied to the forgetful $M_\chi/Mod\Sigma \to Mod\Sigma$. □

Preservation of directed co-limits. This problem can be approached in a similar way to the preservation of direct products. Either establish preservation results for all model reducts directly in concrete institutions, or else develop a general result for a restricted class of model reducts that nevertheless covers concrete first-order quantifications. We illustrate the first method with a $\mathcal{FOL}$ result, which can be replicated with superficial adjustments to other concrete situations. For the second method, we provide a general result in the style of Prop. 6.7.

Proposition 6.8. *In $\mathcal{FOL}$ all model reduct functors lift directed co-limits. Moreover, for the signature morphisms that are surjective on the sorts, the lifting is unique.*

Proof. We first consider the simpler case, when the model reduct functor is the forgetful functor from models to their underlying set carriers. Let $h = (h_{i,j})_{(i\leq j)\in(J,\leq)}$ be a directed diagram of (S,F,P)-model homomorphisms, and let μ be the co-limit of the corresponding diagram of underlying many-sorted sets.

Then

- there exists a *unique* way one can interpret the operations of F on the sets $(M_s)_{s\in S}$ such that μ_i become (S,F)-model homomorphisms. For each $\sigma \in F_{w\to s}$ and each tuple of elements $(m_1,\dots,m_k) \in M_w$, we define

$$M_\sigma(m_1,\dots,m_k) = \mu_j\ (M_j)_\sigma(m_1^j,\dots,m_k^j)$$

  where j and $m_1^j,\dots,m_k^j$ are such that $m_1 = \mu_j m_1^j,\dots,m_k = \mu_j m_k^j$. This is possible because μ is already the co-limit of the underlying set carriers – hence each m_i can be written as $\mu_{j_i} m_i^{j_i}$ for some $j_i \in J$ – and because the length k of the arity w is finite – hence by the directedness of $(J,\leq)$ we can find j such that $j^i \leq j$, $1 \leq i \leq k$. Also, the correctness of the definition is guaranteed by the homomorphism property of each $h_{j^i,j}$. It is easy to check that under this definition of M_σ all μ_i's are (S,F)-homomorphisms and that μ also a co-limit in the category (S,F)-homomorphisms.

- There is also a *minimal* way one can interpret the relations P on the sets $(M_s)_{s\in S}$ such that μ_i are (S,P)-model homomorphisms. For each $\pi \in P$,

$$M_\pi = \bigcup\{\mu_i\ (M_i)_\pi \mid i \in J\}$$

The fact that M_π is the smallest with this property guarantees the co-limit property of the co-cone μ in the category of (S,P)-homomorphisms.

Hence μ is a co-limit of h in $Mod^{\mathcal{FOL}}(S,F,P)$.

In the second part of the proof, we just extend the conclusion of the first part to the case of *any* $\mathcal{FOL}$-signature morphism $\chi\colon\ (S,F,P) \to (S',F',P')$. Let $h' = (h'_{i,j}\colon\ M'_i \to M'_j)_{(i\leq j)\in(J,\leq)}$ be a directed diagram of (S',F',P')-model homomorphisms, and let $h = (h_{i,j}\colon\ M_i \to M_j)_{(i\leq j)\in(J,\leq)}$ be its χ-reduct. Let μ be a co-limit of the later latter diagram. By the first part of the proof we know that μ is a co-limit of h in the category of S-sorted sets. Moreover without any loss of generality, we may assume that for each $i\in J$ and sorts $s1,s2$ such that $\chi s1 = \chi s2$ we have that $(\mu_i)_{s1} = (\mu_i)_{s2}$. This is because $(M_i)_{s1} = (M_i)_{s2}$.

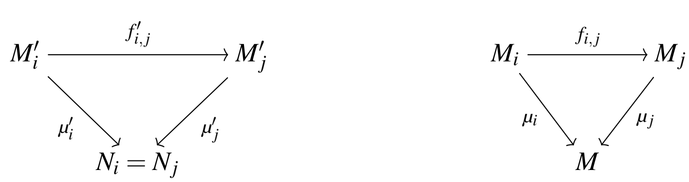
Then for each $i\in J$ we define $(\mu'_i)_{\chi s} = (\mu_i)_s$ for each $s\in S$. If χ^{st} is surjective then μ' is thus uniquely defined and it is a co-limit, if not then we can still extend μ' thus defined outside the image of χ^{st} such that μ' is still a co-limit. Of course, in the non-surjective case, μ' thus defined is not unique. The final step is the application of the first part of the proof for lifting μ' from a co-limit of S'-sorted functions to a co-limit of (S',F',P')-model homomorphisms. It remains to see that for each τ operation or relation symbol of (S,F,P) we have that $M'_{\chi\tau} = M_\tau$. This can be seen directly from the definition of $M'_{\chi\tau}$ according to the first part of the proof of this proposition. □

Note the reliance of the above result to the finiteness of the arities of the symbols of the signatures; this may be typical in other concrete situations too.

The strength of the result of Prop. 6.8 is that it holds for all model reducts, its weakness is that it lacks independence from the underlying institution. The following result is rather complementary to these aspects as it holds for a restricted class of signature morphisms but on the other hand, it is abstract. However, it applies effectively to first-order quantifications situations.

Proposition 6.9. *All model reduct functors corresponding to quasi-representable signature morphisms create directed co-limits of models.*

Proof. Let $\chi\colon\ \Sigma \to \Sigma'$ be a quasi-representable signature morphism, let $f' = (f'_{i,j})_{(i<j)\in(I,\leq)}$ be a directed diagram of Σ'-models, and let $f = (f_{i,j})_{(i<j)\in(I,\leq)}$ be its χ-reduct. Consider a co-limit μ of f.



Because χ is quasi-representable, for each $i \in I$, there exists a unique χ-expansion $\mu'_i : M'_i \to N_i$ of μ_i.

- If $i \leq j$ then both μ'_i and $f'_{i,j};\mu'_j$ are χ-expansions from M'_i of $\mu_i = f_{i,j};\mu_j$ hence by the quasi-representability of χ we get that $\mu'_i = f'_{i,j};\mu'_j$. In particular, this means $N_i = N_j$. Moreover, because the diagram is directed, for any i,k we can establish that $N_i = N_k$ by considering $j \geq i,k$. Hence we have a co-cone μ' for f'.

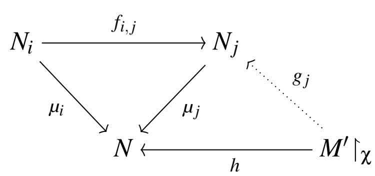
- Now we prove that μ' is a co-limit for f'. Let ν' be a co-cone for f' and let ν be its χ-reduct. Since μ is a co-limit there exists an unique h such that $\mu;h = \nu$. Let M' be the vertex of μ'. By the quasi-representability of χ let h' be the unique χ-expansion of h from M'. Then $\mu'_i;h'$ is a χ-expansion of $\mu_i;h = \nu_i$ from M'_i. By the quasi-representability of χ it follows that $\mu'_i;h' = \nu'_i$.

□

The following is a consequence of Prop. 6.9 and will be used later on.

Corollary 6.10. *Let $\chi : \Sigma \to \Sigma'$ be a finitary quasi-representable signature morphism and M' be a finitely presented Σ'-model. Then its reduct $M'\upharpoonright_\chi$ is finitely presented too.*

Proof. Let us consider a Σ-model homomorphism $h : M'\upharpoonright_\chi \to N$ to the vertex of a co-limit μ of a directed diagram $f = (f_{i,j} : N_i \to N_j)_{(i<j)\in(J,\leq)}$. We first prove that there exists $j \in J$ and $g_j : M'\upharpoonright_\chi \to N_j$ such that $g_j;\mu_j = h$. By the quasi-representability of χ let $h' : M' \to N'$ be the unique χ-expansion of h from M'.



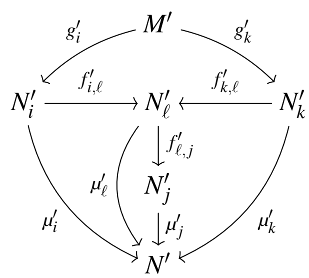
- Because χ is finitary there exists $i \in J$ and $\mu'_i : N'_i \to N'$ a χ-expansion of μ_i.

- By using the quasi-representability of χ we get a χ-expansion $f' = (f'_{j,k} : N'_j \to N'_k)_{(i\leq j<k)\in(J,\leq)}$ of $(f_{j,k} : N_j \to N_k)_{(i\leq j<k)\in(J,\leq)}$ and also a co-cone μ' for f' which is a χ-expansion of $(\mu_j)_{i\leq j\in J}$. Moreover μ' is a co-limit co-cone of f' because
  - by Thm. 2.4 $(\mu_j)_{i\leq j\in J}$ is a co-limit of $(f_{j,k} : N_j \to N_k)_{(i\leq j<k)\in(J,\leq)}$ since this is a final sub-diagram of f, and
  - by Prop. 6.9 $Mod\chi$ creates directed co-limits.

- Because M' is finitely presented there exists $i \leq j$ and $g'_j : M' \to N'_j$ such that $g'_j;\mu'_j = h'$. Then we define $g_j = g'_j\upharpoonright_\chi$.

Now we prove that for any $g_i : M'\upharpoonright_\chi \to N_i$ and $g_k : M'\upharpoonright_\chi \to N_k$ such that $h = g_i;\mu_i = g_k;\mu_k$ there exists $j \geq i,k$ such $g_i;f_{i,j} = g_k;f_{k,j}$.

- By the quasi-representability property for χ we let $g'_i : M' \to N'_i$ / $g'_k : M' \to N'_k$ / $\mu'_i : N'_i \to N'$ / $\mu'_k : N'_k \to N'$ be the χ-expansions of g_i / g_k / μ_i / μ_k, respectively. (That μ'_i and μ'_k share their codomain follows by the uniqueness of the χ-expansions of $g_i;\mu_i$, $g_k;\mu_k$ and also because $g_i;\mu_i = g_k;\mu_k$.)
- Because $(Mod\chi)^{-1}$ preserves directed co-limits (Fact 5.15) it follows that $(Mod\chi)^{-1}\mu$ is a co-limit of $(Mod\chi)^{-1}f$. Hence since $((Mod\chi)^{-1}\mu)N'_i = ((Mod\chi)^{-1}\mu)N'_k (= N')$ there exists $\ell \geq i,k$ such that $((Mod\chi)^{-1}f_{i,\ell})N'_i = ((Mod\chi)^{-1}f_{k,\ell})N'_k$; we denote this by N'_ℓ.
- By the quasi-representability of χ and the directedness of $(J,\leq)$, from N'_ℓ we get an unique χ-expansion $(f'_{j1,j2})_{(l\leq j1<j2)\in(J,\leq)}$ of the final sub-diagram $(f_{j1,j2})_{(l\leq j1<j2)\in(J,\leq)}$ together with a unique χ-expansion $(\mu'_j)_{(l\leq j)\in(J,\leq)}$ of the co-limiting co-cone $(\mu_j)_{(l\leq j)\in(J,\leq)}$.



- Since M' is finitely presented there exists $j \geq \ell$ such that $g'_i;f'_{i,l};f'_{l,j} = g'_k;f'_{k,l};f'_{l,j}$. From this, by reduction by χ, we obtain $g_i;f_{i,j} = g_k;f_{k,j}$.

□

Inventing F-products

With the following general result, we establish classes of signature morphisms for which their model reduct functors invent filtered products. In the applications, these cover the usual first-order quantifications. Both alternative conclusions of Prop. 6.11 below are technically stronger than what is required by Thm. 6.6.

Proposition 6.11. *In any institution, for any class $\mathcal{F}$ of filters, the model reduct functor $Mod\chi$ corresponding to a signature morphism χ*

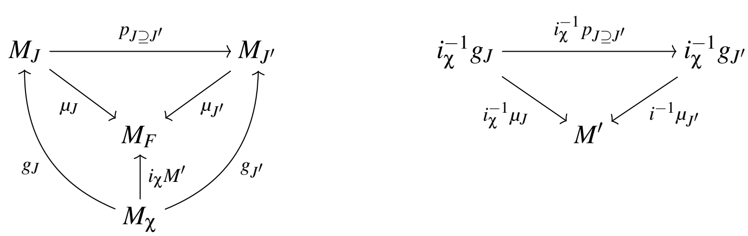
1. *invents strongly $\mathcal{F}$-products if χ is finitary representable and $\mathcal{F}$ is closed under reductions, or*
2. *invents strongly and completely $\mathcal{F}$-products if*
   - χ *is* projectively representable*, i.e., χ is representable such that its representation M_χ is projective, and*
   - *all projections of model products are epis, i.e. for any model product M_I of a family $(M_i)_{i\in I}$ of models and for each $J \subseteq I$, the canonical projection $M_I \to M_J$ is an epi.*

Proof. Let $\chi : \Sigma \to \Sigma'$ and M_χ be a representation of χ. Consider a filter $F \in \mathcal{F}$ over a set I and a χ-expansion M' of M_F where $\{\mu_J : M_J \to M_F \mid J \in F\}$ is an F-product of a family $(M_i)_{i\in I}$ of Σ-models. We first show that in any of the two hypotheses, there exists $J \in F$ and $g_J : M_\chi \to M_J$ such that $g_J;\mu_J = i_\chi M'$. Moreover under the second hypothesis $J = I$.

1. By Corollary 5.17 we know that M_χ is finitely presented. Then there exists $J \in F$ and $g_J : M_\chi \to M_J$ such that $g_J ; \mu_J = i_\chi M'$.
2. Let us prove that $\mu_I : M_I \to M_F$ is epi. Let $f, g : M_F \to N$ such that $\mu_I; f = \mu_I; g$. Because the projections $p_{I\supseteq J}$ are epis it follows that for each $J \in F$ we have $\mu_J ; f = \mu_J ; g$. Because μ is a co-limit, it is an epimorphic family, therefore $f = g$. By using that μ_I is epi, by the projectivity of M_χ there exists $g_I : M_\chi \to M_I$ such that $g_I ; \mu_I = i_\chi M'$.

For each $J \supseteq J' \in F$ let $g_{J'} = g_J ; p_{J\supseteq J'}$.

- By regarding filters as partially ordered sets, $F|_J \subseteq F$ is a final functor. Therefore, by Thm. 2.4 it follows that $(\mu_{J'})_{J'\in F|_J}$ is an $F|_J$-product of $(M_i)_{i\in J}$.



- Since $Mod\chi$ creates direct products (cf. Prop. 6.7) and directed co-limits (cf. Prop. 6.9) it follows that $(i_\chi^{-1}\mu_{J'})_{J'\in F|_J}$ is an $F|_J$-product of $(i_\chi^{-1} p_{J'\supseteq J''})_{J'\supseteq J''\in F|_J}$.

□

Although in most situations Prop. 6.11 can be used effectively for inventing of F-products, the representability of the signature morphism is not always a necessary condition for this. Remarkable such situations can be found in the realm of the 'stratified institutions' of Chapter 12, including institutions based on some form of Kripke semantics.

Exercises

6.11. In $\mathcal{FOL}$ all model reduct functors lift small limits. Moreover, for the signature morphisms that are surjective on the sorts, the lifting is unique. (*Hint:* Follow a similar route to the proof of Prop. 6.8.)

6.12. In $\mathcal{EQL}$ all model reduct functors corresponding to signature morphisms that are surjective on the sorts create small limits.

6.3 Łoś institutions

In this section, we assume institutions that have all ultraproducts of models.

Łoś sentences. A sentence is a *Łoś-sentence* when it is preserved by all ultrafactors and all ultraproducts. The following is a straightforward consequence of Thm. 6.6.

Corollary 6.12. *In any institution I, the sentences accessible from the finitary basic sentences by Boolean connectives and χ-quantification for which χ preserves and invents ultraproducts, are Łoś-sentences.*

Cor. 6.12 can be brought closer to concrete applications by using Prop. 6.9, 6.7 and 6.11:

Corollary 6.13. *In any institution, any sentence which is accessible from the finitary basic sentences by*

– *Boolean connectives,*

– *finitary representable quantification, and*

– *projectively representable quantification (assuming that the institution has epi model projections)*

is a Łoś-sentence.

Łoś-institutions. An institution is a *Łoś-institution* if and only if it has all ultraproducts of models and all its sentences are Łoś-sentences. Note that the condition on the existence of ultraproducts requires the existence of direct products but not necessarily of other F-products. With this terminology the classical Łoś Ultraproducts Theorem can be thus formulated as follows:

Corollary 6.14. *$\mathcal{FOL}$ is a Łoś-institution.*

Note that in the case of Cor. 6.14 we can use any of the two arguments referring to finitary or projectively representable quantifications.

Filtered power embedding

Let F be a filter over a set I. In any institution with F-products of models, for each model M and any F-power $(\mu_J : M_J \to M_F)_{J\in F}$ there is a canonical model homomorphism $d_M^F : M \to M_F$ defined by

$$d_M^F = (M \xrightarrow{\delta_M^I} M_I \xrightarrow{\mu_I} M_F)$$

where $\delta_M^I : M \to M_I$ is the 'diagonal' model homomorphism defined by $\delta_M^I ; p_{I,i} = 1_M$ for each $i \in I$.

Proposition 6.15. *In any institution with diagrams ι, for any filter F over an arbitrary set I such that*

1. *the institution has F-products of models which are preserved by the model reducts corresponding to the elementary extensions, and*
2. *all sentences are preserved by F-products,*

for any model M the canonical homomorphism $d^F_M : M \to M_F$ is ι-elementary.

Proof. Let M_M be the initial model of the diagram (Σ_M, E_M) of a Σ-model M. We have to prove that $i^{-1}_{\Sigma,M} d^F_M \models (M_M)^*$. Let $(\mu'_J : (M_M)_J \to (M_M)_F)_{J\in F}$ be an F-power of M_M. Then

1	$(M_M)_F \models E_M$	$M_M \models E_M$, E_M are preserved by F-products
2	$\mu'\upharpoonright_{\iota_\Sigma M}$ is F-power of M	$Mod(\iota_\Sigma M)$ preserves F-products
3	there exists $\varphi : M_F \to (M_M)_F\upharpoonright_{\iota_\Sigma M}$ isomorphism s.th. $d^F_M;\varphi = d^F_{M_M}\upharpoonright_{\iota_\Sigma M}$	2
4	$i_{\Sigma,M} d^F_{M_M} : i_{\Sigma,M} M_M \to i_{\Sigma,M}(M_M)_F$	definition of $d^F_{M_M}$
5	$d^F_{M_M}\upharpoonright_{\iota_\Sigma M} : 1_M \to i_{\Sigma,M}(M_M)_F$	4
6	$i_{\Sigma,M}(M_M)_F = d^F_{M_M}\upharpoonright_{\iota_\Sigma M}$	5, property (4.1) of $i_{\Sigma,M}$
7	$i^{-1}_{\Sigma,M}(d^F_M;\varphi) = (M_M)_F$	3, 6
8	$i^{-1}_{\Sigma,M}\varphi : i^{-1}_{\Sigma,M} d^F_M \to i^{-1}_{\Sigma,M}(d^F_M;\varphi)$	$\varphi : d^F_M \to d^F_M;\varphi$ in $M/Mod\Sigma$
9	$i^{-1}_{\Sigma,M} d^F_M \cong (M_M)_F$	7, 8, φ isomorphism
10	$(M_M)_F \models (M_M)^*$	F-products preserves all sentences
11	$i^{-1}_{\Sigma,M} d^F_M \models (M_M)^*$	satisfaction invariant under model isomorphisms.

□

The condition of preservation of F-products by the elementary extensions in the above proposition is fulfilled immediately in all institutions for which all signature morphisms preserve direct products and directed co-limits of models. We have seen in Sect. 6.2 that this is an expected situation. Alternatively, we may use the general argument that representable signature morphisms preserve all F-products (cf. Prop. 6.7 and 6.9) and rely on the fact that elementary extensions are usually representable.

By Cor. 6.14 we obtain the following instance of Prop. 6.15.

Corollary 6.16. *Any $\mathcal{FOL}$-model can be elementarily embedded in any of its ultrapowers.*

This classic result in first-order model theory has many important applications. For instance, it is the basis of the *Transfer Principle* of non-standard analysis.

Exercises

6.13. Several concrete institutions (such as $\mathcal{HCL}$, $\mathcal{POA}$, $\mathcal{PA}$, $\mathcal{AUT}$, $\mathcal{MBA}$, $\mathcal{LA}$, $\mathcal{IPL}$, etc.) can be established as Łoś institutions by virtue of Cor. 6.13.

6.14. Ultraproducts in $\mathcal{HNK}$
$\mathcal{HNK}$ has ultraproducts of models. (*Hint:* Consider comorphism $\mathcal{HNK} \to \mathcal{FOEQL}^{\text{th}}$ of Ex. 4.12 and use the fact that (cf. Ex. 6.4) $\mathcal{HNK}$ has direct products of models and that (cf. Cor. 6.14) $\mathcal{FOL}$ is a Łoś institution.) But $\mathcal{HNK}$ does not necessarily have all F-products.

6.15. Borrowing the Łoś property
For any persistently liberal institution comorphism $(\Phi,\alpha,\beta)\colon\ \mathcal{I} \to \mathcal{I}'$, for any $\mathcal{I}$-sentence ρ, if $\alpha(\rho)$ is a Łoś sentence then ρ is a Łoś sentence too. Apply this to the examples of comorphisms of Ex. 4.88 for obtaining the Łoś property for several concrete institutions (such as $\mathcal{POA}$, $\mathcal{PA}$, $\mathcal{AUT}$, $\mathcal{MBA}$, $\mathcal{IPL}$, $\mathcal{LA}$, etc.). By this borrowing method $\mathcal{HNK}$ can also be established as a Łoś institution. (*Hint:* Use the comorphism of Ex. 4.12 and refer also to the result of Ex. 6.14.)

6.16. Horn sentences are preserved by F-products
Let $(\forall\chi)E \Rightarrow E'$ be a finitary universal Horn sentence for a signature Σ in an arbitrary institution that has F-products of models. For each family of Σ-models $(M_i)_{i\in I}$ and filter F over I, show that any F-product of the family of models satisfies $(\forall\chi)E \Rightarrow E'$ when M_i satisfies $(\forall\chi)E \Rightarrow E'$ for each $i \in I$. Apply this for showing that in $\mathcal{HCL}$ any model can be elementarily embedded in any of its F-powers.

6.17. Σ^1_1-sentences
In any institution, e is a Σ^1_1-*sentence* if it is an existential χ-quantification of a Łoś sentence, where χ is any F-product preserving signature morphism. For instance any second-order existential quantification of a $\mathcal{FOL}$ sentence is a Σ^1_1-sentence. In any institution each Σ^1_1-sentence is preserved by ultraproducts.

6.4 Compactness

Compactness is a central theme in logic and model theory. It has two main variants.

1. A consequence theoretic version at the level of abstract institutions is as follows. An institution is *compact* when for each signature Σ, each set E of Σ-sentences and each single Σ-sentence e, if $E \models e$ then there exists a finite subset $E_f \subseteq E$ such that $E_f \models e$.
2. A model-theoretic version which is based on the concept of consistency. In any institution $\mathcal{I}$ a set E of Σ-sentences is *consistent* if $E^* \neq \emptyset$. An institution is *model compact* or *m-compact* for short, if each set of sentences is consistent when all its finite subsets are consistent.

Although the definition of the former version of compactness also involves models (because the concept of semantic consequence relies on models) it can be defined more abstractly for any relation on sets of sentences that satisfy the properties of semantic consequence given in Prop. 3.7.

Consequence-theoretic compactness versus model compactness. The significance of consistency and the distinction between compactness and m-compactness depends on the actual institution. For example, consistency has real significance in $\mathcal{FOL}$, while in $\mathcal{EQL}$ or $\mathcal{HCL}$ it is a trivial property since each set of sentences is consistent. Therefore in some institutions compactness and m-compactness are not necessarily the same concept. For example, any institution in which each set of sentences is consistent is trivially m-compact, but it is not necessarily compact. Below is a simple (counter)example.

Proposition 6.17. *$\mathcal{HCL}_{\infty,\omega}$ (infinitary Horn clause logic) is model compact but it is not compact.*

Proof. That $\mathcal{HCL}_{\infty,\omega}$ is m-compact follows from the fact that each theory in $\mathcal{HCL}_{\infty,\omega}$ is consistent since it has an initial model. This can be established in the same way as for $\mathcal{HCL}$ since in Prop. 4.26 we have not used the fact that the Horn sentences are finitary.

Now let us show that $\mathcal{HCL}_{\infty,\omega}$ is not compact. For this, we consider a signature without sorts, consisting only of an infinite set P of relation symbols of empty arity. Recall that the models of this signature consist of subsets of P. Obviously, each element of P is an atom. Let us pick a $\pi \in P$ and consider $E = P \setminus \{\pi\}$. We have the semantic consequence $E \cup \{\wedge E \Rightarrow \pi\} \models \pi$. We show that for any finite $E_f \subseteq E \cup \{\wedge E \Rightarrow \pi\}$, $E_f \not\models \pi$. For this, we define a model M of E_f which does not satisfy π. This is $M = E_f \setminus \{\wedge E \Rightarrow \pi\}$. □

The following establishes a general relationship between compactness and m-compactness.

Proposition 6.18.

- *Each compact institution having false is m-compact.*
- *Each m-compact institution having negations is compact.*

Proof. Let E be any set of Σ-sentences.

- By *Reductio ad Absurdum* suppose that $E^* = \emptyset$ whilst for each finite $E_f \subseteq E$, $E_f^* \neq \emptyset$. Since $E^* = \emptyset$ means $E \models \mathsf{false}$, by the compactness hypothesis there exists a finite $E_f \subseteq E$ such that $E_f \models \mathsf{false}$. But this means $E_f^* = \emptyset$ which is a contradiction.
- Conversely, by *Reductio ad Absurdum* we consider $E \models e$ such that for each finite $E_f \subseteq E$ we have $E_f \not\models e$. Then there exists a model $M_f \models E_f, e'$ where e' is a negation of e. This implies that $E_f \cup \{e'\}$ is consistent which by the m-compactness hypothesis implies that $E \cup \{e'\}$ is consistent. Let $M \in (E \cup \{e'\})^*$. Then $M \models E$ and $M \not\models e$ thus $E \not\models e$ which is a contradiction.

□

Compactness by ultraproducts

There are two main ways to establish compactness properties as follows:

1. By defining a set of complete finitary proof rules for the institution. This means that for each semantic consequence $E \models e$ we can 'prove' e by a finite process of applying rules starting with premises from E. This would be a syntactic process as it would not involve models at all and it would use only a finite set $E_f \subseteq E$ of premises. Furthermore, if the institution has false then we can also establish m-compactness. There are two drawbacks to this method. It is usually very difficult to establish complete systems of finitary proof rules as in general completeness is a notoriously difficult property to establish. Moreover, if we aim for m-compactness then we need false, which is a form of negation. About proof rules and proof theoretic compactness we will find out in Chap. 11.

2. By using ultraproducts we can establish compactness and m-compactness independently, without the need for a result that relates them, such as Prop. 6.18. This is what we will do now at a fully general institution-independent level.

The following result is the root of establishing both compactness and m-compactness in abstract institutions by model-theoretic means.

Theorem 6.19. *In any institution with ultraproducts of models, let E be a set of sentences preserved by ultraproducts of models. Let I be the set of all finite subsets of E. Consider a model $M_i \in i^*$ for each finite subset $i \in I$. Then there exists an ultrafilter U over I such that for each ultraproduct $(\mu_J : M_J \to M_U)_{J\in U}$, $M_U \models E$.*

Proof. A set $S \subseteq \mathcal{P}I$ *has the finite intersection property* if $J_1 \cap J_2 \cap \cdots \cap J_n \neq \emptyset$ for all $J_1, J_2, \ldots, J_n \in S$. We will use the following classical Ultrafilter Lemma (its proof can be found for example in [42]).

Lemma 6.20. *If $S \subseteq \mathcal{P}I$ has the finite intersection property, then there exists an ultrafilter U over I such that $S \subseteq U$.*

Now let $S = \{\{i \in I \mid \rho \in i\} \mid \rho \in E\}$. S has the finite intersection property because

$$\{\rho_1, \rho_2, \ldots, \rho_n\} \in \{i \in I \mid \rho_1 \in i\} \cap \{i \in I \mid \rho_2 \in i\} \cap \cdots \cap \{i \in I \mid \rho_n \in i\}.$$

By the Ultrafilter Lemma 6.20, let U be an ultrafilter such that $S \subseteq U$. For each $\rho \in E$ we have:

1	$\{i \in I \mid \rho \in i\} \subseteq \{i \in I \mid M_i \models \rho\}$	$M_i \models i$
2	$\{i \in I \mid M_i \models \rho\} \in U$	$\{i \in I \mid \rho \in i\} \in S \subseteq U$, U filter, 1
3	$M_U \models \rho$	2, ρ is preserved by ultraproducts.

Because $\rho \in E$ is arbitrary, it follows that $M_U \models E$. ☐

Corollary 6.21. *Any institution in which each sentence is preserved by ultraproducts is m-compact.*

Corollary 6.22. *Let E be a set of sentences preserved by ultraproducts, and let e be a sentence preserved by ultrafactors such that $E \models e$. Then there exists a finite subset $E_f \subseteq E$ such that $E_f \models e$.*

Proof. Let us assume the contrary, i.e., that for each finite $E_f \subseteq E$, $E_f \not\models e$. Let I be the set of all finite subsets of E. This means that for each $i \in I$ there exist a model M_i such that $M_i \models i$ but $M_i \not\models e$. Let $(\mu_J : M_J \to M_U)_{J\in U}$ be any ultraproduct of $(M_i)_{i\in I}$. Then

1	$M_U \models E$	Thm. 6.19
2	$M_U \models e$	1, $E \models e$
3	$\{i \in I \mid M_i \models e\} \in U$	e is preserved by ultrafactors
4	$\{i \in I \mid M_i \models e\} = \emptyset$	definition of M_i, $i \in I$
5	$\emptyset \in U$	3, 4.

But the conclusion 5 comes in contradiction with the fact that U is ultrafilter. □

By putting together Corollaries 6.22 and 6.21 we obtain:

Corollary 6.23. *Any Łoś-institution is both compact and m-compact.*

Cor. 6.23 constitutes a great source of examples of compact and m-compact institutions. The following well-known concrete result is obtained via Cor. 6.14.

Corollary 6.24. *$\mathcal{FOL}$ is both compact and m-compact.*

Alternatively, this result could be obtained as a direct instance of either of Corollaries 6.21 or 6.22 by relying on the equivalence between compactness and m-compactness in $\mathcal{FOL}$ given by Prop. 6.18.

A note on the hyperreals. In light of the compactness-by-ultraproducts developments above let us have a look at the hyperreals. Let $\mathbb{R}$ be the ordered field of the real numbers and let Σ be the signature of the ordered fields. Obviously

$$\mathbb{R} \not\models (\exists x)(0 < x \wedge \bigwedge_{n\in\omega} x < \tfrac{1}{n}).$$

Let Σ' be the extension of Σ with a new constant x. Let I be the set of the finite subsets of $\{0 < x\} \cup \{x < \frac{1}{n} \mid n \in \omega\}$. Evidently, for each $i \in I$ there exists a Σ'-expansion $\mathbb{R}_i$ of $\mathbb{R}$ such that $\mathbb{R}_i \models i$. By the m-compactness of $\mathcal{FOL}$ there exists an ordered field $\mathbb{R}'$ such that

$$\mathbb{R}' \models 0 < x \wedge \bigwedge_{n\in\omega} x < \tfrac{1}{n}.$$

Then a model ${}^*\mathbb{R}$ of the hyperreals is obtained a Σ-reduct of $\mathbb{R}'$. Indeed

$${}^*\mathbb{R} \models (\exists x)(0 < x \wedge \bigwedge_{n\in\omega} x < \tfrac{1}{n}).$$

This argument can be developed without any involvement of ultraproducts as compactness of $\mathcal{FOL}$ can be established by proof theoretic means as described above. However, this would not tell us much about the nature of the hyperreals. On the other hand, if we would like to get hold of the hyperreals we may use the result of Thm. 6.19 and get $\mathbb{R}'$ as an ultraproduct of $(\mathbb{R}_i)_{i\in I}$ and then by reducing it to Σ get ${}^*\mathbb{R}$ as an ultrapower of $\mathbb{R}$. By Cor. 6.16 we also get that $\mathbb{R}$ is elementarily embedded into ${}^*\mathbb{R}$, which means two things: any real number is indeed a hyperreal, and any first-order property of the hyperreals can be transferred to the reals. Moreover according to Prop. 6.15 we can even do more than that, any sentence that is preserved by ultraproducts is subject to this Transfer Principle.

Exercises

6.18. [161] **Logical compactness versus topological compactness**
Recall that a topology (X,τ) is *compact* when for each family $(U_i)_{i\in I}$ such that $U_i \in \tau$ for each $i \in I$ and such that $\bigcup_{i\in I} U_i = X$, there exists a finite subset $J \in I$ such that $\bigcup_{i\in J} U_i = X$. An institution with negation is (m-)compact if all its semantic topologies (see Ex. 4.5) are compact. Moreover, if the institution has finite conjunctions too, then it is (m-)compact if and only if all its semantic topologies are compact.

6.19. Maximally consistent sets
(a) We say that a set of sentences E for a signature Σ in an arbitrary institution is *maximally consistent* if and only if for any other consistent set E', $E \subseteq E'$ implies $E = E'$. In any institution with negation, a set E of sentences (for a given signature) is maximally consistent only if for each sentence e exactly one of e and $\neg e$ belong to E.
(b) By (a), in any institution with negation, for any signature morphism $\varphi : \Sigma \to \Sigma'$ and each maximally consistent set of Σ'-sentences E', $\varphi^{-1}E'$ is maximally consistent.

6.20. In any institution show that if $E \models e$ where E is a set of Σ^1_1-sentences and e is preserved by ultrafactors, then there exists a finite subset $E' \subseteq E$ such that $E' \models e$.

6.21. Finitely presented theories (alternative to the result of Ex. 4.23)
In any compact institution with signature morphisms that admit the model expansion property, a theory (Σ, E) is finitely presented if Σ is a finitely presented signature and (Σ, E) can be presented by a finite set of sentences.

6.5 Finitely sized models

The method of ultraproducts can be used to prove that elementary equivalence and the finiteness of the size of the models is a sufficient condition for two models to be isomorphic. This is the main topic of this section.

Although saturated models (introduced in Chap. 7 below) provides a more general framework for such types of results in which the finiteness condition on the size of the models can be relaxed to a much softer condition, for finitely sized models this isomorphism result can be achieved by using ultrapower embeddings within the much simpler-minded framework of this section.

ι-finite models. We introduce a concept that captures abstractly the situation when the carrier sets of models have only a finite set of elements. This is stronger than being finitely presented and weaker than having a finite signature and a finite number of elements. In the latter situation, it is rather easy to show that elementary equivalence implies for the latter case (see Ex. 5.10).

In any institution with diagrams ι, a Σ-model M is ι-*finite* if and only if

- the elementary extension $\iota_\Sigma M$ is finitary, and
- it has a finite number of $\iota_\Sigma M$-expansions.

Fact 6.25. *Consider $\mathcal{FOL}$ with the standard system of diagrams ι. Then a model is ι-finite if and only if it has a finite number of elements.*

Note that under the assumption of non-emptiness of the sorts, having a finite number of elements implies also that the number of the sorts of the signature is also finite. However, no other finiteness restriction is implied, such as on the number of operation or relation symbols.

Dense signature morphisms. The following technical concept is required for the developments in this section. A quasi-representable signature morphism $\chi\colon\ \Sigma \to \Sigma'$ is *dense* when $(Mod\chi)^{-1}\colon\ Mod\Sigma \to \mathbb{C}lass$ is faithful. A typical example of a dense signature morphism is as follows.

Proposition 6.26. *In $\mathcal{FOL}$ any injective signature extension $\chi\colon\ \Sigma \to \Sigma'$ with constants such that χ adds at least one new element for each sort of Σ, is dense.*

Proof. Let us consider Σ-homomorphisms $f,g\colon\ M \to N$ such that $(Mod\chi)^{-1}f = (Mod\chi)^{-1}g$. By *Reductio ad Absurdum* suppose that $f \neq g$. Then there exists a sort s and $a \in M_s$ such that $f_s a \neq g_s a$. Let M' be any χ-expansion of M such that $M'_c = a$ from some constant c of sort s which does not belong to Σ. Then $((Mod\chi)^{-1}f)M'$ and $((Mod\chi)^{-1}g)M'$ interpret c differently, as $f_s a$ and $g_s a$, respectively. Hence $(Mod\chi)^{-1}f \neq (Mod\chi)^{-1}g$ which contradicts our assumption. □

Corollary 6.27. *In $\mathcal{FOL}$, considered with its standard system of diagrams ι, the elementary extension $\iota_\Sigma M$ corresponding to any model M without empty sorts is dense.*

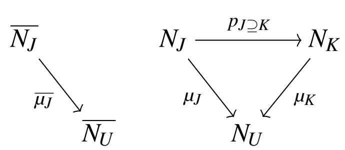
Proposition 6.28. *Let us consider an institution with diagrams ι such that all elementary extensions are quasi-representable dense. For any ι-finite model N, any ultrafilter U over a set I and any ultrapower $(\mu_J\colon\ N_J \to N_U)_{J\in U}$ of N, the canonical homomorphism*

$$d_N^U = (N \xrightarrow{\delta_N^I} N_I \xrightarrow{\mu_I} N_U)$$

is epi.

Proof. Consider two model homomorphisms $f,g\colon\ N_U \to A$ such that $d_N^U;f = d_N^U;g$. We have to prove $f = g$. Because $\iota_\Sigma N$ is quasi-representable and dense, it is enough to prove that for each $\iota_\Sigma N$-expansion $\overline{N_U}$ of N_U, both f and g expand to homomorphisms $\overline{N_U} \to M$.

Because the $\iota_\Sigma N$ is finitary, there exists $J \in U$ and $\overline{\mu_J}\colon\ \overline{N_J} \to \overline{N_U}$ a $\iota_\Sigma N$-expansion of μ_J.



Because $\iota_\Sigma N$ is quasi-representable, for each $K \subseteq J$ in U let us denote

- by $\overline{p_{J\supseteq K}}$ the unique $\iota_\Sigma N$-expansion of the projection $p_{J\supseteq K}$ to a Σ_N-homomorphism $\overline{N_J} \to \overline{N_K}$, and
- by $\overline{\mu_K}$ the unique $\iota_\Sigma N$-expansion of μ_k from $\overline{N_K}$. By the uniqueness property of the quasi-representability applied to $\mu_J = p_{J\supseteq K}\ ;\ \mu_K$ we have that $\overline{\mu_K}\colon\ \overline{N_K} \to \overline{N_U}$.

Let us assume that

1 there exists $K \in U$ such that $K \subseteq J$ and $\overline{\delta_N^K}\colon\ \overline{N} \to \overline{N_K}$ a $\iota_\Sigma N$-expansion of $\delta_N^K\colon\ N \to N_K$.

Then because $d^U_N = \delta^I_N ; \mu_I = \delta^K_N ; \mu_K$ and $\iota_\Sigma N$ is quasi-representable,

2 $\overline{\delta^K_N} ; \overline{\mu_K}$ is the unique $\iota_\Sigma N$-expansion of d^U_N from $\overline{N}$.

By the quasi-representable of $\iota_\Sigma N$ we let $\overline{f} : \overline{N_U} \to M_f$, $\overline{g} : \overline{N_U} \to M_g$ be the unique $\iota_\Sigma N$-expansions from $\overline{N}$ of f, g, respectively. We have only to prove that $M_f = M_g$.

3 $(\overline{\delta^K_N} ; \overline{\mu_K} ; \overline{f})\upharpoonright_{\iota_\Sigma N} = d^U_N ; f,\ (\overline{\delta^K_N} ; \overline{\mu_K} ; \overline{g})\upharpoonright_{\iota_\Sigma N} = d^U_N ; g$ — definitions of $\overline{\delta^K_N}, \overline{\mu_K}, \overline{f}, \overline{g}$

4 $d^U_N ; f = d^U_N ; g$ — by assumption

5 $\overline{\delta^K_N} ; \overline{\mu_K} ; \overline{f} = \overline{\delta^K_N} ; \overline{\mu_K} ; \overline{g}$ — 3, 4, $\iota_\Sigma N$ quasi-representable

6 $M_f = M_g$ — 5.

Modulo the gap represented by the assumption 1 we have just proved the conclusion of the proposition. In order to prove 1, for each $i \in J$ we let $\overline{p_{J,i}} : \overline{N_J} \to N_i$ be the unique $\iota_\Sigma N$-expansion from $\overline{N_J}$ of $p_{J,i} : N_J \to N$. For each $\iota_\Sigma N$-expansion $\overline{N}$ of N we let $J(\overline{N}) = \{i \in J \mid N_i = \overline{N}\}$. Let us prove that

7 there exists $\overline{N}$ such that $J(\overline{N}) \in U$.

By *Reductio ad Absurdum* we assume that 7 is false. Then

8 for each $\overline{N}$, $I \setminus J(\overline{N}) \in U$ — 7 false, U ultrafilter

9 $\bigcap_{\overline{N}} (I \setminus J(\overline{N})) \in U$ — 8, N has a finite number of $\iota_\Sigma N$-expansions, U (ultra)filter

10 $I \setminus \bigcup_{\overline{N}} J(\overline{N}) \in U$ — 9, DeMorgan laws

11 $\bigcup_{\overline{N}} J(\overline{N}) = J$ — definition of $J(\overline{N})$

12 $I \setminus J \in U$ — 10, 11.

But 12 represents a contradiction with $J \in U$ because U is ultrafilter. Hence 7 does hold. Let $K = J(\overline{N})$ from some $\overline{N}$ given by 7. We let $\overline{\delta^K_N} : \overline{N} \to P$ be the unique $\iota_\Sigma N$-expansion from $\overline{N}$ of δ^K_N. If we show that $P = \overline{N_K}$ then 1 is proved. This goes as follows. For each $i \in K$:

13 $\overline{p_{J \supseteq K}}\upharpoonright_{\iota_\Sigma N} = p_{J \supseteq K} = p_{J,i} ; \delta^K_N = \overline{p_{J,i}}\upharpoonright_{\iota_\Sigma N} ; \overline{\delta^K_N}\upharpoonright_{\iota_\Sigma N} = (\overline{p_{J,i}} ; \overline{\delta^K_N})\upharpoonright_{\iota_\Sigma N}$ — $N_i = \overline{N}$

14 $\overline{p_{J \supseteq K}} = \overline{p_{J,i}} ; \overline{\delta^K_N}$ — 13, $\iota_\Sigma N$ quasi-representable

15 $\overline{N_K} = P$ — 14.

□

The following consequence of Prop. 6.28 represents the goal of this section and can be applied easily in actual institutions.

Corollary 6.29. *Consider an institution with diagrams ι such that*

1. *the elementary extensions are quasi-representable and dense,*
2. *it has ultraproducts of models which are preserved by the elementary extensions,*

3. *each sentence is preserved by ultraproducts of models, and*
4. *each elementary homomorphism which is an epi is an isomorphism.*

Then any finitely sized model is isomorphic to any of its ultrapowers. Moreover, if in addition the institution

5. *has finite conjunctions, and*
6. *has existential $\mathcal{D}$-quantification (for a class $\mathcal{D}$ of signature morphisms) such that*
7. *each finitary elementary extension belongs to $\mathcal{D}$,*

then any two elementary equivalent finitely sized models are related by a ι-elementary homomorphism.

Proof. The first part follows immediately from 6.28 and the filtered power embedding result of Prop. 6.15.

For the second part, let $M \equiv N$ be finitely sized models. For each finite $E \subseteq (M_M)^*$ we have

1	$M \models (\exists \iota_\Sigma M) \wedge E$	$M_M \models E$
2	$N \models (\exists \iota_\Sigma M) \wedge E$	1, $M \equiv N$

From 2 we obtain the existence of a $\iota_\Sigma M$-expansion N_E of N such that $N_E \models E$.

By Thm. 6.19, there exists an ultrafilter U on the set I of all finite subsets E of M_M^* and an ultraproduct $(\mu_J : N_J \to N_U)_{J \in U}$ of $(N_E)_{E \in I}$ such that $N_U \models (M_M)^*$. Then

1	$N_U \models E_M$	$E_M \subseteq (M_M)^*$
2	$i_{\Sigma,M} N_U : M \to N_U \upharpoonright_{\iota_\Sigma M}$ is ι-elementary	1, $N_U \models (M_M)^*$
3	$\mu \upharpoonright_{\iota_\Sigma M}$ is ultrapower of N	$\iota_\Sigma M$ preserves ultraproducts, $N_E \upharpoonright_{\iota_\Sigma M} = N$
4	$N \cong N_U \upharpoonright_{\iota_\Sigma M}$	3, first part of this proposition

From 2 and 4 we get a ι-elementary homomorphism $M \to N$. □

A typical concrete application of Cor. 6.29 is the following.

Corollary 6.30. *Any $\mathcal{FOL}$ any finite*[1] *model with non-empty sorts is isomorphic to any of its ultrapowers. Any two elementary equivalent finite $\mathcal{FOL}$-models with non-empty sorts are isomorphic.*

Proof. First recall that according to Fact 6.25 in $\mathcal{FOL}$, considered with the standard system of diagrams ι, a ι-finite model means a model having a finite number of elements. Then only two points may need a bit of additional explanation.

- The first one concerns the fact that in $\mathcal{FOL}$, the elementary embedding homomorphisms which are also epis are in fact isomorphisms. In $\mathcal{FOL}$ the elementary embeddings are injective and closed, and the injective closed epis are isomorphisms.

[1] With a finite number of elements.

- The second point concerns how to derive the isomorphism between two elementary equivalent finite models from the final conclusion of Cor. 6.29. This follows easily by cardinality reasons because the $\mathcal{FOL}$ elementary embeddings are injective.

□

Exercises

6.22. Derive results similar to Cor. 6.30 in actual institutions that have been presented in this book, other than $\mathcal{FOL}$.

Notes. The F-product construction from conventional model theory (see Chap. 4 of [42]) has been introduced in [163] and probably defined categorically for the first time in [171]. Categorical F-products have been intensively used in categorical logic and model theory works such as [9] or [166, 167]. The equivalence between the category-theoretic and the set-theoretic definitions of the F-products appears in [134]. Filtered products are sometimes known under the name of *reduced products*, such as in [42].

The fundamental ultraproducts theorem is the foundation for the method of ultraproducts in conventional model theory [42] and has been stated for the first time in [163]. [18] is an exposition of that part of conventional model theory that can be reached using only ultraproducts. A rather different abstract model-theoretic approach to the fundamental ultraproducts theorem based on satisfaction by injectivity is given in [8]. Our approach originates from [63] and contrasts [8] by making essential use of concepts central to institution theory, such as signature morphisms and model reducts. This multi-signature framework, very characteristic of institution theory, leads to higher generality and simpler proofs. The notion of 'inventing' of F-products was formulated first time in [63] under the name 'lifting', a terminology that clashes with the well-established concept of the lifting of (co-)limits.

The institution-independent results on compactness by ultraproducts essentially constitute a generalization of similar ultraproduct-based compactness results from conventional model theory [42]. The ι-finite models have been introduced in an institution-independent setting and their isomorphism criterion has been developed in [202].

The original reference on the hyperreals and their associated non-standard analysis is [209]. An excellent exposition on this subject, which is also more recent is [132].

Chapter 7

Saturated Models

A lot of deep results in model theory can be reached by the method of saturated models. Two of the most useful properties of saturated models are their existence and their uniqueness. The existence means that each model can be elementarily extended to a saturated model, while uniqueness holds when the model is 'sufficiently' small. The main topic of this chapter is the axiomatic investigation of general frameworks supporting these two properties, and of some important applications.

The existence property of saturated models requires that directed co-limits of diagrams of elementary homomorphisms are still elementary. This is treated in the first section of this chapter. This important preservation property of elementary homomorphisms, which is due to Tarski in the conventional concrete setting of $\mathcal{FOL}$, will be used for several results in other chapters too.

An important class of applications can be developed in conjunction with the method of ultraproducts. In the last section, we show that for certain ultrafilters, the corresponding ultraproducts of models are always saturated. Assuming the Generalized Continuum Hypothesis, this leads to one of the most beautiful applications of saturated models in first-order model theory, namely the Keisler-Shelah isomorphism theorem saying that "two models are elementary equivalent if and only if they have isomorphic ultrapowers". Apart from its theoretical significance this has several important applications, such as to axiomatizability and interpolation. We develop a general institution-independent version of this result.

7.1 Elementary co-limits

For this section, we assume an arbitrary institution with a designated (sub-)category $\mathcal{D}$ of quasi-representable signature morphisms.

Sentences preserved (reflected) by directed co-limits of $\mathcal{D}$-elementary homomorphisms

The preservation and the reflection of sentences by directed co-limits of $\mathcal{D}$-elementary homomorphisms have the flavour of dual properties. Both constitute auxiliary concepts that enable the development of the institution-independent version of Tarski's elementary chain theorem. We say that a Σ-sentence ρ is *preserved (reflected) by directed co-limits of $\mathcal{D}$-elementary homomorphisms* if for each directed diagram of $\mathcal{D}$-elementary Σ-model homomorphisms $(f_{i,j})_{(i<j)\in(I,\leq)}$ with co-limit μ,

$$\begin{array}{ccc} M_i & \xrightarrow{f_{i,j}} & M_j \\ & {\scriptstyle \mu_i}\searrow \quad \swarrow{\scriptstyle \mu_j} & \\ & M & \end{array} \tag{7.1}$$

$M_i \models \rho$ implies $M \models \rho$ ($M \models \rho$ implies $M_i \models \rho$) for each $i \in |I|$. Note that in the case of preservation the mere existence of an i such that $M_i \models \rho$ guarantees that $M_j \models \rho$ for each $j \geq i$ and that $M \models \rho$. On the other hand, in the case of reflection if $M \models \rho$ then *all* models M_i of the diagram satisfy ρ.

Theorem 7.1. *The set of sentences $\mathcal{D}$-elementary preserved by directed co-limits of $\mathcal{D}$-elementary homomorphisms:*

1. *contains all basic sentences,*
2. *is closed under (possibly infinite) conjunctions and disjunctions,*
3. *is closed under existential $\mathcal{D}$-quantifications, and*
4. *is closed under finitary universal $\mathcal{D}$-quantifications.*

Proof. Without any loss of generality, we can fix a co-limit μ of a directed diagram of $\mathcal{D}$-elementary Σ-homomorphisms like in (7.1).

1. Consider ρ a basic Σ-sentence having M_ρ as a basic model. If $M_i \models \rho$ from some $i \in |I|$ then there exists a homomorphism $M_\rho \to M_i$, which implies that there exists a homomorphism $M_\rho \to M_i \xrightarrow{\mu_i} M$. Hence $M \models \rho$.
2. Consider E a set of Σ-sentences that are preserved by μ and e' a conjunction of E. If $M_i \models e'$ from some $i \in |I|$ then

1	for each $e \in E$, $M_i \models e$	e' is a conjunction of E
2	for each $e \in E$, $M \models e$	1, e preserved by μ
3	$M \models e'$	2, e' is a conjunction of E.

The proof for disjunctions is similar.

3. Consider $(\chi\colon \Sigma \to \Sigma') \in \mathcal{D}$, $e' \in Sen\Sigma'$ which is preserved by directed co-limits of $\mathcal{D}$-elementary homomorphisms, and ρ an existential χ-quantification of ρ'. Assume $M_i \models \rho$ for some $i \in I$. We have to prove that $M \models \rho$.

1	there exists M'_i s.th. $M'_i \models \rho'$, $M'_i\upharpoonright_\chi = M_i$	$M_i \models \rho$, ρ universal χ-quantification of ρ'
2	there exists f' χ-expansion of $(f_{j,k})_{i\leq j<k}$	χ quasi-representable, $(I,\leq)$ directed
3	each $f'_{j,k}$ $\mathcal{D}$-elementary	each $f_{j,k}$ $\mathcal{D}$-elementary, Prop. 5.31(3.)
4	there exists μ' χ-expansion of $(\mu_j)_{i\leq j}$	χ quasi-representable, $(I,\leq)$ directed
5	$(f_{j,k})_{i\leq j<k}$ final sub-diagram of f	
6	μ' co-limit of f'	5, Thm. 2.4 (co-limit of final functors), Prop. 6.9 (creation of directed co-limits by quasi-representable signature morphisms).

Let M' be the vertex of μ'. Then

7	$M' \models \rho'$	$M'_i \models \rho'$, 3, 6, ρ' preserved by μ'
8	$M \models \rho$	$M'\upharpoonright_\chi = M$, ρ existential χ-quantification of ρ'.

4. Consider $(\chi\colon \Sigma \to \Sigma') \in \mathcal{D}$ finitary, $e' \in Sen\Sigma'$ which is preserved by directed co-limits of $\mathcal{D}$-elementary homomorphisms, and ρ an universal χ-quantification of ρ'. Assume that $M_i \models \rho$ for some $i \in I$. In order to prove $M \models \rho$ we consider any χ-expansion M' of M and prove that $M' \models \rho'$.

9	$\exists j \geq i, \mu'_j\colon M'_j \to M', \mu'_j\upharpoonright_\chi = \mu_j$	χ finitary quasi-representable, $(I,\leq)$ directed
10	$M_j \models \rho$	$i \leq j$, $M_i \models \rho$, $f_{i,j}$ $\mathcal{D}$-elementary, $1_\Sigma \in \mathcal{D}$
11	there exists f' χ-expansions of $(f_{j,\ell})_{j\leq k<\ell}$, μ' of $(\mu_k)_{j\leq k}$ s.th. μ' co-limit of f'	9; like in 2, 4, 5, 6
12	f' consists of $\mathcal{D}$-elementary homomorphisms	like in 3
13	$M'_j \models \rho'$	10, $M'_j\upharpoonright_\chi = M_j$, ρ universal χ-quantification of ρ'
14	$M' \models \rho'$	13, μ' directed co-limit of f' (11), 12, ρ' preserved by μ'.

□

Negations turn preservations to reflection and vice versa as shown by the following result that enables the extension of the result of Thm. 7.1 to institutions with negations.

Theorem 7.2. *If the institution has negations, then*

1. *each finitary basic sentence is reflected by directed co-limits of $\mathcal{D}$-elementary homomorphisms,*

2. *the set of sentences reflected by directed co-limits of $\mathcal{D}$-elementary homomorphisms are closed under (possibly infinite) conjunctions, and*

3. *if a sentence e is preserved (reflected) by directed co-limits of $\mathcal{D}$-elementary homomorphisms then any of its negations e' is preserved (reflected) by directed co-limits of $\mathcal{D}$-elementary homomorphisms.*

Proof. Like in the proof of Thm. 7.1 we fix a co-limit μ of a directed diagram $(f_{i,j})_{(i<j)\in(I,\leq)}$ of $\mathcal{D}$-elementary homomorphisms.

1. Consider a finitary basic Σ-sentence ρ and M_ρ a basic model for ρ. Assume that $M \models \rho$ and consider any $i \in I$. We prove that $M_i \models \rho$.

1	there exists a homomorphism $M_\rho \to M$	$M \models \rho$
2	there exists $j \geq i$, $h: M_\rho \to M_j$	M_ρ finitely presented, $(I,\leq)$ directed
3	$M_j \models \rho$	2.

By *Reductio ad Absurdum* suppose that $M_i \not\models \rho$. Because the institution has negations there exists a sentence ρ' which is a negation of ρ. Then

4	$M_i \models \rho'$	$M_i \not\models \rho$, ρ' negation of ρ
5	$M_j \models \rho'$	4, $f_{i,j}$ $\mathcal{D}$-elementary, $1_\Sigma \in \mathcal{D}$.

Since 3 and 5 are contradictory we conclude that $M_i \models \rho$.

2. Straightforward by *Reductio ad Absurdum* and by using negation.
3. Straightforward by *Reductio ad Absurdum*.

□

Elementary co-limit theorem

The following institution-independent version of the corresponding Tarski's result in first-order model theory falls immediately from the concept of preservation alone. This means that the concept of reflection is secondary to that of preservation, its role is to support preservation in the presence of negations.

Proposition 7.3. *Assume that all sentences of the institution are preserved by directed co-limits of $\mathcal{D}$-elementary homomorphisms. Then, for each signature Σ, any co-limit of a directed diagram of $\mathcal{D}$-elementary Σ-homomorphisms is $\mathcal{D}$-elementary.*

Proof. Let $(f_{i,j})_{(i\leq j)\in(I,\leq)}$ be a directed diagram of $\mathcal{D}$-elementary Σ-homomorphisms and let μ be its co-limit. For each $k \in I$, in order to prove that μ_k is $\mathcal{D}$-elementary, let $\chi \in \mathcal{D}$ and let $\mu'_k : M'_k \to M'$ be a χ-expansion of μ_k. Let ρ' be a Σ'-sentence such that $M'_k \models \rho'$. We have to show that $M' \models \rho'$.

As in the proof of Thm. 7.1(3.) we can χ-expand the final sub-diagram $(f_{i,j})_{k\leq i<j}$ of f to f' and μ to a co-limit μ' of f'. Also like there f' consists of $\mathcal{D}$-elementary homomorphisms. Thus ρ' is preserved by μ', hence $M' \models \rho'$. □

Altogether Prop. 7.3 and Thm.s 7.1 and 7.2 lead to the following result that represents an institution-independent version of Tarski's elementary chain theorem in first-order logic. This is less abstract that Prop. 7.3 and can be applied almost without any additional effort to actual institutions.

Corollary 7.4. *Assume the institution satisfies one of the following:*

1. *each sentence is accessible from the basic ones by (possibly infinite) conjunctions, disjunctions, universal $\mathcal{D}$-quantifications, and finitary existential $\mathcal{D}$-quantifications, or*

2. *the institution has negations and each sentence is accessible from the finitary basic ones by (possibly infinite) conjunctions, negations, and finitary $\mathcal{D}$-quantifications.*

Then any co-limit of a directed diagram of $\mathcal{D}$-elementary homomorphisms is $\mathcal{D}$-elementary.

When in addition the institution has $\mathcal{D}$-normal diagrams such that all elementary extensions are in $\mathcal{D}$, by Cor. 5.35, in the above Cor. 7.4 we may replace '$\mathcal{D}$-elementary' just by 'ι-elementary'.

A typical concrete instance of Cor. 7.4 is obtained by taking $\mathcal{D}$ to the class of all $\mathcal{FOL}$-signature injective extensions with constants.

Corollary 7.5. *In $\mathcal{FOL}$, $\mathcal{EQLN}$, $\mathcal{FOL}^+$, and $\mathcal{EQL}$, the class of elementary homomorphisms is closed under directed co-limits.*

Exercises

7.1. Develop a general consequence of Prop. 7.3 that establishes the existence of directed co-limits in the category of $\mathcal{D}$-elementary homomorphisms.

7.2 Existence of saturated models

In this section, we introduce the concept of a saturated model and develop the fundamental existence theorem for saturated models. We start with a brief survey of some basic set-theoretic notions required by the concept of saturated models.

Some set theory

For a gentle introduction to (axiomatic) set theory, we recommend [240]. In the remaining part of this chapter, we will involve some concepts and results from set theory that are beyond what is required in most mathematical works. The theory of saturated models is 'guilty' of this. Especially the results stated in Prop. 7.6 below are non-trivial and we cannot explain them here. We have to take them for granted, but of course, the keen reader may go to the literature and study them.

Ordinals. We skip the formal lists of axioms for set theories such as Zermelo, Zermelo-Fraenkel, Bernays, or Bernays-Morse which can be found in the rather rich set theory literature, and just recall the concept of ordinal from the point of view of formal set theory:

- $0 = \emptyset$,
- $n+1 = n \cup \{n\}$ for each natural number n,
- $\omega = \{0,1,2,\dots\}$ the set of all natural numbers,
- $\omega+1 = \omega \cup \{\omega\} = \{0,1,2,\dots,\omega\}$, etc.

All these are examples of ordinals. Formally, an *ordinal* is a set X such that X and each member of X is $\in$*-transitive*, i.e., every member of x is a subset of x. Although this definition might seem quite artificial, it has become fairly standard in the literature. The underlying intuition is that an ordinal is a special kind of ordering (by the membership relation $\in$), its definition as a certain kind of set being just a trick.

One of the important properties of ordinals is that they are well ordered, i.e., totally ordered and any non-empty class of ordinals has a least element. Ordinals support the following *Principle of Transfinite (or Ordinal) Induction:*

$$[(\forall\alpha)((\forall\beta < \alpha)P(\beta)) \Rightarrow P(\alpha)] \Rightarrow (\forall\alpha)P(\alpha)$$

for each property P on ordinals.

For each ordinal α, let $\alpha+1 = \alpha \cup \{\alpha\}$ be its *successor ordinal*. If α is neither a successor ordinal nor 0, we say that α is a *limit ordinal*.

Cardinals. Cardinal numbers are essentially equivalence classes, or representatives of equivalence classes, of sets under the bijection relation. For each set X, let $card(X)$ denote its *cardinality*. An ordering between cardinals can be defined by $card(X) \leq card(Y)$ if and only if there exists an injective function $X \to Y$.

When we take the point of view of cardinals as representatives of equivalence classes, we can formally define *cardinals* as the smallest ordinals α which are in bijective correspondence to $card(\alpha)$. For example, ω is a cardinal while $\omega+1$ is not. Infinite cardinals are always limit ordinals.

Basic arithmetic operations on cardinals can be defined by

- $\alpha+\beta = card(\alpha \uplus \beta)$,
- $\alpha \cdot \beta = card(\alpha \times \beta)$, and
- $\alpha^\beta = card\{f \text{ function} \mid f : \beta \to \alpha\}$.

For each ordinal α the least cardinal greater than α is denoted by α^+. The *Generalized Continuum Hypothesis* (abbreviated *GCH*) states that for every infinite cardinal α, $2^\alpha = \alpha^+$. The ordinary Continuum Hypothesis represents the particular case of GCH when $\alpha = \omega$. Since $card(\mathbb{R}) = 2^\omega$ this means that between ω and $card(\mathbb{R})$ there is no other cardinal. The following is a list of well-known cardinal arithmetic properties which will be used in this chapter. More on cardinal arithmetic can be found in [151].

Proposition 7.6 (Cardinal arithmetic)**.**

- *if* $\omega \leq \alpha$ *then* $\alpha \cdot \alpha = \alpha$*,*
- *if* $2 \leq \alpha \leq \beta$ *and* $\omega \leq \beta$ *then* $\alpha^{\beta} = 2^{\beta}$*,*
- *if* $\alpha \leq \beta^{+}$ *then* $\alpha^{\beta} \leq \beta^{+}$ *(requires GCH).*

Saturated models

Now we introduce the institution-independent concept of saturated model as an abstraction of the corresponding concept from classical first-order model theory.

Chains. In any category $\mathbb{C}$, for any ordinal λ, a λ*-chain* f is a (commutative) diagram, or functor, $\lambda \to \mathbb{C}$, written $(f_{i,j} : A_i \to A_j)_{i<j\leq\lambda}$, such that for each limit ordinal $\zeta \leq \lambda$, $(f_{i,\zeta})_{i<\zeta}$ is the co-limit of $(f_{i,j})_{i<j<\zeta}$. Note that the commutativity of the chain, which is implicit by functoriality, just means that $f_{i,j}; f_{j,k} = f_{i,k}$ for all $i < j < k \leq \lambda$.

For any class of arrows $\mathcal{D} \subseteq \mathbb{C}$, a $(\lambda, \mathcal{D})$*-chain* is any λ-chain $(f_{i,j})_{i<j\leq\lambda}$ such that $f_{i,i+1} \in \mathcal{D}$ for each $i < \lambda$. We say that an *arrow* h *is a* $(\lambda, \mathcal{D})$*-chain* if there exists a $(\lambda, \mathcal{D})$-chain $(f_{i,j})_{i<j\leq\lambda}$ such that $h = f_{0,\lambda}$. In that case we may denote $f_{i,j}$ by $h_{i,j}$, for any $i < j \leq \lambda$.

The definition of the concrete concept of saturated model from the first-order model theory uses extensions of signatures with sequences of variables that are not necessarily countable. Our $(\lambda, \mathcal{D})$-chains realize that in the abstract institution-independent context by abstracting (an extension by a) single variables by signature morphisms of $\mathcal{D}$. From this perspective, it is clear that gradually we will have to assume some properties for $\mathcal{D}$ that will get this closer to the first order variables, such as quasi-representability and finiteness. The following technical result represents a trivial situation in first-order model theory, but at the abstract level is not only far from being trivial but it also reveals a dependence on the amalgamation properties of the institution.

Proposition 7.7. *Consider an institution I which has inductive co-limits of signatures and which is inductive-exact. Let $\mathcal{D}$ be a class of quasi-representable signature morphisms. Then each signature morphism φ which is a $(\lambda, \mathcal{D})$-chain is quasi-representable.*

Proof. Let $\varphi = \varphi_{0,\lambda}$ where $(\varphi_{i,j} : \Sigma_i \to \Sigma_j)_{i<j\leq\lambda}$ is a $(\lambda, \mathcal{D})$-chain. We prove by Ordinal Induction that for any $\alpha \leq \lambda$, $\varphi_{0,\alpha}$ is quasi-representable. Let us asume that for each $\beta < \alpha$, $\varphi_{0,\beta}$ is quasi-representable. We have two cases:

- α is a succesor ordinal, i.e., $\alpha = \beta + 1$. Then $\varphi_{0,\alpha} = \varphi_{0,\beta} ; \varphi_{\beta,\beta+1}$. Since both $\varphi_{0,\beta}$ and $\varphi_{\beta,\beta+1}$ are quasi-representable (the former by the induction hypothesis and the latter by hypothesis), by Prop. 5.12 (1.) we get that their composition is quasi-representable too.
- α is a limit ordinal. Since φ is a $(\lambda, \mathcal{D})$-chain, $(\varphi_{i,\alpha})_{i<\alpha}$ is a co-limit of $(\varphi_{i,j})_{i<j<\alpha}$. By the induction hypothesis $\varphi_{0,i}$ and $\varphi_{0,j}$, $i < j < \alpha$, are quasi-representable, hence by Prop. 5.12 (4. applied to the situation $\varphi_{0,i} ; \varphi_{i,j} = \varphi_{0,j}$) we obtain that $\varphi_{i,j}$ is

quasi-representable too. The last part of this argument relies on the hypothesis that I is inductive-exact. Then by Prop. 5.12(3.) it follows that each $\varphi_{i,\alpha}$, $i < \alpha$, is quasi-representable, hence in particular $\varphi_{0,\alpha}$ is quasi-representable.

□

λ-small signature morphisms. This concept represents a sort of generalisation of the concept of finitary signature morphism to infinite cardinals. In any institution a signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ is λ-*small* for a cardinal λ when for each λ-chain $(f_{i,j}\colon\ M_i \to M_j)_{i<j\leq\lambda}$ of Σ-homomorphisms and each φ-expansion M' of M_λ, there exists $i < \lambda$ and a φ-expansion $f'_{i,\lambda}\colon\ M'_i \to M'$ of $f_{i,\lambda}$. The following is an obvious example:

Fact 7.8. *Finitary signature morphisms are λ-small for each infinite cardinal λ.*

The following result constitutes one of the causes of the existence of saturated models.

Proposition 7.9. *Consider an institution I which has inductive co-limits of signatures and which is inductive-exact. Let $\mathcal{D}$ be a class of finitary quasi-representable signature morphisms. Then for each infinite ordinal λ, each $(\lambda,\mathcal{D})$-chain of signature morphisms is λ^+-small.*

Proof. Consider a $(\lambda,\mathcal{D})$-chain of signature morphisms $\varphi\colon\ \Sigma \to \Sigma'$ and consider a λ^+-chain of Σ-model homomorphisms $h_{i,j}\colon\ M_i \to M_j)_{i<j\leq\lambda^+}$. Let $M^\lambda_{\lambda^+}$ be a φ-expansion of M_{λ^+}. For each $0 \leq i < j \leq \lambda$, let $\varphi_{i,j}\colon\ \Sigma_i \to \Sigma_j$ denote the segment in the $(\lambda,\mathcal{D})$-chain φ determined by i and j.

By Ordinal Induction on $\alpha \leq \lambda$ we define an increasing sequence of ordinals $(i_\alpha)_{\alpha\leq\lambda}$, strictly bounded by λ^+, and an inductive diagrams $(h^\alpha_{j,k}\colon\ M^\alpha_j \to M^\alpha_k)_{i_\alpha\leq j<k\leq\lambda^+}$ in $Mod\Sigma_\alpha$ such that $M^\alpha_{\lambda^+}\upharpoonright_{\varphi_{\beta,\alpha}} = M^\beta_{\lambda^+}$ and $h^\alpha_{j,k}\upharpoonright_{\varphi_{\beta,\alpha}} = h^\beta_{j,k}$ for all $0 \leq \beta < \alpha \leq \lambda$ and $i_\alpha \leq j < k$.

$$\begin{array}{ccccccccccc} \Sigma_\beta & & M^\beta_{i_\beta} & \xrightarrow{h^\beta_{i_\beta,i_\alpha}} & M^\beta_{i_\alpha} & \xrightarrow{h^\beta_{i_\alpha,j}} & M^\beta_j & \xrightarrow{h^\beta_{j,k}} & M^\beta_k & \xrightarrow{h^\beta_{k,\lambda^+}} & M^\beta_{\lambda^+} \\ {\scriptstyle\varphi_{\beta,\alpha}}\downarrow & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \Sigma_\alpha & & & & M^\alpha_{i_\alpha} & \xrightarrow[h^\alpha_{i_\alpha,j}]{} & M^\alpha_j & \xrightarrow[h^\alpha_{j,k}]{} & M^\alpha_k & \xrightarrow[h^\alpha_{k,\lambda^+}]{} & M^\alpha_{\lambda^+} \end{array}$$

If we achieved that then our problem is solved because $h^\lambda_{\lambda,\lambda^+}\colon\ M^\lambda_\lambda \to M^\lambda_{\lambda^+}$ is a φ-expansion of $h_{\lambda,\lambda^+}\colon\ M_\lambda \to M_{\lambda^+}$. The construction goes as follows.

- $i_0 = 0$ and $h^0_{j,k} = h_{j,k}$ for all $j < k \leq \lambda^+$.
- Assume that $\alpha = \beta + 1$ is a successor ordinal. Then

1 $(h_{i,\lambda^+})_{i_\beta\leq i<\lambda^+}$ co-limit of $(h_{j,k})_{i_\beta\leq j<k<\lambda^+}$ $\quad$ $[i_\beta,\lambda^+)$ final sub-poset of $[0,\lambda^+)$, Thm. 2.4

2 $\varphi_{0,\beta}$ quasi-representable $\quad$ Prop. 7.7

3	$h^{\beta}_{j,k}\restriction_{\varphi_{0,\beta}} = h_{j,k}$	induction hypothesis
4	$(h^{\beta}_{i,\lambda^+})_{i_\beta \leq i < \lambda^+}$ co-limit of $(h^{\beta}_{j,k})_{i_\beta \leq j < k < \lambda^+}$	1, 2, 3, Prop. 6.9.

We define $M^{\alpha}_{\lambda^+} = M^{\lambda}_{\lambda^+}\restriction_{\varphi_{\alpha,\lambda}}$. Then

5	$M^{\alpha}_{\lambda^+}\restriction_{\varphi_{\beta,\alpha}} = M^{\lambda}_{\lambda^+}\restriction_{\varphi_{\alpha,\lambda}}\restriction_{\varphi_{\beta,\alpha}} = M^{\lambda}_{\lambda^+}\restriction_{\varphi_{\beta,\lambda}} = M^{\beta}_{\lambda^+}$	definition of $M^{\alpha}_{\lambda^+}$, induction hypothesis
6	$\varphi_{\beta,\alpha}$ finitary	$\varphi_{\beta,\alpha} \in \mathcal{D}$ because $\alpha = \beta + 1$
7	there exists $i_\beta \leq i_\alpha < \lambda^+$, $h^{\alpha}_{i_\alpha,\lambda^+} : M^{\alpha}_{i_\alpha} \to M^{\alpha}_{\lambda^+}$ s.th. $h^{\alpha}_{i_\alpha,\lambda^+}\restriction_{\varphi_{\beta,\alpha}} = h^{\beta}_{i_\alpha,\lambda^+}$	4, 5, 6.

By the quasi-representability of $\varphi_{\beta,\alpha}$, by Ordinal Induction, $h^{\alpha}_{i_\alpha,\lambda^+}$ determines an unique $\varphi_{\beta,\alpha}$-expansion $(h^{\alpha}_{j,k})_{i_\alpha \leq j < k < \lambda^+}$ of $(h^{\beta}_{j,k})_{i_\alpha \leq j < k < \lambda^+}$.

- Assume that α is a limit ordinal. It is straightforward to check that $\bigcup_{\beta<\alpha} i_\beta$ is an ordinal. We define $i_\alpha = \bigcup_{\beta<\alpha} i_\beta$. Then

8	forall $\beta < \alpha$, *card* $i_\beta \leq$ *card* λ	$i_\beta < \lambda^+$ (induction hypothesis)
9	*card* $i_\alpha \leq$ *card*$\alpha \cdot$ *card*λ	definition of i_α, 8
10	*card*$\lambda \cdot$ *card*$\lambda =$ *card*$\lambda < \lambda^+$	λ infinite, Prop. 7.6
11	*card* $i_\alpha < \lambda^+$	9, 10, $\alpha \leq \lambda$.

Hence $i_\alpha < \lambda^+$. Now for all $i_\alpha \leq j < k \leq \lambda^+$, by inductive-exactness we define $h^{\alpha}_{j,k}$ to be the amalgamation of $(h^{\beta}_{j,k})_{\beta<\alpha}$.

□

$(\lambda, \mathcal{D})$-saturated models. For each signature morphism $\chi\colon\ \Sigma \to \Sigma'$, a Σ-model M χ-*realizes* a set E' of Σ'-sentences (denoted $M \models [\exists\chi]E'$), if there exists a χ-expansion M' of M which satisfies E'. It χ-*realizes* E' *finitely* (denoted $M \models [\exists\chi]_f E'$) if it realizes every finite subset of E'.

A Σ-model M is $(\lambda, \mathcal{D})$-*saturated* for λ a cardinal and $\mathcal{D}$ a class of signature morphisms when for each ordinal $\alpha < \lambda$ and each $(\alpha, \mathcal{D})$-chain of signature morphisms $(\varphi_{i,j}\colon\ \Sigma_i \to \Sigma_j)_{i<j\leq\alpha}$ with $\Sigma_0 = \Sigma$, for each $(\chi\colon\ \Sigma_\alpha \to \Sigma') \in \mathcal{D}$, each $\varphi_{0,\alpha}$-expansion of M χ-realizes any set of Σ'-sentences if it χ-realizes it finitely.

$$\Sigma \xrightarrow{\varphi_{0,\alpha}} \Sigma_\alpha \xrightarrow{\chi} \Sigma'$$

$$M \xleftarrow[Mod\varphi_{0,\alpha}]{} M_\alpha \qquad M_\alpha \models [\exists\chi]E' \text{ if } M_\alpha \models [\exists\chi]_f E'.$$

An immediate example of saturated models is given by the finite $\mathcal{FOL}$-models (which according to Fact 6.25 are the ι-finite models in $\mathcal{FOL}$).

Proposition 7.10. *Let $\mathcal{D}$ be the class of $\mathcal{FOL}$ injective signature extensions with a finite number of constants, and let λ be an infinite cardinal.*

1. *Any $\mathcal{FOL}$ model with non-empty sorts which has a finite number of elements is $(\lambda,\mathcal{D})$-saturated.*
2. *For each $(\lambda,\mathcal{D})$-saturated $\mathcal{FOL}$-model M and for each sort s, if M_s is infinite then $card(M_s) \geq \lambda$.*

Proof. 1. Let $\varphi : \Sigma_0 \to \Sigma_\alpha$ be a $(\alpha,\mathcal{D})$-chain for $\alpha < \lambda$, and let M_α be a φ-expansion of a Σ_0-model M with non-empty sorts. Let $(\chi : \Sigma_\alpha \to \Sigma') \in \mathcal{D}$. Assume that $M_\alpha \models [\exists\chi]_f E'$ for $E' \subseteq Sen\Sigma'$ and for each finite $i \subseteq E'$ let M^i be a χ-expansion of M_α such that $M^i \models_{\Sigma'} i$. By Thm. 6.19, there exists an ultrafilter U on the set $\mathcal{P}_\omega E'$ of the finite subsets of E' and an ultraproduct $(\mu_J : M_J \to M_U)_{J\in U}$ of $(M^i)_{i\in\mathcal{P}_\omega E}$ such that $M_U \models E'$. Then

1	$\mu\!\upharpoonright_\chi$ is ultrapower of $(M_\alpha)_{i\in\mathcal{P}_\omega E}$	$Mod\chi$ pres. ultraproducts (Prop. 6.7, 6.9), $M^i\!\upharpoonright_\chi = M_\alpha$
2	$M_U\!\upharpoonright_\chi \cong M_\alpha$	1, M_α finite, Cor. 6.30
3	there exists M', $M'\!\upharpoonright_\chi = M_\alpha$, $M_U \cong M'$	2, χ quasi-representable
4	$M_\alpha \models [\exists\chi]E'$	3, $M_U \models E'$, satisfaction invariant under model isomorphisms.

2. Let Σ be the signature of M. By *Reductio ad Absurdum* let us assume that $card(M_s) < \lambda$ for some sort s for which M_s is infinite. Then we take the $(card(M_s),\mathcal{D})$-chain given by the signature extension with constants $\Sigma \hookrightarrow \Sigma + M_s$, and let χ be the extension of $\Sigma + M_s$ with one new constant x. Consider
   - $E' = \{x \neq m \mid m \in M_s\}$ and
   - M' the $(\Sigma + M_s)$-expansion of M such that $M'_m = m$ for each $m \in M_s$.

   Then $M' \models [\exists\chi]_f E'$ but $M' \not\models [\exists\chi]E'$, which contradicts the fact that M is $(\lambda,\mathcal{D})$-saturated.

□

Existence theorem

Let us say that an institution *has $\mathcal{D}$-saturated models* if for any cardinal λ and for each Σ-model M there exists a Σ-homomorphism $M \to N$ such that $M \equiv N$ and N is $(\lambda,\mathcal{D})$-saturated.

Thm. 7.11 below on the existence of saturated models comes up with a rather long set of conditions of mixed degrees of difficulty in the applications. While some of these are straightforward in actual institutions of interest, others require some more substantial justification. After Thm. 7.11 we will discuss in detail its underlying hypotheses, one by one.

Theorem 7.11 (Existence of saturated models)**.** *Consider an institution I with a designated sub-category $\mathcal{D}$ of signature morphisms that contains all isomorphisms. Let us assume that:*

1. *$M \equiv N$ if there exists a model homomorphism $M \to N$.*
2. *I has finite conjunctions and existential $\mathcal{D}$-quantifications.*
3. *I has inductive co-limits of signatures and is inductive-exact.*
4. *For each I-signature Σ, the category of Σ-models has inductive co-limits.*
5. *For each signature morphism $(\chi\colon\ \Sigma \to \Sigma') \in \mathcal{D}$ and set E' of Σ'-sentences, if $M \models [\exists\chi]_f E'$ then there exists a model homomorphism $M \to N$ such that $N \models [\exists\chi]E'$.*
6. *For each signature morphism $(\chi\colon\ \Sigma \to \Sigma') \in \mathcal{D}$ and each Σ-model M, the class of the χ-expansions of M form a set.*
7. *Each signature morphism from $\mathcal{D}$ is quasi-representable.*
8. *The category Sig^I is $\mathcal{D}$-co-well-powered.*
9. *For each ordinal λ there exists a cardinal α such that each signature morphism that is a $(\lambda,\mathcal{D})$-chain is α-small.*

Then I has $\mathcal{D}$-saturated models.

Proof. We develop a stepwise proof.

- First we prove that for each Σ-model M

1 There exists a Σ-homomorphism $h\colon\ M \to N$ such that for each $(\lambda,\mathcal{D})$-chain $\varphi\colon\ \Sigma \to \Sigma'$, each $(\chi\colon\ \Sigma' \to \Sigma'') \in \mathcal{D}$, each φ-expansion M' of M, and each set E'' of Σ''-sentences such that $M' \models [\exists\chi]_f E''$ then $N' \models [\exists\chi]E''$, where $h'\colon\ M' \to N'$ is the unique φ-expansion of h from M' (as given by Prop. 7.7).

$$
\begin{array}{ccc}
\Sigma & \qquad & M \xrightarrow{\ h\ } N \\
\varphi\big\downarrow & & \\
\Sigma' & & M' \xrightarrow{\ h'\ } N' \\
\chi\big\downarrow & & \\
\Sigma'' & & M'' \xrightarrow{\ h''\ } N''
\end{array}
$$

  – For fixed Σ and M, by (φ, M', χ, E'') let us denote tuples where $\varphi\colon\ \Sigma \to \Sigma'$ is a $(\lambda,\mathcal{D})$-chain, M' is a φ-expansion of M, $(\chi\colon\ \Sigma' \to \Sigma'') \in \mathcal{D}$, and E'' is a set of Σ''-sentences such that $M' \models [\exists\chi]_f E''$. Two such tuples $(\varphi^1, M'^1, \chi^1, E''^1)$ and $(\varphi^2, M'^2, \chi^2, E''^2)$ are *isomorphic* when there exists an isomorphism $\theta\colon\ \varphi^1;\chi^1 \Rightarrow \varphi^2;\chi^2$ of $(\lambda+1)$-chains

$$
\begin{array}{ccccccccc}
\Sigma = \Sigma^1_0 \ldots & \longrightarrow & \ldots \Sigma^1_i & \xrightarrow{\varphi^1_{i,j}} & \Sigma^1_j & \longrightarrow \ldots \longrightarrow & \Sigma^1_\lambda = \Sigma'^1 & \xrightarrow{\chi^1} & \Sigma''^1 \\
{\scriptstyle 1_\Sigma = \theta_0}\big\downarrow & & \big\downarrow{\scriptstyle \theta_i} & & \big\downarrow{\scriptstyle \theta_j} & & \big\downarrow{\scriptstyle \theta_\lambda = \theta'} & & \big\downarrow{\scriptstyle \theta''} \\
\Sigma = \Sigma^2_0 \ldots & \longrightarrow & \ldots \Sigma^2_i & \xrightarrow[\varphi^2_{i,j}]{} & \Sigma^2_j & \longrightarrow \ldots \longrightarrow & \Sigma^2_\lambda = \Sigma'^2 & \xrightarrow[\chi^2]{} & \Sigma''^2
\end{array}
$$

such that $M'^2\upharpoonright_{\theta'} = M'^1$ and $\theta''(E''^1) = E''^2$. By the conditions of the theorem (*Sig* being $\mathcal{D}$-co-well-powered and all χ-expansions of a model forming a set), the isomorphism classes of tuples (φ, M', χ, E'') form a set; let us denote this by $L_\lambda(M)$. If k is the cardinal of $L_\lambda(M)$, we may consider $\{(\varphi^i, M'^i, \chi^i, E''^i) \mid i < k\}$ a complete system of independent representatives for $L_\lambda(M)$.

– The homomorphism $h\colon\ M \to N$ of claim 1 will be now obtained as $h = h_{0,k}$ where $(h_{i,j}\colon\ M_i \to M_j)_{i<j\leq k}$ is a chain of Σ-homomorphisms (thus in particular $N = M_k$) constructed inductively as follows.
  . $M_0 = M$
  . For each successor ordinal $j+1$ let $h'_{0,j}\colon\ M'^j \to M'_j$ be the unique φ^j-expansion of $h_{0,j}\colon\ M \to M_j$ from M'^j. Then for any $E''^j_f \subseteq E''^j$ finite:

2 there exists Σ''^j-model M''^j s.th. $M''^j\upharpoonright_{\chi^j} = M'^j, M''^j \models E''^j_f$ — $M'^j \models [\exists\chi^j]_f E''^j$

3 there exists $h''_{0,j}\colon\ M''^j \to M''_j$ s.th. $h''_{0,j}\upharpoonright_{\chi^j} = h'_{0,j}$ — χ^j quasi-representable

4 $M''_j \models E''^j_f$ — $M''^j \models E''^j_f$, hypothesis 1. applied to $h''_{0,j}$

5 $M'_j \models [\exists\chi^j]_f E''_j$ — 4, E''^j_f was considered arbitrary

$$
\begin{array}{ccccccc}
\Sigma & & M & \xrightarrow{h_{0,j}} & M_j & \xrightarrow{h_{i,j}} & M_{j+1} \\
\varphi^j\big\downarrow & & \big\uparrow & & \big\uparrow & & \big\uparrow \\
\Sigma'^j & & M'^j & \xrightarrow[h'_{0,j}]{} & M'^j & \xrightarrow[f']{} & P' \\
\chi^j\big\downarrow & & \big\uparrow & & \big\uparrow & & \\
\Sigma''^j & & M''^j & \xrightarrow[h''_{0,j}]{} & M''_j & &
\end{array}
$$

6 there exists $f'\colon\ M'_j \to P'$ s.th. $P' \models [\exists\chi^j]E''^j$ — 5, hypothesis 5.

Then we define $M_{j+1} = P'\upharpoonright_{\varphi^j}$ and $h_{j,j+1} = f'\upharpoonright_{\varphi^j}$.

  . For each limit ordinal j we take the co-limit of the chain before j.

– Now we prove that h satisfies the property claimed at 1. Keeping the above notations, consider (φ, M', χ, E''). If $j < k$ is the isomorphism class of (φ, M', χ, E''), we may assume without any loss of generality that $(\varphi, M', \chi, E'') = (\varphi^j, M'^j, \chi^j, E''^j)$. We have to show that $N' \models [\exists\chi^j]E''^j$.

7 there exists Σ''-model M''_{j+1} s.th. $M''_{j+1}\upharpoonright_{\chi^j} = M'_{j+1}$, $M''_{j+1} \models E''^j$ — 6, $P' = M'_{j+1}$

8 φ^j ; χ^j quasi-rep. — φ^j quasi-rep. (cf. Prop. 7.7), χ^j quasi-rep. (hyp. 7.), Prop. 5.12(1.)

9 there exists $h''_{j+1,k}\colon\ M''_{j+1} \to N''$ s.th. $h''_{j+1,k}\upharpoonright_{(\varphi^j;\chi^j)} = h_{j+1,k}$ — 8

10 $N'' \models E''^j$ — 9, hypothesis 1.

11 $N''\upharpoonright_{\chi^j} = N'$ — 9, φ^j quasi-representable

12 $N' \models [\exists\chi^j]E''^j$ — 10, 11.

- In the final part of the proof we exploit property 1 for showing that for each Σ-model M and for each cardinal λ' there exists a Σ-homomorphism $M \to N$ such that N is $(\lambda', \mathcal{D})$-saturated. By hypothesis 9. there exists a cardinal α such that each $(\lambda', \mathcal{D})$ is α-small. By Ordinal Induction we construct an α-chain $(f_{i,j} : N_i \to N_j)_{i<j\leq\alpha}$ such that $N_0 = M$ and each $f_{j,j+1}$ play the role of h of property 1. We want to show that N_α is $(\lambda', \mathcal{D})$-saturated, therefore the desired model homomorphism $M \to N$ would be $f_{0,\alpha} : M \to N_\alpha$. Let λ be any ordinal such that $\lambda < \lambda'$. Since any $(\lambda, \mathcal{D})$-chain can be completed trivially to a $(\lambda', \mathcal{D})$-chain just by considering only identities beyond λ, we have that each $(\lambda, \mathcal{D})$-chain is α-small too.

  Assume $N'_\alpha \models [\exists\chi]_f E''$, where $(\varphi, N'_\alpha, \chi, E'') \in L_\lambda(N_\alpha)$. We have to prove that $N'_\alpha \models [\exists\chi]E''$.

13 there exists $j < \alpha$, $f'_{j,\alpha} : N'_j \to N'_\alpha$ such that $f'_{j,\alpha}\restriction_\varphi = f_{j,\alpha}$ — φ α-small

14 there exists $f'_{j,j+1} : N'_j \to N'_{j+1}$ s.th. $f'_{j,j+1}\restriction_\varphi = f_{j,j+1}$ — 13, φ quasi-rep. (Prop. 7.7)

15 there exists $f'_{j+1,\alpha} : N'_{j+1} \to N'_\alpha$ s.th. $f'_{j+1,\alpha}\restriction_\varphi = f_{j+1,\alpha}$ — 13, 14, φ quasi-rep. (Prop. 7.7)

By hypotheses 1. and 2., for any finite $E''_f \subseteq E''$ there exists $\rho(E''_f) \in Sen\Sigma'$ and existential χ-quantification of a conjunction of the sentences of E''_f. Then:

16 $N'_\alpha \models \rho(E''_f)$ — $N'_\alpha \models [\exists\chi]E''$

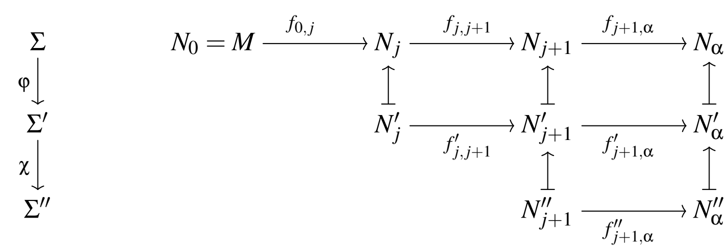
17 $N'_j \models \rho(E''_f)$ — 13, 16, hypothesis 1. applied to $f'_{j,\alpha}$

18 $N'_j \models [\exists\chi]_f E''$ — 17, E''_f has been considered arbitrary

19 $N'_{j+1} \models [\exists\chi]E''$ — 18, $f_{j,j+1}$ has property 1 by definition / construction, 14

20 there exists N''_{j+1} s.th. $N''_{j+1}\restriction_\chi = N'_{j+1}$, $N''_{j+1} \models E''$ — 19

21 there exists $f''_{j+1,\alpha} : N''_{j+1} \to N''_\alpha$ s.th. $f''_{j+1,\alpha}\restriction_\chi = f'_{j+1,\alpha}$ — 20, χ quasi-representable



22 $N''_\alpha \models E''$ — 20, 21, hypothesis 1. applied to $f''_{j+1,\alpha}$

23 $N'_\alpha \models [\exists\chi]E''$ — $N'_\alpha = N''_\alpha\restriction_\chi$ (21), 22.

□

Existence of saturated models, concretely

In what follows we discuss the applicability of Thm. 7.11 by analyzing its hypotheses, what they mean in concrete situations. We will take $\mathcal{FOL}$ as a benchmark example.

1. This is a strong hypothesis that is not satisfied by $\mathcal{FOL}$ or by many other standard institutions. When the respective institution has negations this hypothesis amounts to all model homomorphisms being elementary. When the institution does not have negations the hypothesis is stronger than elementariness. To develop the existence of saturated models in $\mathcal{FOL}$ we consider $I = E(\mathcal{FOL}q)$, which means the sub-institution of $\mathcal{FOL}$ such that

   - its signature morphisms are the $\mathcal{FOL}$ injective signature extensions with constants; this specific sub-institution is called $\mathcal{FOL}q$, and
   - its model homomorphisms are the elementary homomorphisms.

   While the latter restriction represents the obvious way to satisfy hypothesis 1., the reason for the former one will become apparent in what follows. The parameter $\mathcal{D}$ is taken to be the class of the injective signature extensions with a finite number of constants.

2. The availability of finite conjunctions and of existential $\mathcal{D}$-quantifications does not require any further comments, perhaps only that evidently $\mathcal{FOL}$ has them.

3. Generally speaking, the inductive co-limits are a special case of directed co-limits which are a special case of small general co-limits. In the case of $E(\mathcal{FOL}q)$, due to the very particular signature morphisms, inductive co-limits are straightforward, the only interesting aspect here is that having infinitely large signatures is crucial. The inductive-exactness hypothesis has two aspects of different weight as follows:

   - On the models any exactness property in $E(\mathcal{FOL}q)$ is inherited from $\mathcal{FOL}$, which is exact for any kind of co-limit of signature morphisms (Prop. 4.7).
   - On the model homomorphisms the situation is less straightforward because of the elementariness condition. Let us consider a λ-chain of signature morphisms $(\varphi_{i,j} : \Sigma_i \to \Sigma_j)_{i<j\leq\lambda}$ of $\mathcal{FOL}q$ signature morphisms and a family $(h_i \in Mod\Sigma_i)_{i\leq\lambda}$ of $\mathcal{FOL}$ model homomorphisms such that $h_j\restriction_{\varphi_{i,j}} = h_i$ when $i < j$. In this case, the inductive-exactness means that h_λ is elementary when h_i is elementary for each $i < \lambda$. This is solved by Prop. 5.31(3.); in fact it is enough one $i < \lambda$ such that h_i is elementary (N.B. when applied to $\mathcal{FOL}q$, the '$\mathcal{D}$' of Prop. 5.31 is not the same with the '$\mathcal{D}$' in Thm. 7.11, the former representing injective signature extensions with any number of constants while the latter allows only such extensions with a finite number of constants.).

4. This hypothesis relies on Tarski's Elementary Chain Theorem, Cor. 7.4 representing its institution-independent version, while the more concrete Cor. 7.5 includes the $\mathcal{FOL}$ case. However this is not quite enough, it is also required that the mediating homomorphisms from the elementary co-limits to elementary co-cones are elementary

too. This is achieved as follows. Let $(h_{i,j} : M_i \to M_j)_{i<j<\lambda}$ be a chain of elementary homomorphisms and let $(\mu_i : M_i \to M)_{i<\lambda}$ be a co-limit of this chain in $\mathcal{FOL}$. Let $(\nu_i : M_i \to N)_{i<\lambda}$ be a co-cone consisting of elementary homomorphisms and let $f : M \to N$ be the unique mediating homomorphism given by the co-limit property. Let $\chi \in \mathcal{D}$ and $f' : M' \to N'$ be any χ-expansion of f. Since any signature morphism in $\mathcal{D}$ is finitary, and χ in particular, there exists $j < \lambda$ and a corresponding χ-expansion μ'_j of μ_j, which determines an expansion ν'_j of ν_j defined by $\nu'_j = \mu'_j; f'$. By the elementariness of μ_j and ν_j we get that $M'^* = M'^*_j = N'^*$, hence f is elementary.

5. This can be regarded as a form of compactness. It can be addressed successfully at the general level by some results from Chapter 6 in a similar way we did in the proof of the first part of Prop. 7.10. Thus for each finite $i \subseteq E'$, let M'_i be the χ-expansion of M such that $M'_i \models i$. By compactness Thm. 6.19, there exists an ultrafilter U on $\mathcal{P}_\omega E'$ (the set of all finite subsets of E') and an ultraproduct $(\mu_J : M'_J \to M'_U)_{J\in U}$ such that $M'_U \models E'$. We define $N = M'_U \upharpoonright_\chi$. Since $Mod\chi$ preserves ultraproducts (cf. Prop. 6.7 and 6.9), $M'_U \upharpoonright_\chi \cong M_U$, an ultrapower of M. By Cor. 6.16, M can be elementarily embedded into M_U, hence into N too. This solved hypothesis 5. in $E(\mathcal{FOL}q)$. Most of this argument consists of institution-independent facts, so it can be easily replicated in situations when the role of $\mathcal{FOL}$ is played by any other Łoś institution such that the model reducts preserve the ultraproducts.
6. In $\mathcal{FOL}$ / $E(\mathcal{FOL}q)$, with $\mathcal{D}$ being the class of the finitary injective signature extensions with constants, this is an obvious fact. Moreover, this property holds in any institution where models consist of interpretations of the symbols of the signatures in set-theoretic universes, for those signature morphisms which do not add new sorts.
7. From Prop. 5.13 / Cor. 5.14 we know that in $\mathcal{FOL}$ all injective signature extensions with constants are quasi-representable.
8. For each $\mathcal{FOL}$ signature Σ there exists only a set of isomorphism classes of finitary signature extensions of Σ with constants, its cardinal being ω. If $\mathcal{D}$ consisted of extensions with arbitrarily large sets of constants the $\mathcal{D}$-co-well-powered property of the category of the signatures would have been lost.
9. This can be addressed at the general institution-independent level by Prop. 7.9. Note that in the case of $E(\mathcal{FOL}q)$ the conditions of Prop. 7.9 are included in the set of the hypotheses of Thm. 7.11 that have already been discussed for this particular concrete case, with only one exception: that $\mathcal{D}$ contains only finitary signature morphisms. However the setup of $\mathcal{D}$ in the definition of $E(\mathcal{FOL}q)$ guarantees this via Cor. 5.18 (each injective signature extension with a finite number of constants is finitary in the institution-independent sense).

Based on the above analysis of the conditions underlying Thm. 7.11 we obtain the classical standard existence of saturated models in $\mathcal{FOL}$. Below we formulate this using the concepts of this chapter.

Corollary 7.12. *$\mathcal{FOL}$ has $\mathcal{D}$-saturated models for $\mathcal{D}$ the class of the injective signature extensions with a finite number of constants.*

Let us now sum up the development of the result of the existence of saturated models. We have a general theorem which comes with a list of hypotheses. The theorem itself is not easy, but it can be applied relatively easily to concrete institutions by addressing its hypotheses in the respective context. However both the proof of the general theorem and solving the hypotheses require a significant quantity of concepts and results that have been previously developed in this book. In most cases these belong to the 'basic stuff', but in a few cases they are more advanced than that. It is important that the process of concrete validation of the hypotheses of Thm. 7.11 does not involve any particular difficulties.

This result is emblematic of the institution-independent method of developing model theory. First, there is a clarification of the concepts at an appropriate level of abstraction. Then there is a clarification of the causes for a certain result to happen which leads to the formulation of hypotheses. Then there is a proof of the result at the abstract level. Finally, we have to address the issue of the applicability of the result, when and how its causes arise. The application stage should involve a relatively straightforward technical validations, or at least these should have a degree of difficulty that is significantly lower than that of the main result itself.

Borrowing saturated models along institution comorphisms

Perhaps the most limiting hypothesis of Thm. 7.11 is the requirement on conjunctions and on existential quantifications. We can get rid of this by the following general borrowing result and thus obtain the existence of saturated models in sub-institutions with less expressive power of the sentences.

Proposition 7.13. *Let* $(\Phi,\alpha,\beta):\ I \to I'$ *be an institution comorphism and* $\mathcal{D} \subseteq Sig$, $\mathcal{D}' \subseteq Sig'$ *be classes of signature morphisms such that*

1. (Φ,α,β) *has the model expansion property and has weak model amalgamation,*
2. Φ *preserves inductive co-limits, and*
3. $\Phi\mathcal{D} \subseteq \mathcal{D}'$.

Then I *has* $\mathcal{D}$*-saturated models whenever* I' *has* $\mathcal{D}'$*-saturated models.*

Proof. We first show that

1 β maps $(\lambda,\mathcal{D}')$-saturated models to $(\lambda,\mathcal{D})$-saturated models.

Consider M' a $(\lambda,\mathcal{D}')$-saturated $\Phi\Sigma$-model. Let $M = \beta_\Sigma M'$ and consider a $(k,\mathcal{D})$-chain $\varphi:\ \Sigma \to \Sigma_k$ for $k < \lambda$, $(\chi:\ \Sigma_k \to \overline{\Sigma}) \in \mathcal{D}$, M_k a φ-expansion of M, and E a set of $\overline{\Sigma}$-sentences such that $M_k \models [\exists\chi]_f E$.

2 $\Phi\varphi$ is a $(k,\mathcal{D})$-chain — hypotheses 2., 3.

3 $\exists M'_k \in |Mod'(\Phi\Sigma_k)|$ s.th. $M'_k\restriction_{\Phi\varphi} = M'$, $\beta_{\Sigma_k} M'_k = M_k$ — (Φ,α,β) has model amalgamation

$$\begin{array}{ccccc}
M' & \xleftarrow{Mod'(\Phi\varphi)} & M'_k & \xleftarrow{Mod'(\Phi\chi)} & \overline{N}' \models \alpha_\Sigma E_f \\
\downarrow \beta_\Sigma & & \downarrow \beta_{\Sigma_k} & & \downarrow \beta_{\overline{\Sigma}} \\
M & \xleftarrow[Mod\varphi]{} & M_k & \xleftarrow[Mod\chi]{} & \overline{N} \models E_f
\end{array}$$

4 $\forall \overline{N}, \overline{N}\restriction_\chi = M_k$, $\exists \overline{N}'$ s.th. $\overline{N}'\restriction_{\Phi\chi} = M'_k$, $\beta_{\overline{\Sigma}}\overline{N}' = \overline{N}$ — (Φ,α,β) has model amalgamation

5 $M'_k \models [\exists \Phi\chi]_f \alpha_{\overline{\Sigma}} E$ — $M_k \models [\exists\chi]_f E$, $M'_k\restriction_{\Phi\varphi} = M'$, 3, 4, Satisfaction Condition of (Φ,α,β)

6 $M'_k \models [\exists \Phi\chi]\alpha_{\overline{\Sigma}} E$ — M' $(\lambda,\mathcal{D}')$-saturated, $M'_k\restriction_{\Phi\varphi} = M'$, 5

7 there exists $\overline{M'}$ s.th. $\overline{M'}\restriction_{\Phi\chi} = M'_k$, $\overline{M'} \models \alpha_{\overline{\Sigma}} E$ — 6

8 $\beta_{\overline{\Sigma}}\overline{M'} \models E$ — 7, Satisfaction Condition of (Φ,α,β)

9 $(\beta_{\overline{\Sigma}}\overline{M'})\restriction_\chi = M_k$ — β natural, 3

10 $M_k \models [\exists\chi]E$ — 8, 9.

Now that we have established 1 we may proceed to the final part of the proof. Let M be any Σ-models.

11 there exists a $\Phi\Sigma$-model M' s.th. $\beta_\Sigma M' = M$ — (Φ,α,β) has model expansion

12 there exists $h' : M' \to N'$ s.th. $M' \equiv N'$, N' $(\lambda,\mathcal{D})$-saturated — I' has $\mathcal{D}'$-saturated models

13 $\beta N'$ $(\lambda,\mathcal{D})$-saturated — 1

14 $\beta h' : M \to \beta N'$ — $h' : M' \to N'$ (12), $M = \beta M'$

15 $M \equiv \beta N'$ — $M' \equiv N'$, $M \equiv \beta M'$, Satisfaction Condition of (Φ,α,β)

The conclusions 14 and 15 say that I has $(\lambda,\mathcal{D})$-saturated models. □

A concrete application of Prop. 7.13 is the following.

Corollary 7.14. *$\mathcal{EQL}$ and $\mathcal{HCL}$ have $\mathcal{D}$-saturated models for the usual $\mathcal{D}$ consisting of the injective signature extensions with a finite number of constants.*

Exercises

7.2. Lindenbaum Theorem
We say that an institution has the *Lindenbaum Property* if and only if each consistent set of sentences can be extended to a maximal consistent set of sentences. Each m-compact institution has the Lindenbaum Property. (*Hint:* Let β be the cardinal of $Sen\Sigma$ and arrange $Sen\Sigma = (e_\alpha)_{\alpha<\beta}$. Define $E_0 = E$, and for each successor ordinal $\alpha+1$ define $E_{\alpha+1} = E_\alpha \cup \{e_\alpha\}$ if $E_\alpha \cup \{e_\alpha\}$ is consistent, otherwise $E_{\alpha+1} = E_\alpha$, and for each limit ordinal α' define $E_{\alpha'} = \bigcup_{\alpha<\alpha'} E_\alpha$. Then $E_\beta = \bigcup_{\alpha<\beta} E_\alpha$ is the desired maximally consistent set.)

7.3. Let $\mathcal{D}$ be the class of $\mathcal{FOL}^1$ signature extensions with a finite number of constants. Consider the $\mathcal{FOL}^1$ signature having only one binary relation symbol $<$. The model $\mathbb{Q}$ of the rational numbers interpreting $<$ as the 'strictly less than' relation is $(\omega,\mathcal{D})$-saturated but it is not $(\lambda,\mathcal{D})$-saturated for cardinals $\lambda > \omega$. (*Hint:* For each finite n the elementary equivalence relation between the expansions of $\mathbb{Q}$ with n constants determines a finite partition, i.e., has a finite number of equivalence classes, whose cardinal is less than $n!$. Each such equivalence class is determined by the mutual position of the constants concerning $<$.)

7.4. In any semi-exact institution, the model reduct functors preserve the $(\lambda, \mathcal{D})$-saturated models if $\mathcal{D}$ is stable under pushouts. (*Hint:* $(\lambda, \mathcal{D})$-chains are stable under pushouts.)

7.5. Establish the existence of saturated models in several concrete institutions presented as examples in this book (such as $\mathcal{PA}$, $\mathcal{POA}$, etc.) as instances of the general institution-independent Thm. 7.11.

7.6. Saturated models for theories
For any class $\mathcal{D}$ of signature morphisms in an institution I, let $\mathcal{D}^{\mathrm{th}}$ denote the class of the presentation morphisms $\chi : (\Sigma, E) \to (\Sigma', E')$ for which $\chi \in \mathcal{D}$.

1. Any model of the institution I^{th} is $(\lambda, \mathcal{D}^{\mathrm{th}})$-saturated if it is $(\lambda, \mathcal{D})$-saturated in I.
2. I^{th} has $\mathcal{D}^{\mathrm{th}}$-saturated models if I has $\mathcal{D}$-saturated models.
3. The following two institutions have $\mathcal{D}$-saturated models which are 'borrowed' through Prop. 7.13:
   (a) $\mathcal{HNK}$ has $\mathcal{D}$-saturated models for $\mathcal{D}$ the class of the injective signature extensions with a finite number of constants (*Hint:* use the comorphism $\mathcal{HNK} \to \mathcal{FOEQL}^{\mathrm{th}}$ of Ex. 4.12.)
   (b) $\mathcal{IPL}$ has $\mathcal{D}$-saturated models for $\mathcal{D}$ the class of the injective signature extensions with a finite number of symbols (*Hint:* use the comorphism $\mathcal{IPL} \to \mathcal{FOEQL}^{\mathrm{th}}$ of Ex. 4.11.)

7.3 Uniqueness of saturated models

The uniqueness property of saturated models is subject to a set of conditions which are introduced and discussed in the first part of this section. The most important one, in the sense that it is the only one with a special significance is a limit to the 'size' of the models. A general concept of size may be defined when the institution has diagrams. Another condition is that the diagrams satisfy a certain rather natural property. Another property required is that sentences are finitary; in concrete terms this means that they can contain only a finite number of symbols.

For this section, we assume an institution endowed with a system ι of diagrams and with a designated class $\mathcal{D}$ of signature morphisms.

Simple diagrams. By the functoriality property of ι, we know that for all Σ-models M and N and for each $\iota_\Sigma N$-expansion M' of M the following square of signature morphisms commutes:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\iota_\Sigma N} & \Sigma_N \\ {\scriptstyle \iota_\Sigma M}\downarrow & & \downarrow{\scriptstyle \iota_{\Sigma_N} M'} \\ \Sigma_M & \xrightarrow[\iota_{\iota_\Sigma N} 1_M]{} & (\Sigma_N)_{M'} \end{array} \qquad (7.2)$$

(In the explanation of the functoriality of ι in Sec. 4.4 take $\varphi = \iota_\Sigma N$ and $h = 1_M : M \to M'\restriction_{\iota_\Sigma N}$.) Then the diagrams ι are *simple* when the commutative squares (7.2) are pushout squares.

Despite the frightening formulae in the definition of simple diagrams above, in common concrete situations this concept is almost trivial. When the elementary extensions just add the elements of the model as new constants to its signature, the diagrams ι are simple because the squares (7.2) are of the form

$$\begin{array}{ccc} \Sigma & \longrightarrow & \Sigma + |N| \\ \downarrow & & \downarrow \\ \Sigma + |M| & \longrightarrow & \Sigma + |N| + |M| \end{array}$$

where $|M|$, $|N|$ denote the underlying carrier sets of M and N, respectively.

Sizes of models. In Sect. 6.5 we have introduced a concept of 'finite' size for abstract models based on the categorical finiteness property of the associated elementary extension. Now we take this idea further for infinite cardinals. For any cardinal number λ, we say that a model M has $\mathcal{D}$-ι-*size* λ when $\iota_\Sigma M = \varphi_{0,\lambda}$ for some $(\lambda, \mathcal{D})$-chain $(\varphi_{i,j})_{i<j\leq\lambda}$. When there is no ambiguity about ι we may say '$\mathcal{D}$-size' rather than '$\mathcal{D}$-ι-size'. Note that this concept of 'size' is a relation between models and cardinals rather than a function from models to cardinals.

Fact 7.15. *Let $\mathcal{D}$ be the class of the $\mathcal{FOL}$ injective signature extensions with a finite number of constants. An infinite $\mathcal{FOL}$-model M has $\mathcal{D}$-size λ if and only if card$|M| \leq \lambda$ (where $|M| = \biguplus_{s\in S} M_s$, with S being the set of the sorts of Σ). Moreover any finite $\mathcal{FOL}$-model has $\mathcal{D}$-size λ for any cardinal λ.*

By Prop. 7.10 we can further establish the following:

Corollary 7.16. *For any infinite cardinal λ, for each $(\lambda, \mathcal{D})$-saturated $\mathcal{FOL}$ model M of $\mathcal{D}$-size λ such that M_s is infinite for at least one sort, card$|M| = \lambda$.*

Finitary sentences. In any institution a Σ-sentence ρ is *finitary* if and only if it can be written as $\varphi\rho_0$ where $\varphi : \Sigma_0 \to \Sigma$ is a signature morphism such that Σ_0 is a finitely presented signature and ρ_0 is a Σ_0-sentence. An institution *has finitary sentences* when all its sentences are finitary. This concept is a categorical expression of the fact that a sentence contains only a finite number of symbols. This is illustrated by the following typical example.

Fact 7.17. *A $\mathcal{FOL}$ signature (S, F, P) is finitely presented if and only if S, F, and P are finite. (Here F 'finite' means that $\{(w,s) \mid F_{w\to s} \neq \emptyset\}$ is finite and each non-empty $F_{w\to s}$ is also finite. The same applies to P.) Consequently, $\mathcal{FOL}$ has finitary sentences.*

Here we have to warn the reader about some possible terminology confusion which may arise in relation to the term 'finitary' when used in conjunction with basic sentences.

Therefore by 'finitary basic' set of sentences E we will always mean that at least one of its basic models M_E is finitely presented (like in Sect. 5.5) and not that the set of sentences is 'finitary' and 'basic'.

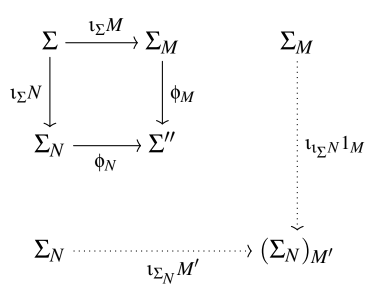
Uniqueness theorem

Theorem 7.18. *Assume that the institution*

1. *has pushouts and inductive co-limits of signatures,*
2. *is semi-exact and inductive-exact on models,*
3. *has simple diagrams* ι*,*
4. *has existential* $\mathcal{D}$*-quantification for a (sub)category* $\mathcal{D}$ *of signature morphisms which is stable under pushouts,*
5. *has negations and finite conjunctions, and*
6. *has finitary sentences.*

Then any two elementary equivalent $(\lambda, \mathcal{D})$*-saturated* Σ*-models of* $\mathcal{D}$*-size* λ *are isomorphic.*

Proof. Let M,N be $(\lambda,\mathcal{D})$-saturated Σ-models of $\mathcal{D}$-size λ such that $M \equiv N$. For any pushout square of signature morphisms as follows:



let us assume that

1 there exists Σ''-models M'', N'' s.th. $M''\upharpoonright_{\phi_M} = M_M,\ N''\upharpoonright_{\phi_N} = N_N,\ M'' \equiv N''$.

Under this assumption, we can prove the theorem as follows. Define $M' = M''\upharpoonright_{\phi_N}$ and $N' = N''\upharpoonright_{\phi_M}$. Because the diagrams are simple and pushouts are unique up to isomorphism, we may assume without any loss of generality that $\Sigma'' = (\Sigma_N)_{M'}, \phi_M = \iota_{\iota_\Sigma N} 1_M$ and $\phi_N = \iota_{\Sigma_N} M'$. Then

2 $M'_{M'}\upharpoonright_{\phi_N} = M'$ $\qquad\qquad$ $\phi_N = \iota_{\Sigma_N} M'$

3 $M'_{M'}\upharpoonright_{\phi_M} = M_M$ $\qquad\qquad$ $\phi_M = \iota_{\iota_\Sigma N} 1_M$, naturality of i applied to $M'_{M'}$ (see the diagram below)

$$\begin{array}{ccc}
Mod((\Sigma_N)_{M'}, E_{M'}) & \xrightarrow{i_{\Sigma_N, M'}} & M'/Mod\Sigma_N \\
{\scriptstyle Mod\ \iota_{\Sigma_N} 1_M}\downarrow & & \downarrow{\scriptstyle Mod\ \iota_{\Sigma_N}} \\
Mod(\Sigma_M, E_M) & \xrightarrow[i_{\Sigma,M}]{} & M/Mod\Sigma
\end{array}$$

4 $M'' = M'_{M'}$ — definition of M', 1, 2, 3, hypothesis 2. (semi-exactness)

5 $M'' \models E_{M'}$ — 4

6 $N'' \models E_{M'}$ — $M'' \equiv N''$

7 there exists $h: M'' \to N''$ — 4, 6, $M'_{M'}$ initial.

Now, by the symmetry between M and N we can swap their role in all of the above reasoning steps and get that

8 there exists $h': N'' \to M''$.

Then

9 $h;h' = 1_{M''}$ and $h';h = 1_{N''}$ — 7, 8, M'', N'' are initial (see 4)

10 $M \cong N$ — 9, $M = M''\!\upharpoonright_{\iota_\Sigma M;\phi_M}$, $N = N''\!\upharpoonright_{\iota_\Sigma N;\phi_N}$, $\iota_\Sigma M;\phi_M = \iota_\Sigma N;\phi_N$.

Thus under assumption 1 we have proved the theorem. It remains to prove 1. We do that in two steps as follows:

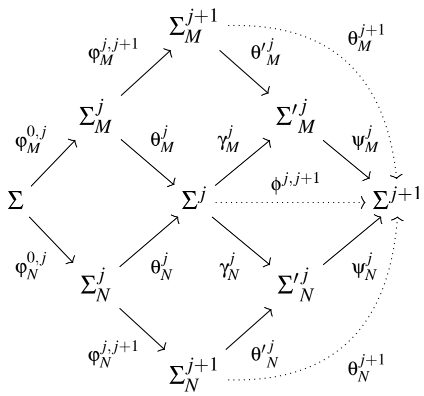
- Since both M,N have $\mathcal{D}$-size λ there are $(\lambda,\mathcal{D})$-chains $(\varphi_M^{i,j}: \Sigma_M^i \to \Sigma_M^j)_{i<j\leq\lambda}$ and $(\varphi_N^{i,j}: \Sigma_N^i \to \Sigma_N^j)_{i<j\leq\lambda}$ such that $\iota_\Sigma M = \varphi_M^{0,\lambda}$ and $\iota_\Sigma N = \varphi_N^{0,\lambda}$. By Ordinal Induction we define

  1. a $(\lambda,\mathcal{D})$-chain $(\phi^{i,j}: \Sigma^i \to \Sigma^j)_{i<j\leq\lambda}$ such that $\Sigma^0 = \Sigma$, $\Sigma^\lambda = \Sigma''$, and
  2. two natural transformations $\theta_M: \varphi_M \Rightarrow \phi$, $\theta_N: \varphi_M \Rightarrow \phi$ such that $\theta_M^\lambda = \phi_M$, $\theta_N^\lambda = \phi_N$ and for each $j \leq \lambda$ the commutative square

$$\begin{array}{ccc}
\Sigma & \xrightarrow{\varphi_M^{0,j}} & \Sigma_M^j \\
{\scriptstyle \varphi_N^{0,j}}\downarrow & & \downarrow{\scriptstyle \theta_M^j} \\
\Sigma_N^j & \xrightarrow[\theta_N^j]{} & \Sigma^j
\end{array} \qquad (7.3)$$

  is a pushout square.

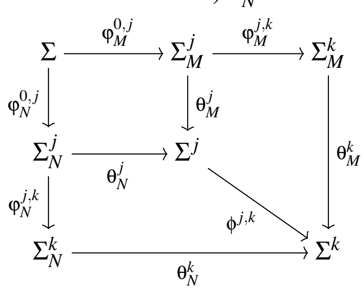
  – For each successor ordinal $j+1$ we construct the following system consisting of four pushout squares in three steps:



The left side square is given as a pushout square by the induction hypothesis. Then the upper and the lower squares are defined as pushout squares. In the third step, we define the right square as a pushout square. Thus θ_M^{j+1}, θ_N^{j+1}, $\phi^{j,j+1}$ are defined and the outer square is a pushout square by composition of pushout squares, which represents the pushout property for (7.3) for $j+1$.

To establish that ϕ will be a $(\lambda, \mathcal{D})$-chain we show at this stage that $\phi^{j,j+1} \in \mathcal{D}$. Since $\mathcal{D}$ is stable under pushouts (hypothesis 4.), by following the stepwise construction of $\phi^{j,j+1}$ we successively obtain that $\gamma_M^j, \gamma_N^j \in \mathcal{D}$, then that $\psi_M^j, \psi_N^j \in \mathcal{D}$, and finally that $\phi^{j,j+1} \in \mathcal{D}$.

– For each limit ordinal $k \leq \lambda$, by using that $(\theta_M^j ; \phi^{j,k})_{j<k}$ is a co-cone for $(\varphi_M^{i,j})_{i<j<k}$, we define $\theta_M^k : \Sigma_M^k \to \Sigma^k$ as the unique signature morphism such that $\varphi_M^{j,k} ; \theta_M^k = \theta_M^j ; \phi^{j,k}$ for each $j < k$. (Note that $\phi_M^{j,k}$ are already available as the components of the inductive co-limit at k.) θ_N^k is defined similarly.



That the outer square of the above diagram is a pushout square ((7.3) for k) can be established as follows. Consider

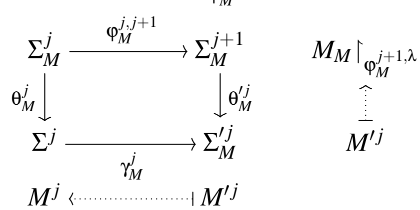
. $f_M^k : \Sigma_M^k \to \Omega$, $f_N^k : \Sigma_N^k \to \Omega$ such that $\varphi_M^{0,k} ; f_M^k = \varphi_N^{0,k} ; f_N^k$.

. For each $j < k$ we let $f_M^j = \varphi_M^{j,k} ; f_M$, $f_N^j = \varphi_N^{j,k} ; f_N$ and

. $f^j : \Sigma^j \to \Omega$ be the unique morphism such that $\theta_M^j ; f^j = f_M^j$ and $\theta_N^j ; f^j = f_N^j$ (by the pushout property of (7.3)).

Then the unique mediating morphism $f^k : \Sigma^k \to \Omega$ such that $\theta^k_M; f^k = f^k_M$ and $\theta^k_N; f^k = f^k_N$, is given by the co-limit property of the k-chain $(\phi^{i,j})_{i<j\leq k}$.

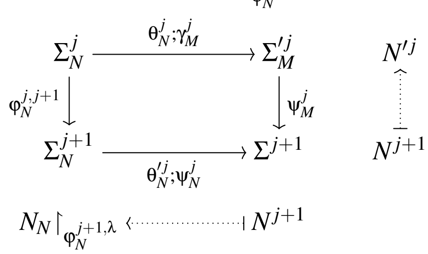
- In the second part of the proof, by Ordinal Induction we define for each $j \leq \lambda$, Σ^j-models $M^j \equiv N^j$ such that $M^j\restriction_{\phi^{i,j}} = M^i$, $N^j\restriction_{\phi^{i,j}} = N^i$ for each $i \leq j$, and $M^j\restriction_{\theta^j_M} = M_M\restriction_{\varphi^{j,\lambda}_M}$ and $N^j\restriction_{\theta^j_N} = N_N\restriction_{\varphi^{j,\lambda}_N}$. Then 1 gets proved by taking $M'' = M^\lambda$ and $N'' = N^\lambda$.
  - Let $M^0 = M$ and $N^0 = N$.
  - Let $j+1$ be a successor ordinal.
    . By semi-exactness applied to the pushout square below we define M'^j as the amalgamation of M^j and $M_M\restriction_{\varphi^{j+1,\lambda}_M}$.



    . Now we find a γ^j_M-expansion N'^j such that $N'^j \equiv M'^j$.

11	$M^j \models (\exists\gamma^j_M) \wedge E'_f$ for each finite $E'_f \subseteq (M'^j)^*$	$M'^j\restriction_{\gamma^j_M} = M^j$, $\gamma^j_M \in \mathcal{D}$, I has conjunctions and existential $\mathcal{D}$-quantification
12	$N^j \models (\exists\gamma^j_M) \wedge E'_f$	$M^j \equiv N^j$, 11
13	$N^j \models [\exists\gamma^j_M]_f (M'^j)^*$	12, E'_f arbitrary
14	$N^j \models [\exists\gamma^j_M](M'^j)^*$	13, $\phi^{0,j}$ $(j,\mathcal{D})$-chain, $\gamma^j_M \in \mathcal{D}$, N $(\lambda,\mathcal{D})$-saturated
15	$\exists N'^j$, $N'^j\restriction_{\gamma^j_M} = N^j$, $N'^j \models (M'^j)^*$	14
16	$N'^j \equiv M'^j$	15, I has negations.

    . By semi-exactness applied to the pushout square below we define N^{j+1} as the amalgamation of N'^j and $N_N\restriction_{\varphi^{j+1,\lambda}_N}$.



    . We can replicate the reasoning that 'produced' N'^j on the basis of M'^j in order to 'produce' M^{j+1} on the basis of N^{j+1}. For this we use that M is $(\lambda,\mathcal{D})$-saturated, that M'^j is the $\phi^{0,j};\gamma_M^j$-expansion of M (and $\phi^{0,j};\gamma_M^j$ is a $(j+1,\mathcal{D})$-chain), and that $\psi_M^j \in \mathcal{D}$. This gets us a ψ_M^j-expansion M^{j+1} of M'^j such that $M^{j+1} \equiv N^{j+1}$.

– Let k be a limit ordinal. By the inductive-exactness hypothesis let M^k, N^k be the unique Σ^k-models such that $M^k\restriction_{\phi^{j,k}} = M^j$ and $N^k\restriction_{\phi^{j,k}} = N^j$ for each $j<k$. We have to prove that $M^k \equiv N^k$. Let $\rho^k \in Sen\Sigma^k$.

17 there exists Ω finitely presented signature, $\zeta^k\colon\ \Omega \to \Sigma^k$, $\rho \in Sen\Omega$ s.th. $\rho^k = \zeta^k\rho$ — I has finitary sentences

18 there exists $j<k$, $\zeta^j\colon\ \Omega \to \Sigma^j$ s.th. $\zeta^j;\phi^{j,k} = \zeta^k$ — Ω finitely presented

19 $(M^k \models \rho^k) = (M^j \models \zeta^j\rho)$ — 18, $\rho^k = \zeta^k\rho$ (17), Satisfaction Condition

20 $(N^k \models \rho^k) = (N^j \models \zeta^j\rho)$ — 18, $\rho^k = \zeta^k\rho$ (17), Satisfaction Condition

21 $(M^k \models \rho^k) = (N^k \models \rho^k)$ — 19, 20, $M^j \equiv N^j$ (induction hypothesis).

Finally, we prove that $M^k\restriction_{\theta_M^k} = M_M\restriction_{\varphi_M^{k,\lambda}}$ and $N^k\restriction_{\theta_N^k} = N_N\restriction_{\varphi_N^{k,\lambda}}$. As both equalities get similar proofs, we do it for one of them only.

22 $M^k\restriction_{\theta_M^k}\restriction_{\varphi_M^{j,k}} = M^k\restriction_{\phi^{j,k}}\restriction_{\theta_M^j} = M^j\restriction_{\theta_M^j}$ — $\varphi_M^{j,k};\theta_M^k = \theta_M^j;\phi^{j,k}$ (naturality of θ), M^k definition

23 $M^j\restriction_{\theta_M^j} = M_M\restriction_{\varphi_M^{j,\lambda}}$ — induction hypothesis

24 $M^k\restriction_{\theta_M^k}\restriction_{\varphi_M^{j,k}} = M_M\restriction_{\varphi_M^{j,\lambda}} = M_M\restriction_{\varphi_M^{k,\lambda}}\restriction_{\varphi_M^{j,k}}$ — 22, 23

25 $M^k\restriction_{\theta_M^k} = M_M\restriction_{\varphi_M^{k,\lambda}}$ — 24, $(\varphi_M^{j,k})_{j<k}$ co-limit, inductive-exactness.

□

The well-known uniqueness property of saturated models in $\mathcal{FOL}$ is a concrete instance of the general uniqueness Thm. 7.18.

Corollary 7.19. *In $\mathcal{FOL}$, any two elementarily equivalent $(\lambda,\mathcal{D})$-saturated models of cardinality λ are isomorphic.*

Because finitely sized $\mathcal{FOL}$ models are saturated (cf. Prop. 7.10) the above consequence of our general uniqueness result can be further applied to obtain that in $\mathcal{FOL}$ for any two finite models with non-empty sorts being elementary equivalent is the same as being isomorphic (Cor. 6.30).

Exercises

7.7. Establish the uniqueness of saturated models in several concrete institutions presented as examples in this book (such as $\mathcal{PA}$, $\mathcal{POA}$, etc.) as instances of the general institution-independent Thm. 7.18.

7.8. In $\mathcal{FOL}$ any two elementary equivalent models admit a common elementary extension.

7.4 Saturated ultraproducts

For this section, we assume the Generalized Continuum Hypothesis. We also need some more set-theoretic concepts and results.

Good ultrafilters. Let $(P,\leq)$ and $(P',\leq)$, respectively, be partial orders with binary least upper bounds $\vee$ and greatest lower bounds $\wedge$, respectively. A function $f: P \to P'$ is

– *contra-monotonic* if $x < y$ implies $fx > fy$, and

– *contra-additive* if $f(x \vee y) = fx \wedge fy$.

For any functions $f, g: P \to P'$, $f \leq g$ if $fx \leq gx$ for all $x \in P$.

An ultrafilter U is λ-*good* for a cardinal λ if for each $\alpha < \lambda$ and each contra-monotonic function $f: \mathcal{P}_\omega\alpha \to U$ there exists an contra-additive function $g: \mathcal{P}_\omega\alpha \to U$ such that $g \leq f$.

Countably incomplete ultrafilters. An ultrafilter U over I is *countably incomplete* if there exists an ω-chain $I = I_0 \supset I_1 \supset \cdots \supset I_n \supset \ldots$ such that $I_n \in U$ and $I_\omega = \bigcap_{n\in\omega} I_n = \emptyset$.

The proof of the following theorem consists of combinatorial set-theoretic arguments, and can be found in [42].

Theorem 7.20. *For any set I of cardinality λ, there exists a λ^+-good countably incomplete ultrafilter over I.*

Stable sentence functors. In any institution endowed with a designated class $\mathcal{D}$ of signature morphisms, its sentence functor Sen is $\mathcal{D}$-*stable* when $card(Sen\Sigma') \leq card(Sen\Sigma)$ for each $\chi: \Sigma \to \Sigma'$ in $\mathcal{D}$. The stability of sentence functors is a rather common property of institutions, the following being a typical example.

Proposition 7.21. *The $\mathcal{FOL}$ sentence functor is $\mathcal{D}$-stable for $\mathcal{D}$ the class of all injective signature extensions with a finite number of constants.*

Proof. Let $(\chi: \Sigma \to \Sigma') \in \mathcal{D}$. The function $Sen\Sigma' \to Sen\Sigma$ which maps each Σ'-sentence ρ' to $(\exists\chi)\rho'$ is an injection, hence $card(Sen\Sigma') \leq card(Sen\Sigma)$. □

The following shows that the stability of the sentence functor guarantees a limit to the growth of the cardinality of the sets of sentences. It also constitutes an important technical step in the proof of the main result of this section.

Proposition 7.22. *Consider an institution endowed with a designated class $\mathcal{D}$ of signature morphisms such that*

1. *it has finitary sentences, and*
2. *the sentence functor Sen is $\mathcal{D}$-stable.*

Then for each $(\alpha,\mathcal{D})$-chain $\varphi: \Sigma \to \Sigma'$, $card(Sen\Sigma') \leq card\alpha \cdot card(Sen\Sigma)$.

Proof. Let us denote the segment of the chain φ between i and j by $\varphi_{i,j} : \Sigma_i \to \Sigma_j$. Then $\Sigma = \Sigma_0$ and $\Sigma' = \Sigma_\alpha$. We prove the proposition by Ordinal Induction on α.

- If $\alpha + 1$ is a successor ordinal we successively have that

$$\begin{aligned} card(Sen\Sigma_{\alpha+1}) &\leq card(Sen\Sigma_\alpha) && \text{Sen $\mathcal{D}$-stable} \\ &\leq card\alpha \cdot card(Sen\Sigma) && \text{induction hypothesis} \\ &\leq card(\alpha+1) \cdot card(Sen\Sigma). \end{aligned}$$

- If α is a limit ordinal it is infinite and $\alpha = \bigcup_{\beta<\alpha} \beta$. Let us assume

1 $card(Sen\Sigma_\alpha) \leq card(\biguplus_{\beta<\alpha} Sen\Sigma_\beta)$

and prove this case as follows.

2 for each $\beta < \alpha$, $card(Sen\Sigma_\beta) \leq card\beta \cdot card(Sen\Sigma)$ — induction hypothesis

3 for each $\beta < \alpha$, $card\beta < card\alpha$ — $\beta < \alpha$, α limit ordinal

4 for each $\beta < \alpha$, $card(Sen\Sigma_\beta) \leq card\alpha \cdot card(Sen\Sigma)$ — 2, 3

5 $card(Sen\Sigma_\alpha) \leq card\alpha \cdot card\alpha \cdot card(Sen\Sigma)$ — 1, 4

6 $card(Sen\Sigma_\alpha) \leq card\alpha \cdot card(Sen\Sigma)$ — $card\alpha \cdot card\alpha = card\alpha$ (Prop. 7.6).

It remains to prove 1. We do this by defining an injection $f : Sen\Sigma_\alpha \to \biguplus_{\beta<\alpha} Sen\Sigma_\beta$. Because the institution has finitary sentences, like in the proof of Thm. 7.18, for each $\rho \in Sen\Sigma_\alpha$ there exists $\beta < \alpha$ and $\rho' \in Sen\Sigma_\beta$ such that $\varphi_{\beta,\alpha}\rho' = \rho$. We define $f\rho = \rho'$. In order to prove the injectivity of f we consider $\rho_1, \rho_2 \in Sen\Sigma_\alpha$ such that $f\rho_1 = f\rho_2$. Then there are $\beta_k < \alpha$, $k = 1,2$, such that $f\rho_k \in Sen\Sigma_{\beta_k}$ and $\varphi_{\beta_k,\alpha}(f\rho_k) = \rho_k$. Since in $\biguplus_{\beta<\alpha} Sen\Sigma_\beta$, $Sen\Sigma_{\beta_1}$ and $Sen\Sigma_{\beta_2}$ are taken disjointly when $\beta_1 \neq \beta_2$, we necessarily have that $\beta_1 = \beta_2$ hence $\rho_1 = \rho_2$.

□

Ultraproducts that are saturated

The following gives sufficient conditions for ultraproducts to be saturated.

Theorem 7.23. *Consider a Łoś institution endowed with a designated class $\mathcal{D}$ of signature morphisms such that*

1. *its sentences are finitary,*
2. *it has finite conjunctions and existential $\mathcal{D}$-quantifications,*
3. *the sentence functor Sen is $\mathcal{D}$-stable,*

4. *the model reduct functors corresponding to signature morphisms in $\mathcal{D}$ preserve ultraproducts of models and lifts isomorphisms (i.e. if $\chi \in \mathcal{D}$ and $M \cong N'\upharpoonright\chi$ then there exists $M' \cong N'$ such that $M'\upharpoonright_{\chi} = M$), and*
5. *each $(\alpha, \mathcal{D})$-chain invents completely ultraproducts of models.*

For any infinite cardinal λ and each countably incomplete λ-good ultrafilter U over I, for any signature Σ such that $card(Sen\Sigma) < \lambda$, for any ultraproduct $(\mu_J : M_J \to M_U)_{J\in U}$ of any family $(M_i)_{i\in I}$ of Σ-models, M_U is $(\lambda, \mathcal{D})$-saturated.

Proof. Consider an $(\alpha, \mathcal{D})$-chain $(\varphi_{i,j} : \Sigma_i \to \Sigma_j)_{i<j\leq\alpha}$ with $\alpha < \lambda$ such that $\Sigma_0 = \Sigma$, a $\varphi_{0,\alpha}$-expansion M^{α} of M_U, $(\chi : \Sigma_\alpha \to \Sigma') \in \mathcal{D}$, and a set E' of Σ'-sentences such that $M_\alpha \models [\exists\chi]_f E'$. We have to find a χ-expansion M' of M_α such that $M' \models E'$.

- We define a contra-monotonic function $f : (\mathcal{P}_\omega E', \subset) \to (U, \subset)$ as follows:
  - Since U is countably incomplete it contains a descending ω-chain $I = I_0 \supset I_1 \supset \ldots \supset I_N \supset \ldots I_\omega = \emptyset$ with $I_n \in U$ for each $n \in \omega$.
  - Since $\varphi_{0,\alpha}$ invents completely ultraproducts there are $\varphi_{0,\alpha}$-expansions N_i of M_i, $i \in I$, and an ultraproduct $(\nu_J : N_J \to N_U)_{J\in U}$ such that $N_U = M_\alpha$.
  - For each finite $E'_0 \subset E'$, we let $fE'_0 = I_{cardE'_0} \cap \{i \in I \mid N_i \models (\exists\chi) \wedge E'_0\}$. (By $(\exists\chi) \wedge E'_0$ we designate any existential χ-quantification of any conjunction of the sentences in E'_0; these exist by hypothesis 2.)

  Then f is well defined (i.e. $fE'_0 \in U$) because

1	$I_n \in U$ for each $n \in \omega$	
2	$N_U \models (\exists\chi)E'_0$	$N_U = M_\alpha$, $M_\alpha \models [\exists\chi]_f E'$
3	$\{i \in I \mid N_i \models (\exists\chi) \wedge E'_0\} \in U$	2, $(\exists\chi) \wedge E'_0$ preserved by ultrafactors
4	$fE'_0 \in U$	1, 3, definition of f.

  That f is contra-monotonic can be checked immediately.

- Now, by using the λ-good hypothesis on U we establish that there exists a contra-additive function $g : (\mathcal{P}_\omega E', \subset) \to (U, \subset)$ such that $g \leq f$. For this, we have only to establish that $cardE' < \lambda$. This goes as follows.

5	$cardE' \leq card(Sen\Sigma')$	$E' \leq Sen\Sigma'$
6	$card(Sen\Sigma') \leq card(Sen\Sigma_\alpha)$	Sen $\mathcal{D}$-stable, $(\chi : \Sigma_\alpha \to \Sigma') \in \mathcal{D}$
7	$card(Sen\Sigma_\alpha) \leq card\alpha \cdot card(Sen\Sigma)$	Prop. 7.22
8	$cardE' \leq card\alpha \cdot card(Sen\Sigma) < \lambda \cdot \lambda = \lambda$	5, 6, 7, $\alpha < \lambda$, $card(Sen\Sigma) < \lambda$, Prop. 7.6.

- For each $i \in I$ we let $E'_i = \{\rho \in E' \mid i \in g\{\rho\}\}$. We establish that E'_i is finite by *Reductio ad Absurdum*. Suppose that $cardE'_i \geq \omega$ and consider any $n \in \omega$. Thus consider $\{\rho_1, \ldots, \rho_n\} \subset E'_i$. We have that:

9 for each $k = \overline{1,n}$, $i \in g\{\rho_k\}$ — definition of E'_i

10 $i \in \bigcap_{k \leq n} g\{\rho_k\} = g\{\rho_1, \ldots, \rho_n\} \subset f\{\rho_1, \ldots, \rho_n\} \subset I_n$ — 9, g contra-additive, $g \leq f$, definition of f

11 $i \in \bigcap_{n \in \omega} I_n$ — 10.

Since $\bigcap_{n \in \omega} I_n = \emptyset$, 11 represents a contradiction, hence E'_i is finite.

- Since E'_i is finite we can consider an existential χ-quantification of a conjunction of the sentences in E'_i; let us denote it by $(\exists \chi) \wedge E'_i$. We have that:

12 $i \in \bigcap_{\rho \in E'_i} g\{\rho\} = g(\bigcup_{\rho \in E'_i} \{\rho\}) = gE'_i \subset fE'_i$ — like at 10

13 $N_i \models (\exists \chi) \wedge E'_i$ — 12, definition of f

- Based on 13, let N'_i be a χ-expansion of N_i such that $N'_i \models E'_i$. Consider an ultraproduct $(\mu'_J : N'_J \to N'_U)_{J \in U}$ of $(N'_i)_{i \in I}$. Then

14 for each $\rho \in E'$, $g\{\rho\} \subset \{i \mid N'_i \models \rho\}$ — definition of E'_i

15 $\{i \mid N'_i \models \rho\} \in U$ — 14, $g\{\rho\} \in U$

16 $N'_U \models \rho$ — 15, ρ preserved by ultraproducts.

Hence $N'_U \models E'$. Since $Mod\chi$ preserves ultraproducts it follows that $N'_U \upharpoonright_\chi \cong N_U = M_\alpha$. By the hypothesis of lifting isomorphisms, there exists a χ-expansion M' of M_α such that $M' \cong N'_U$.

Thus we have found a χ-expansion M' of M_α such that $M' \models E'$. □

As we have always done in similar situations, let us analyse the conditions of Thm. 7.23 in terms of their applicability. We have to bear in minds that in common concrete institutions, including $\mathcal{FOL}$, $\mathcal{D}$ is usually taken to be the class of the injective signature extensions with a finite number of constants.

- The first two hypotheses of Thm. 7.23 do not require further comments.
- The $\mathcal{D}$-stability of the sentence functor has been already discussed in the context of $\mathcal{FOL}$ (see Prop. 7.21). In general, it is expected that for infinite signatures but finitary sentences the respective set of sentences has the same cardinality with the signature.
- Previously in the book we have discussed several times the preservation of ultraproducts (or filtered products, more generally); it is usually a mild condition and it is more so under our choices for $\mathcal{D}$. The lifting of isomorphisms condition has a mild appearance. Since it refers only to signature morphisms from $\mathcal{D}$, under our typical choices for $\mathcal{D}$ it becomes almost trivial. At the general level when χ is representable this is also the case.

- For the last hypothesis Prop. 6.11 gives a general solution which applies well in concrete situations, such as in $\mathcal{FOL}$. In $\mathcal{FOL}$, the $(\alpha, \mathcal{D})$-chains are just injective signature extensions with constants, which means that both alternatives provided by Prop. 6.11 can be applied with only a minimal effort.
- In light of the previous points it seems that the heavy weight of Thm. 7.23 no refers to the existence of countably incomplete λ-good ultrafilters, but this has nothing to do with the application of the general result. Finally, the boundary on the cardinality of $Sen\Sigma$ is very easy in the applications. As discussed above it typically follows from the boundaries on the size of the signatures.

Another remark is that in some institutions Thm. 7.23 together with Prop. 6.15 (and its $\mathcal{FOL}$ Cor. 6.16, saying that each model can be elementarily embedded in any of its ultrapowers) may provide an alternative way to reach essentially the existence of saturated models (Thm. 7.11). However, the mathematical effort in this case may be significantly higher than that in Thm. 7.11: the Łoś property for the institution, and especially the rather difficult result of the existence of good countably incomplete ultrafilters (Thm. 7.20).

Keisler-Shelah isomorphism theorem

The following is an amazing application of the uniqueness property of saturated models and of the existence of saturated ultraproducts.

Corollary 7.24. *Consider an institution which satisfies the hypotheses of Thm.s 7.18 and 7.23 and such that each model M has a $\mathcal{D}$-size such that if M has a $\mathcal{D}$-size λ, then each ultrapower M_U for an ultrafilter U over I has $\mathcal{D}$-size $\lambda^{card(I)}$. Then any two elementarily equivalent models have isomorphic ultrapowers (for the same ultrafilter).*

Proof. Let $M \equiv N$ be elementarily equivalent Σ-models. Consider a cardinal λ such that both M and N have $\mathcal{D}$-size λ^+ and such that $card(Sen\Sigma) \leq \lambda$. We can do this because once a model has a certain $\mathcal{D}$-size it also has any bigger $\mathcal{D}$-size. Cf. Thm. 7.20 let U be a countably incomplete λ^+-good ultrafilter over λ. We consider λ-ultrapowers M_U, N_U of M, N, respectively. Then both M_U and N_U have $\mathcal{D}$-size $(\lambda^+)^\lambda = \lambda^+$ (cf. Prop. 7.6 on cardinal arithmetic). By Thm. 7.23 both ultrapowers are $(\lambda^+, \mathcal{D})$-saturated. By the uniqueness Thm. 7.18 they are therefore isomorphic. □

The famous corresponding $\mathcal{FOL}$ result, due to Keisler and Shelah, comes now as an instance of Cor. 7.24.

Corollary 7.25. *In $\mathcal{FOL}$, any two elementarily equivalent models have isomorphic ultrapowers.*

Proof. While hypotheses of Thm.s 7.23 and 7.18 in the framework of $\mathcal{FOL}$ have been discussed above, if we define the sizes of models by their cardinality, then the specific condition about sizes of Cor. 7.24 holds obviously as each ultrapower M_U is the quotient of the power M_I. □

Now we can take a breath and contemplate this result a bit. It is the pinnacle of a big development effort which involved several important methods (diagrams, ultraproducts, saturated models) and a lot of results, some of them mathematically difficult. Also, it says something deep and perhaps surprising, that in a certain sense the first-order syntax 'gets evaporated' by the ultraproduct construction, a merely algebraic construction. Models satisfying the same first-order properties are characterized just by 'convergence' to the same model.

Keisler-Shelah institutions. An institution with ultraproducts of models satisfying the property that any two elementary equivalent models have isomorphic ultrapowers is called a *Keisler-Shelah institution*.

Exercises

7.9. Establish the Keisler-Shelah property in concrete institutions presented as examples in this book (such as $\mathcal{PA}$, $\mathcal{POA}$, etc.) by one of the methods below:

1. directly the general result of Cor. 7.24, or
2. by 'borrowing' it from $\mathcal{FOL}$ via various institution comorphisms presented in this book.

Which of the two methods suggested above does apply to $\mathcal{HNK}$?

Notes. The $\mathcal{FOL}^1$ special case of Cor. 7.5 for chains instead of any directed co-limit was proved by Tarski and Vaught in [234] and received high notoriety in conventional model theory under the name 'Elementary Chain Theorem', while Theorems 7.1 and 7.2 are due to [139].

The concept of a saturated model can be traced back to the η_α-sets of [144]. A good reference for cardinal arithmetic is [151]. The theory of saturated models at the institution-independent level was developed in [98] where both the existence and the uniqueness Theorems 7.11 and 7.18 appear. The $\mathcal{FOL}^1$ instances of these results were proved in [180].

Our definition of countably incomplete ultrafilters is formulated slightly differently but equivalently to the standard one in [42]. The existence of saturated ultraproducts (Thm. 7.23) is due to [98] and generalizes the corresponding $\mathcal{FOL}^1$ result which can be traced back to [154]. The Keisler-Shelah isomorphism theorem in $\mathcal{FOL}^1$ (Cor. 7.25) was proved in [223] without assuming GCH.

Chapter 8

Preservation and Axiomatizability

Axiomatizability results express a rather subtle relationship between semantics and syntax. They give complete characterizations of certain classes of theories in purely semantic terms, formulated as closure properties of classes of models under some categorical operators. Perhaps the most famous example is the Birkhoff Variety Theorem of equational logic: a class of algebras for a signature is closed under products, sub-algebras, and homomorphic images (i.e. quotients) if and only if it is the class of algebras of an equational theory. This result is considered as one of the developments that represented the dawn of model theory.

Axiomatizability results have been traditionally considered to have mostly theoretical significance. But this may not correspond to the truth as they do have important applications such as interpolation and definability. Some of these applications have been discovered and understood properly only relatively recently.

Preservation results are halfway to axiomatizability results in the sense that, assuming a theory, then it can be presented by a certain kind of sentences whenever it is 'preserved' by some semantic operators. Thus in the case of preservation results the respective class of models is already axiomatised and a preservation result just says that it can be axiomatised by using only a specific kind of sentences. A typical example is the following: a $\mathcal{FOL}$ theory can be presented by a set of universal sentences if and only if it is 'preserved' by sub-models. Some axiomatizability results can be obtained via their preservation correspondents.

We start this chapter with the development of a general institution-independent preservation result by using the saturated models of Chap. 7. Then we develop some axiomatisability results by using ultraproducts. In another section, we develop the interdependence between quasi-varieties and the existence of initial models of theories, a result of great relevance for logic-based computing languages. In the next section, we show that quasi-varieties are exactly the class of models of Horn sentences. Then we develop a general institution-independent replica of the Birkhoff Variety Theorem. The chapter ends with an abstract formulation for Birkhoff-style axiomatizability; this captures uniformly all axiomatizability results of this chapter and much more. The current institution-independent model theory literature covers only a relatively small subset of the preservation and axiomatizability results that are known in first order model theory. It is worth extending the collection of preservation and axiomatizability results of this chapter with new ones, possibilities in this research direction being vast.

8.1 Preservation by saturation

In this section, we develop a general preservation result as an application of saturated models. It can be regarded as a high generalisation of the $\mathcal{FOL}$ preservation example mentioned in the introduction to this chapter.

The framework. For this section we consider

1. an institution with diagrams $I = (Sig, Sen, Mod, \models, \iota)$ together
2. with a sub-functor $Sen^0 \subseteq Sen$ (i.e., a natural transformation $Sen^0 \Rightarrow Sen$ such that all its components are set inclusions),

$$
\begin{array}{ccccc}
\Sigma & \quad & Sen^0\Sigma & \xrightarrow{\subseteq} & Sen\Sigma \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle Sen^0\varphi} & & \downarrow{\scriptstyle Sen\varphi} \\
\Sigma' & & Sen^0\Sigma' & \xrightarrow[\subseteq]{} & Sen\Sigma'
\end{array}
$$

   and

3. a sub-category $\mathcal{D} \subseteq Sig$ of signature morphisms,

such that

- for each Σ-model M, (the diagram) $E_M \subseteq Sen^0\Sigma_M$,
- $I^0 = (Sig, Sen^0, Mod, \models)$ has finite conjunctions, finite disjunctions, and existential $\mathcal{D}$-quantification.

Universal and existential sentences in $\mathcal{FOL}$. As a typical example for this framework, we may take I to be $\mathcal{FOL}$ and $Sen^0\Sigma$ to be the set of all *existential* Σ*-sentences* which are existential quantifications of quantifier-free sentences. Note that the existential sentences in $\mathcal{FOL}$ are indeed closed under conjunctions and disjunctions (Ex. 5.6 gave a general institution-independent version of this). In this context recall from Sect. 3.2 that *universal sentences* are the negations of the existential sentences, which means that they are universal quantifications of quantifier-free sentences. While universal sentences are closed under finite conjunctions and disjunctions they are not closed under existential quantifications.

Sen^0**-extensions.** For any Σ-models M and N, let us establish the notation

$$M[Sen^0]N \text{ if and only if } M^* \cap Sen^0\Sigma \subseteq N^* \cap Sen^0\Sigma.$$

We say that M *is a* Sen^0*-submodel of* N when there exists a Σ-model homomorphism $h\colon M \to N$ such that $M_M[Sen^0]N_h$. (Recall that by N_h we mean $i^{-1}_{\Sigma,M}h$, the mapping of h by the canonical isomorphism $M/Mod(\Sigma) \to Mod(\Sigma_M, E_M)$.) Alternatively we may say that N *is a* Sen^0*-extension of* M. Let us denote this relation by $M \xrightarrow{Sen^0} N$.

In particular situations, it is often possible to express the Sen^0-extension relationship in purely semantic terms. The following is a typical example.

Proposition 8.1. *In $\mathcal{FOL}$, let* $\mathsf{Exist} \subseteq Sen^{\mathcal{FOL}}$ *be the sub-functor of the existential sentences. Then $M \xrightarrow{\mathsf{Exist}} N$ if and only if there exists a closed injective model homomorphism $M \to N$.*

Proof. • Consider $M \xrightarrow{\mathsf{Exist}} N$ for a $\mathcal{FOL}$-signature Σ. By definition, there exists a model homomorphism $h\colon M \to N$ such that $M_M[\mathsf{Exist}]N_h$. Then

– h is injective because for all $m_1 \neq m_2 \in M_s$ we have that

1	$M_M \models \neg(m_1 = m_2)$	$m_1 \neq m_2$
2	$N_h \models \neg(m_1 = m_2)$	1, $(M_M)^* \cap \mathsf{Exist}\Sigma_M \subseteq (N_h)^* \cap \mathsf{Exist}\Sigma_M$, $\neg(m_1 = m_2) \in \mathsf{Exist}\Sigma_M$
3	$hm_1 \neq hm_2$	2, $(N_h)_{m_1} = hm_1$, $(N_h)_{m_2} = hm_2$.

– h is closed because for each relation symbol π in Σ and each m an appropriate sequence of arguments such that $hm \in N_\pi$, if we assumed that $m \notin M_\pi$ then we reach a contradiction as follows:

4	$m \notin (M_M)_\pi$	$m \notin M_\pi$, $(M_M)_\pi = M_\pi$
5	$M_M \models \neg\pi m$	4
6	$N_h \models \neg\pi m$	5, $(M_M)^* \cap \mathsf{Exist}\Sigma_M \subseteq (N_h)^* \cap \mathsf{Exist}\Sigma_M$, $\neg\pi m \in \mathsf{Exist}\Sigma_M$
7	$(N_h)_m \notin (N_h)_\pi$	6
8	$hm \notin N_\pi$	7, $(N_h)_m = hm$, $(N_h)_\pi = N_\pi$.

Since 8 contradicts $hm \in N_\pi$ we necessarily have $m \in M_\pi$.

• Now we show that the existence of any closed injective model homomorphism $h\colon M \to N$ implies that $M \xrightarrow{\mathsf{Exist}} N$. We consider any sentence $(\exists X)\rho' \in \mathsf{Exist}\Sigma_M$ such that $M_M \models (\exists X)\rho'$. Let us denote the extension of Σ_M with X by $\chi\colon \Sigma_M \to \Sigma_M + X$. First, we prove that

9 for each quantifier-free Σ_M-sentence ρ, $(M_M \models \rho) = (N_h \models \rho)$.

We prove 9 by induction on the structure of the sentences as follows.

– First, we consider an atomic equation:

$$\begin{aligned}(M_M \models t = t') &= ((M_M)_t = (M_M)_{t'}) && \text{definition of satisfaction of equations}\\ &= (i^{-1}_{\Sigma,M}h)(M_M)_t = (i^{-1}_{\Sigma,M}h)(M_M)_{t'}) && i^{-1}_{\Sigma,M}h \text{ injective homomorphism (property inherited from } h)\\ &= ((N_h)_t = (N_h)_{t'}) && \text{homomorphisms preserve evaluation of terms}\\ &= (N_h \models t = t') && \text{definition of satisfaction of equations.}\end{aligned}$$

– Then we consider a relational atom:

10 $M_M \models \pi(t_1,\dots,t_n) = ((M_M)_{t_1},\dots,(M_M)_{t_n}) \in (M_M)_\pi$ — satisfaction of relations

11 $(i^{-1}_{\Sigma,M}h)(M_M)_{t_k} = (N_h)_{t_k},\ k = \overline{1,n}$ — homomorphisms preserve evaluation of terms

12 $((N_h)_{t_1},\dots,(N_h)_{t_n}) \in (N_h)_\pi = ((M_M)_{t_1},\dots,(M_M)_{t_n}) \in (M_M)_\pi$ — 11, $i^{-1}_{\Sigma,M}h$ closed homomorphism (property inherited from h)

13 $N_h \models \pi(t_1,\dots,t_n) = ((N_h)_{t_1},\dots,(N_h)_{t_n}) \in (N_h)_\pi$ — satisfaction of relations

14 $M_M \models \pi(t_1,\dots,t_n) \ = \ N_h \models \pi(t_1,\dots,t_n)$ — 10, 12,13.

The two cases above proved the induction base. These were the interesting parts of this induction proof as the induction step corresponding to the Boolean connectives is completely straightforward. Thus 9 has been proved and we can proceed further with our proof.

15 there exists $M' \in |Mod(\Sigma_M + X)|$, $M'\restriction_\chi = M_M$, $M' \models \rho'$ — $M_M \models (\exists X)\rho'$

Let $\theta\colon\ \chi \to 1_{\Sigma_M}$ be the Σ_M-substitution defined by $\theta x = M'_x$ for each $x \in X$. (At this point it may be worth recalling the concept of institution-independent substitution applied to $\mathcal{FOL}$ as presented in Sec. 5.3.) We have that:

16 $(Mod\theta)M_M = M'$ — definition of θ

17 $M_M \models (Sen\theta)\rho'$ — $M' \models \rho'$, 16, Satisfaction Condition of θ

18 $N_h \models (Sen\theta)\rho'$ — 9, $(Sen\theta)\rho'$ quantifier-free (ρ' quantifier-free)

19 $(Mod\theta)N_h \models \rho'$ — 18, Satisfaction Condition of θ

20 $(Mod\chi)((Mod\theta)N_h) = N_h$ — θ Σ_M-substitution $\chi \to 1_{\Sigma_M}$

21 $N_h \models (\exists\chi)\rho'$ — 19, 20.

□

Preservation by saturation

We say that a set of sentences E is *preserved by* Sen^0*-extensions* when for any two models M and N such that $M \xrightarrow{Sen^0} N$, $M \models E$ implies $N \models E$. Dually, E *is preserved by* Sen^0*-submodels* when $N \models E$ implies $M \models E$.

The following notation will ease our presentation. For any set of sentences Γ we let $\neg\Gamma$ denote the set of the negations of the sentences in Γ.

Theorem 8.2 (Preservation by saturation)**.** *In addition to the framework of this section let us also assume the following conditions:*

1. *I has inductive weak model amalgamation,*
2. *I is compact and Boolean complete,*
3. *Sen^0 consists of finitary sentences,*
4. *each model has a $\mathcal{D}$-size,*
5. *I has $\mathcal{D}$-saturated models.*

Then for any consistent Σ-theory E

- *E is preserved by Sen^0-extensions if and only if $E^{**} \cap Sen^0\Sigma \models E$, and*
- *E is preserved by Sen^0-submodels if and only if $E^{**} \cap \neg Sen^0\Sigma \models E$.*

Proof. • First we will prove a Lemma:

Lemma 8.3. *In any m-compact Boolean complete institution, for any consistent theory E and set Δ of sentences closed under finite disjunctions, the following are equivalent:*

- *$E^{**} \cap \Delta \models E$, and*
- *for all models M, N, $M \models E$ and $N \models M^* \cap \Delta$ implies $N \models E$.*

- For the direct implication we consider models M, N such that $M \models E$ and $N \models M^* \cap \Delta$. We have that:

1	$E^{**} \subseteq M^*$	$E \subseteq M^*$, M^* closed theory
2	$E^{**} \cap \Delta \subseteq M^* \cap \Delta \subseteq N^*$	1, $M^* \cap \Delta \subseteq N^*$
3	$(E^{**} \cap \Delta)^{**} \subseteq N^*$	2, N^* closed theory
4	$E \subseteq (E^{**} \cap \Delta)^{**}$	hypothesis $E^{**} \cap \Delta \models E$
5	$E \subseteq N^*$	3, 4.

Let us note that the proof of the direct implication has not used any of the hypotheses of the Lemma, so it holds in general.

- For the inverse implication we consider $N \in (E^{**} \cap \Delta)^*$ and prove that $N \models E$. First, we show that

6 $E^{**} \cup (N^* \cap \neg\Delta)$ consistent.

Consider any finite sets $E_0 \subseteq E^{**}$, $\Delta_0 \subseteq N^* \cap \neg\Delta$.

7	$\neg\Delta_0 \cap N^* = \emptyset$	$\Delta_0 \subseteq N^*$
8	$\neg\Delta_0 \subseteq \Delta$	$\Delta_0 \subseteq \neg\Delta$
9	$E^{**} \cap \neg\Delta_0 \subseteq N^*$	8, $E^{**} \cap \Delta \subseteq N^*$
10	$E^{**} \cap \neg\Delta_0 = \emptyset$	7, 9

11 $E_0^{**} \cap \neg\Delta_0 = \emptyset$ — 10, $E_0 \subseteq E^{**}$

12 $E_0^{**} \not\models \vee\neg\Delta_0$ (the disjunction of the sentences of $\neg\Delta_0 \subseteq \Delta$) — 11

13 $E_0 \cup \Delta_0$ consistent — 12.

From 13 by the m-compactness of the institution we obtain 6, hence there exists M a model such that $M \models E^{**} \cup (N^* \cap \neg\Delta)$. Then

14 $\neg M^* \cap \neg\Delta \cap N^* \subseteq \neg M^* \cap M^* = \emptyset$ — $N^* \cap \neg\Delta \subseteq M^*$

15 $\neg M^* \cap \neg\Delta \subseteq \neg N^*$ — 14

16 $M^* \cap \Delta \subseteq N^*$ — 15.

Then the conclusion that $N \models E$ follows from 16, from $M \models E^{**}$, and by the hypothesis (of the inverse implication).

We have thus proved both implications of the conclusion of Lemma 8.3.

- Let us now prove that for any Σ-models M and N

17 if M has $\mathcal{D}$-size λ, N is $(\lambda^+, \mathcal{D})$-saturated, and $M[Sen^0]N$, then $M \xrightarrow{Sen^0} N$.

Let $(\varphi_{i,j} : \Sigma_i \to \Sigma_j)_{0 \leq i < j \leq \lambda}$ be a $(\lambda, \mathcal{D})$-chain such that $\iota_\Sigma M = \varphi_{0,\lambda}$. By Ordinal Induction we define $(N_i)_{0 \leq i \leq \lambda}$ such that $M_i[Sen^0]N_i$ and $N_i \upharpoonright_{\varphi_{0,i}} = N$, where $M_i = (M_M)\upharpoonright_{\varphi_{i,\lambda}}$. If this were achieved then $M \xrightarrow{Sen^0} N$ because

- on the one hand $M_M[Sen^0]N_\lambda$ as $M_M = M_\lambda$, and
- on the other hand, by letting $h = i_{\Sigma,M} N_\lambda$ (which is possible because $N_\lambda \models E_M$ as $E_M \subseteq (M_M)^* \cap Sen^0\Sigma_M \subseteq (N_\lambda)^* \cap Sen^0\Sigma_M$) we have $N_\lambda = N_h$.

Let us now do the promised Ordinal Induction construction.

- For any successor ordinal $\alpha + 1 \leq \lambda$, we consider any finite set $\Gamma \subseteq M^*_{\alpha+1} \cap Sen^0\Sigma_{\alpha+1}$. Let γ be a conjunction of the sentences of Γ (as I^0 has finite conjunctions). Then:

  18 $\gamma \in M^*_{\alpha+1} \cap Sen^0\Sigma_{\alpha+1}$ — I^0 has finite conjunctions

  19 let γ' be an existential $\varphi_{\alpha,\alpha+1}$-quantification of γ — $\varphi_{\alpha,\alpha+1} \in \mathcal{D}$, I^0 has existential $\mathcal{D}$-quantifications

  20 $\gamma' \in M^*_\alpha \cap Sen^0\Sigma_\alpha$ — 18, 19, $M_\alpha = M_{\alpha+1}\upharpoonright_{\varphi_{\alpha,\alpha+1}}$

  21 $N_\alpha \models \gamma'$ — 20, $M_\alpha[Sen^0]N_\alpha$ (induction hypothesis)

  22 $N_\alpha \models [\exists\varphi_{\alpha,\alpha+1}]_f(M^*_{\alpha+1} \cap Sen^0\Sigma_{\alpha+1})$ — 21, Γ arbitrary

  23 $N_\alpha \models [\exists\varphi_{\alpha,\alpha+1}](M^*_{\alpha+1} \cap Sen^0\Sigma_{\alpha+1})$ — 22, $\alpha + 1 < \lambda^+$, N $(\lambda^+, \mathcal{D})$-saturated

  By 23 we obtain $N_{\alpha+1}$ such that $N_{\alpha+1}\upharpoonright_{\varphi_{\alpha,\alpha+1}} = N_\alpha$ and $N_{\alpha+1} \models M^*_{\alpha+1} \cap Sen^0\Sigma_{\alpha+1}$.

- For α a limit ordinal, by the inductive weak model amalgamation hypothesis, let N_α be an amalgamation of $(N_\beta)_{0 \leq \beta < \alpha}$. Let $\rho_\alpha \in M^*_\alpha \cap Sen^0\Sigma_\alpha$. We have to prove that $N_\alpha \models \rho_\alpha$. This goes as follows.

24	there exists $\beta < \alpha$, $\rho_\beta \in Sen^0\Sigma_\beta$ such that $\varphi_{\beta,\alpha}\rho_\beta = \rho_\alpha$	like in the proof of Thm. 7.18 as Sen^0 consists of finitary sentences
25	$M_\beta \models \rho_\beta$	$M_\beta = M_\alpha \upharpoonright_{\varphi_{\alpha,\alpha+1}}$, $M_\alpha \models \rho_\alpha$, 24, Satisfaction Condition
26	$N_\beta \models \rho_\beta$	25, $M_\beta[Sen^0]N_\beta$ (induction hypothesis)
27	$N_\alpha \models \rho_\alpha$	26, $N_\beta = N_\alpha \upharpoonright_{\varphi_{\beta,\alpha}}$, 24, Satisfaction Condition.

- Now let us prove that

  28 if $M[Sen^0]N$ then there exists N' such that $M \xrightarrow{Sen^0} N' \equiv N$.

  Let λ be a $\mathcal{D}$-size for M. Because I has $\mathcal{D}$-saturated models there exists a homomorphism $N \to N'$ such that N' is $(\lambda^+, \mathcal{D})$-saturated and $N' \equiv N$. By 17 we also have that $M \xrightarrow{Sen^0} N'$.

- Now we proceed to the proof of the first conclusion of the theorem.
  - For the direct implication we consider a consistent Σ-theory E preserved by Sen^0-extensions. By Lemma 8.3, with Δ set to $Sen^0\Sigma$, because $Sen^0\Sigma$ is closed under finite disjunctions it is enough to show that for arbitrary Σ-models, $M \models E$ and $N \models M^* \cap Sen^0\Sigma$ imply $N \models E$. We do this as follows:

29	$M[Sen^0]N$	$N \models M^* \cap Sen^0\Sigma$
30	there exists N' such that $M \xrightarrow{Sen^0} N' \equiv N$	29, 28
31	$N \models E$	30, E preserved by Sen^0-extensions.

  - For the reverse implication we consider models M, N such that $M \xrightarrow{Sen^0} N$ (with $h\colon M \to N$ homomorphism) and $M \models E$. We prove that $N \models E$.

32	$M \models E^{**}$	$M \models E$
33	$M_M \models (\iota_\Sigma M)E^{**}$	32, $M = M_M \upharpoonright_{\iota_\Sigma M}$, Satisfaction Condition
34	$(\iota_\Sigma M)(E^{**} \cap Sen^0\Sigma) \subseteq (M_M)^* \cap Sen^0\Sigma_M$	33, $(\iota_\Sigma M)Sen^0\Sigma \subseteq Sen^0\Sigma_M$
35	$(\iota_\Sigma M)(E^{**} \cap Sen^0\Sigma) \subseteq (N_h)^* \cap Sen^0\Sigma_M$	34, $M_M[Sen^0]N_h$
36	$N_h \models (\iota_\Sigma M)(E^{**} \cap Sen^0\Sigma) \models (\iota_\Sigma M)E$	35, hypothesis of the reverse implication, 'translation' property of $\models$
37	$N \models E$	36, $N_h \upharpoonright_{\iota_\Sigma M} = N$, Satisfaction Condition.

The direct implication of the first conclusion may be considered as the important implication, a true preservation result. In some sense the other implication is straightforward. This evaluation is consonant with what is required to obtain each of these results. While the former implication uses Lemma 8.3 (in fact its hard implication) and all hypotheses of the theorem, the latter implication needs only the basic framework and properties and the respective definitions.

- The proof of the second part of the conclusion, that refers to preservation by Sen^0-submodels, is similar to the proof of the first part of the conclusion given above. In this case the Δ in Lemma 8.3 is set to $\neg Sen^0\Sigma$. Another helpful fact is that $M[\neg Sen^0]N$ means $N[Sen^0]M$.

□

From the conditions of the preservation Thm. 8.2, except for the condition on the existence of saturated models, all others are rather common conditions whose applicability has already been discussed elsewhere in the book (for instance in Chap. 7). In applications, the existence of saturated models is handled by Thm. 7.11. The following is a typical concrete instance of the general result of Thm. 8.2.

Corollary 8.4. *A $\mathcal{FOL}$ theory is preserved by closed sub-models, respectively extensions, if and only if it is presented by a set of universal, respectively existential, sentences.*

Proof. $\mathcal{FOL}$ has model amalgamation (Prop. 4.6), is compact (Cor. 6.24), has only finitary sentences (Fact 7.17), each model has a $\mathcal{D}$-size given by the cardinality of its carrier sets (Fact 7.15) for $\mathcal{D}$ the class of the injective signature extensions with a finite number of constants, and has $\mathcal{D}$-saturated models (Cor. 7.12). The conclusion follows by Thm. 8.2 and by Prop. 8.1 when taking Sen^0 to be Exist, the existential sentences. □

Exercises

8.1. The result of Prop. 8.1 can be changed in various ways by weakening the requirements on the model homomorphisms as follows. Let Sen^0 consist of the existential quantifications of sentences accessible by disjunction and conjunction from a class of sentences B (which is a parameter of the problem). For any $\mathcal{FOL}$ models M and N (of the same signature) $M \xrightarrow{Sen^0} N$ if and only if there exists a homomorphism $h : M \to N$ which is

- just plain, when B consists of all the atoms,
- injective, when B consists of all the atoms and the negations of equational atoms, and
- closed, when B consists of all the atoms and the negations of relational atoms.

By instantiating the general preservation Thm. 8.2, formulate variants of the preservation results of Cor. 8.4 corresponding to the three situations above.

8.2. Preservation in $\mathcal{PA}$
A $\mathcal{PA}$ sentence is existential, respectively universal, when it is an existential, respectively universal, quantification of a sentence which is accessible by Boolean connectives from existence equations. A $\mathcal{PA}$ theory is preserved by closed sub-algebras (see Ex. 4.68), respectively extensions, if and only if it is presented by a set of universal, respectively existential, sentences. Provide variants of these results which correspond to the three situations from Ex. 8.1.

8.2 Axiomatizability by ultraproducts

Recall that a class of models $\mathbb{M}$ for a signature is called *elementary* when it is closed, i.e., $\mathbb{M}^{**} = \mathbb{M}$. In other words, elementary classes of models are the classes of models of theories.

Theorem 8.5. *Consider an institution with ultraproducts of models such that its sentences are preserved by ultraproducts. Then*

1. *Any elementary class of models is closed under elementary equivalence and ultraproducts.*
2. *If the institution has finite conjunctions and negations then the reverse of the above holds too: any class of models that is closed under elementary equivalence and ultraproducts is elementary.*

Proof. 1. Follows immediately from the hypothesis.

2. This is the 'interesting' implication. Consider a class of Σ-models $\mathbb{M}$ closed under ultraproducts and elementary equivalence. Let $E = \mathbb{M}^*$. We have to prove that $\mathbb{M} = E^*$, i.e. that $N \in \mathbb{M}$ whenever $N \in E^*$.

   - First we show that

1	for all $N \in E^*$ and $i \in \mathcal{P}_\omega N^*$ there exists $M_i \in \mathbb{M} \cap i^*$.	

   Let i' be a negation of a conjunction of all sentences of i. Then

2	$N \in \mathbb{M}^{**} \setminus i'^*$	$N \in E^* = \mathbb{M}^{**}, N \models i$
3	$\mathbb{M} \setminus i'^* \neq \emptyset$	2, $\mathbb{M} \subseteq i'^*$ implies $\mathbb{M}^{**} \subseteq i'^*$
4	$\mathbb{M} \cap i^* \neq \emptyset$	3, $i^* \cap i'^* = \emptyset$, $i^* \cup i'^* = \lvert Mod\Sigma \rvert$

   - From 1, by the compactness Thm. 6.19 there exists an ultrafilter over $\mathcal{P}_\omega N^*$ such that for any ultraproduct $(\mu_J : M_J \to M_U)_{J \in U}$ of $(M_i)_{i \in \mathcal{P}_\omega N^*}$ we have that $M_U \models N^*$. Thus

5	$N^* \subseteq (M_U)^*$	$M_U \models N^*$
6	$(M_U)^* \subseteq N^*$	5, the institution has negations
7	$M_U \equiv N$	5, 6
8	$M_U \in \mathbb{M}$	$(M_i)_{i \in \mathcal{P}_\omega N^*} \subseteq \mathbb{M}$ (1), $\mathbb{M}$ closed under ultraproducts
9	$N \in \mathbb{M}$	7, 8, $\mathbb{M}$ closed under elementary equivalence.

□

Finitely elementary classes. A class of models of a signature is *finitely elementary* when it is the class of models of a finite theory.

Theorem 8.6. *Let $\mathbb{M}$ be an elementary class of Σ-models in an arbitrary institution. Then*

1. *If the institution has negations and finite conjunctions and $\mathbb{M}$ is finitely elementary then $\lvert Mod\Sigma \rvert \setminus \mathbb{M}$ is elementary.*
2. *If the institution has ultraproducts of models and its sentences are preserved by ultraproducts, and $\lvert Mod\Sigma \rvert \setminus \mathbb{M}$ is elementary then $\mathbb{M}$ is finitely elementary.*

Proof. 1. There exists a finite set E Σ-sentences such that $E^* = \mathbb{M}^*$. Then the complement of $\mathbb{M}$ is e'^* where e' is any negation of any conjunction of the sentences of E.

2. By *Reductio ad Absurdum* we show that $\mathbb{M}^*$ is presented by a finite theory. Thus suppose that for each finite $i \in \mathbb{M}^*$, $\mathbb{M} \neq i^*$. Then

1	$i^* \setminus \mathbb{M} \neq \emptyset$	$\mathbb{M} \neq i^*$, $\mathbb{M} = \mathbb{M}^{**} \subseteq i^*$

Let $M_i \in i^* \setminus \mathbb{M} = i^* \cap (Mod\Sigma \setminus \mathbb{M})$. By compactness Thm. 6.19 there exists an ultrafilter U over $\mathcal{P}_\omega \mathbb{M}^*$, an ultraproduct $(\mu_J : M_J \to M_U)_{J \in U}$ over $(M_i)_{i \in \mathcal{P}_\omega \mathbb{M}^*}$ such that $M_U \models \mathbb{M}^*$. Then on the one hand:

2	$\mathbb{M}^* \subseteq (M_U)^*$	$M_U \models \mathbb{M}^*$
3	$(M_U)^{**} \subseteq \mathbb{M}^{**} = \mathbb{M}$	2, $\mathbb{M}$ elementary
4	$M_U \in \mathbb{M}$	3, $M_U \in (M_U)^{**}$.

On the other hand:

5	$M_U \in \lvert Mod\Sigma \rvert \setminus \mathbb{M}$	$(M_i)_{i \in \mathcal{P}_\omega \mathbb{M}^*} \subseteq \lvert Mod\Sigma \rvert \setminus \mathbb{M}$, $\lvert Mod\Sigma \rvert \setminus \mathbb{M}$ closed under ultraproducts ($\lvert Mod\Sigma \rvert \setminus \mathbb{M}$ elementary).

But 4 and 5 together represent a contradiction. Hence there exists a finite $i \subseteq \mathbb{M}^*$ such that $\mathbb{M} = i^*$.

□

Axiomatizability in Keisler-Shelah institutions

Institutions admitting the Keisler-Shelah property allow for a purely algebraic characterization of elementary equivalence as a consequence of Thm. 8.5.

Ultraradicals. Let us say that a model M is an *ultraradical* of a model N if N is an ultrapower of M.

Corollary 8.7. *In any Keisler-Shelah institution with negations and finite conjunctions and such that each sentence is preserved by ultraproducts, for any class $\mathbb{M}$ of Σ-models the following are equivalent:*

1. *$\mathbb{M}$ is elementary.*
2. *$\mathbb{M}$ is closed under ultraproducts and ultraradicals.*
3. *$\mathbb{M}$ is closed under ultraproducts and its complementary is closed under ultraradicals.*

Moreover, $\mathbb{M}$ is finitely elementary if and only if both $\mathbb{M}$ and its complement are closed under ultraproducts.

Proof. Let N be any ultrapower of M. Then

1 $M \equiv N$ — $M^* \subseteq N^*$ (each sentence is preserved by ultraproducts), the institution has negations.

- 1. implies 2.: The closure under ultraproducts follows from the hypothesis that each sentence of the institution is preserved by ultraproducts. Let M be an ultraradical of a model $N \in \mathbb{M}$. By 1 $M \equiv N$ hence $M \in \mathbb{M}$.
- 2. implies 3.: Let $M \in |Mod\Sigma| \setminus \mathbb{M}$ and let M_U be any ultrapower of M. By *Reductio ad Absurdum*, if $M_U \in \mathbb{M}$ then since $\mathbb{M}$ is closed under ultraradicals it follows that $M \in \mathbb{M}$, which contradicts $M \in |Mod\Sigma| \setminus \mathbb{M}$. Hence $M_U \in |Mod\Sigma| \setminus \mathbb{M}$.
- 3. implies 1.: By Thm. 8.5(2.) it would suffice to prove that $\mathbb{M}$ is closed under elementary equivalence. Let $M \in \mathbb{M}$ and $N \equiv M$. By *Reductio ad Absurdum* we suppose $N \notin \mathbb{M}$. Then by the Keisler-Shelah property, there exists an ultrafilter U and ultrapowers M_U, N_U of M, N, respectively, such that $M_U \cong N_U$. Then

  2 $N_U \notin \mathbb{M}$ — $N \notin \mathbb{M}$, $|Mod\Sigma| \setminus \mathbb{M}$ closed under ultrapowers

  3 $M_U \notin \mathbb{M}$ — 2, $M_U \cong N_U$

  4 $M_U \in \mathbb{M}$ — $M \in \mathbb{M}$, $\mathbb{M}$ closed under ultraproducts.

  Since 3 and 4 together represent a contradiction, we conclude that $N \in \mathbb{M}$.
- For the last conclusion of the corollary, by Thm. 8.6 it follows that $\mathbb{M}$ is finitely elementary if and only if both $\mathbb{M}$ and $|Mod\Sigma| \setminus \mathbb{M}$ are elementary. From this, the conclusion follows by the equivalence between 1. and 3. above, applied to both $\mathbb{M}$ and its complement simultaneously.

□

Cor. 8.7 applies well to $\mathcal{FOL}$.

Axiomatizability by universal sentences

The Keisler-Shelah property makes it possible to convert the general preservation result of Thm. 8.2 into an axiomatizability result.

Corollary 8.8. *Further to the framework of Sect. 8.1 and the conditions of the preservation Thm. 8.2 let us also assume that*

1. *the institution has ultraproducts of models which are preserved by the model reducts corresponding to the elementary extensions,*
2. *all sentences of the institution are preserved by ultraproducts, and*
3. *the institution has the Keisler-Shelah property.*

Then the following are equivalent for a non-empty class $\mathbb{M}$ of models of a signature:

- *$\mathbb{M}$ is closed under ultraproducts and Sen^0-submodels, and*

- $\mathbb{M}$ *is the class of models of a* $\neg Sen^0$*-theory.*

Proof. • We prove the direct implication in two steps, first that $\mathbb{M}$ is elementary and then that it can be axiomatized by a $\neg Sen^0$-theory.

- By Thm. 8.5 it is enough to show that $\mathbb{M}$ is closed under elementary equivalence. Let $M \equiv N$ with $N \in \mathbb{M}$. We show that $M \in \mathbb{M}$ too as follows:

1	M, N have isomorphic ultrapowers $M_U \cong N_U$	$M \equiv N$, Keisler-Shelah property
2	$N_U \in \mathbb{M}$	$N \in \mathbb{M}$, $\mathbb{M}$ closed under ultraproducts
3	$M_U \in \mathbb{M}$	1, 2
4	there exists ι-elementary homomorphism $h: M \to M_U$	Prop. 6.15
5	$(M_M)^* \cap Sen^0\Sigma_M \subseteq (N_h)^* \cap Sen^0\Sigma_M$	$(M_M)^* \subseteq (N_h)^*$ (4)
6	$M \in \mathbb{M}$	3, 6, $\mathbb{M}$ closed under Sen^0-submodels.

- Now let $E = \mathbb{M}^*$. Because $\mathbb{M}$ is elementary we have $\mathbb{M} = E^*$ and thus E is preserved by Sen^0-submodels. Now we can apply the 'hard' implication of the second conclusion of the preservation by saturation Thm. 8.2 and get that E can be presented only by sentences from $\neg Sen^0\Sigma$.

• The inverse implication falls immediately from the 'easy' part of the second conclusion of Thm. 8.2.

□

By taking Sen^0 to be the functor Exist of the existential sentences in $\mathcal{FOL}$, we get the following concrete 'axiomatizability by universal sentences' result.

Corollary 8.9. *A class of $\mathcal{FOL}$ models is the class of models of a universal theory (i.e., a theory presented by universal sentences) if and only if it is closed under ultraproducts and closed submodels.*

Exercises

8.3. Develop instances of the universal axiomatizability Cor. 8.8 in $\mathcal{FOL}$, different from Cor. 8.9, based upon the preservation results of Ex. 8.1. Develop similar universal axiomatizability results in $\mathcal{PA}$ and other concrete institutions presented in the book.

8.3 Quasi-varieties and initial models

In this section, we establish a mutual interdependency relationship between the existence of initial models for theories and the closure of the class of models of the theory under direct products and 'sub-models'. This is very significant in the context of the so-called 'initial semantics' in computing science, especially in algebraic specification and in logic and functional programming. As initial models are usually associated with computability properties this result is an important step for understanding the logical limits of computability from an axiomatic model-theoretic perspective. We can understand what kind of logical theories may serve the purpose of the above-mentioned computational paradigms.

In the first part of this section, we will introduce the concepts of 'sub-model' and 'quotient model' in the abstract context of categories endowed with inclusion systems, and then we will establish the conditions for the equivalence between initial semantics and quasi-varieties. In the process will notice the following asymmetric situation: while the fact that quasi-varieties of models admit initial models holds at the very general level of abstract categories, the other way around requires not only a substantial model-theoretic infrastructure but also applies only to elementary classes of models.

Subobjects in categories with inclusion systems. We have already introduced and used several notions of 'submodel', such as plain $\mathcal{FOL}$ submodels or closed $\mathcal{FOL}$ submodels (see Sect. 4.5). Both the simple and the closed concepts of $\mathcal{FOL}$ submodels are examples of the following general concept of 'subobject'.

In any category $\mathbb{C}$ with an inclusion system $\langle I, \mathcal{E}\rangle$, we say that an object A is an I*-subobject* of another object B if there exists an abstract inclusion $(A \hookrightarrow B) \in I$. When the inclusion system is fixed then we may simply say 'subobject' instead of 'I-subobject'.

An object A of $\mathbb{C}$ is I*-reachable* if and only if it has no I-subobjects which are different from A. The same as above, when $\langle I, \mathcal{E}\rangle$ is fixed we may simply say 'reachable' rather than 'I-reachable'. By varying the inclusion system of a category, one obtains different notions of reachability. For example, in the category of the Σ-models for a $\mathcal{FOL}$ signature, a reachable model in the strong inclusion system is reachable in the closed inclusion system too, but the other way around is not true.

Fact 8.10. *In any category* $\mathbb{C}$ *with a given inclusion system and which has an initial object* $0_{\mathbb{C}}$ *the following hold:*

- *Each object A is reachable if and only if the unique arrow* $0_{\mathbb{C}} \to A$ *is an abstract surjection.*
- *Each object has exactly one reachable subobject.*

Quotient objects in abstract categories with inclusion systems. The concept of quotient object can be seen as dual to that of subobject. In any category $\mathbb{C}$ with an inclusion system $\langle I, \mathcal{E}\rangle$, an object B is an $\mathcal{E}$*-quotient representation* of A if there exists an abstract surjection $A \to B$. An $\mathcal{E}$*-quotient* of A is an isomorphism class of $\mathcal{E}$-quotient representations. When the inclusion system is fixed we may simply say 'quotient' instead of '$\mathcal{E}$-quotient'. An inclusion system $\langle I, \mathcal{E}\rangle$ is *co-well-powered* if the category $\mathbb{C}$ is $\mathcal{E}$-co-well-powered. Recall from Sect. 2.1 that this means the class of $\mathcal{E}$-quotients of each object is a *set*.

Quasi-varieties and varieties in abstract categories. In any category $\mathbb{C}$ endowed with a designated inclusion system and with small products, a class of objects $Q \subseteq |\mathbb{C}|$

- is a *quasi-variety* when it is closed under small products and subobjects, and
- is a *variety* if it is a quasi-variety closed under quotient representations.

Note that from this definition we get that any quasi-variety is closed under isomorphisms just by considering the direct products of one object.

Initial models of quasi-varieties

Proposition 8.11. *Consider a category $\mathbb{C}$ with an initial object $0_{\mathbb{C}}$, small products, and with a co-well-powered epic inclusion system. Each quasi-variety Q of $\mathbb{C}$ has a reachable initial object.*

Proof. Let $\{A_i \mid i \in I\}$ be the class of all reachable subobjects of all objects of Q. Then we consider a subclass of indices $I' \subseteq I$ such that there are no isomorphic objects in $\{A_i \mid i \in I'\}$ and for each $i \in I$ there exists $j \in I'$ such that $A_i \cong A_j$. I' is a *set* because the inclusion system of $\mathbb{C}$ is co-well-powered and because we know that for each reachable object B the unique arrow $0_{\mathbb{C}} \to B$ is abstract surjection (Fact 8.10). Let $A_{I'}$ be a direct product of $(A_j)_{i \in I'}$. Let 0_Q be the reachable subobject of the product $A_{I'}$ (cf. Fact 8.10). We prove that 0_Q is initial in Q.

$$0_{\mathbb{C}} \longrightarrow 0_Q \longrightarrow A_{I'} \xrightarrow{p_j} A_j \cong A_i \longrightarrow A.$$

1 for each $A \in Q$ there exists $i \in I$ s.th. A_i is reachable subobject of A — definition of I

2 there exists $j \in I'$ s.th. $A_i \cong A_j$ — definition of I'

3 there exists arrow $A_{I'} \to A$ — 2, 1, $p_j : A_{I'} \to A_j$.

This gives an arrow $0_Q \to A$.

4 the unique arrow $0_{\mathbb{C}} \to 0_Q$ is abstract surjection — 0_Q reachable, Fact 8.10

5 $0_{\mathbb{C}} \to 0_Q$ epi — 4, epic inclusion system.

The uniqueness of the arrow $0_Q \to A$ follows from 5. □

The initial object of a quasi-variety exists in dependence on the existence of an initial object at the level of the whole category, which is the trivial quasi-variety. However in the applications this condition is mild, a typical example being the initial models in the categories of models of $\mathcal{FOL}$ signatures (Prop. 4.27).

Initial semantics of Horn theories in $\mathcal{FOL}$. Prop. 8.11 provides a rather convenient way of showing the existence of initial models of Horn theories. The example below extends the corresponding $\mathcal{FOL}$ result of Cor. 4.28 to infinitary Horn sentences. Recall that an infinitary universal Horn Σ-sentence is a sentence of the form $(\forall X)H \Rightarrow C$ where X is a set of first-order variables for Σ (i.e., new constants), H is the conjunction of any set of $(\Sigma + X)$-atoms and C is a $(\Sigma + X)$-atom.

Corollary 8.12. *For any $\mathcal{FOL}$ signature Σ, any set Γ of infinitary universal Horn Σ-sentences has an initial model.*

Proof. Because there exists the initial Σ-model (cf. Prop. 4.27), by Prop. 8.11 it is enough to show that Γ^* is a quasi-variety. For this, we consider the closed inclusion system for the categories of $\mathcal{FOL}$ models in which the abstract surjections are the surjective homomorphisms and the abstract inclusions are the closed submodels (see Sect. 4.5).

- For the preservation of Γ^* by direct products, unfortunately, we cannot use any of the general results from Chap. 6, so we have to do this from scratch. Let M_I be a direct product of a family of Σ-models $(M_i)_{i\in I} \subseteq \Gamma^*$. Let $(\forall X)H \Rightarrow C$ in Γ and let M' be any $(\Sigma+X)$-expansion of M_I such that $M' \models H$. Then

1	the product $(p_i : \ M_I \to M_i)_{i\in I}$ lifts uniquely to a product $(p'_i : \ M' \to M'_i)_{i\in I}$ of $(\Sigma+X)$-models	
2	for each $i \in I$, $M'_i \models H$	$M' \models H$, p'_i as homomorphism preserves sat. of conjunctions of atoms
3	for each $i \in I$, $M'_i \models H \Rightarrow C$	$M_i \models (\forall X)H \Rightarrow C$, $M'_i\upharpoonright_\Sigma = M_i$
4	for each $i \in I$, $M'_i \models C$	3, 2
5	$M' \models C$	4, atoms, as basic sentences are preserved by direct products (Thm. 6.6).

  Hence $M_I \models \Gamma$.

- For the preservation of Γ^* by closed sub-models we consider $M \in \Gamma^*$ and $N \hookrightarrow M$ a closed sub-model of M. Let $(\forall X)H \Rightarrow C$ in Γ and let N' be any $(\Sigma+X)$-expansion of N such that $N' \models H$. Let M' be the $(\Sigma+X)$-expansion of M such that $M'_x = N'_x$ for each $x \in X$. Then

6	$N' \to M'$ is a closed sub-model	$N \to M$ is a closed sub-model
7	$M' \models H$	$N' \models H$, satisfaction of conjunctions of atoms is preserved by homomorphisms
8	$M' \models C$	7, $M \models (\forall X)H \Rightarrow C$, $M = M'\upharpoonright_\Sigma$
9	$N' \models C$	satisfaction of atoms is preserved by closed sub-models.

  Hence $N \models \Gamma$.

$\square$

Note how the proof of the existence of initial models of Horn theories given by Cor. 8.12 is simpler than the proof provided by Cor. 4.28 as it avoids the construction of the congruence $=_\Gamma$ and of the quotient of the initial Σ-model by $=_\Gamma$. However, this is heavily disguised in the abstract construction of 0_Q as the image of the arrow $0_{\mathbb{C}} \to A_{I'}$. In general, obtaining initial semantics via quasi-varieties is technically easier than constructing it concretely, although in some areas (such as in specification and in programming) the concrete construction of initial models matters a lot.

Liberality via quasi-varieties. Cor. 4.30 showed that the existence of initial models of theories is the essential factor for the liberality of institutions. By using Prop. 8.11 it can be reformulated as follows:

Corollary 8.13. *Consider a semi-exact institution with pushouts of signatures and with diagrams such that for each signature its category of models has an initial model, small direct products, and a co-well-powered epic inclusion system. If the class of models of each theory is a quasi-variety, then the institution is liberal.*

The equivalence between quasi-varieties and existence of initial models. In essence, Thm. 8.14 below represents the reverse of the result of Prop. 8.11. Together they provide adequate conditions for the equivalence between the class of the models of a theory being a quasi-variety and having an initial model. This equivalence represents a crucial intermediate step for a general syntactic characterisation of the theories that admit initial semantics.

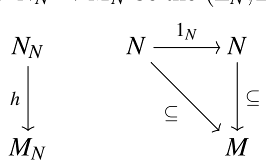
Theorem 8.14. *Consider an institution with diagrams* ι *such that*

1. *for each signature* Σ *the category of* Σ*-models has an initial object* 0_Σ*, small products, and a co-well-powered epic inclusion system, and*
2. *the model reduct functors corresponding to the elementary extensions preserve the abstract inclusions and the abstract surjections.*

In this institution, all theories have reachable initial models if and only if the class of models of each theory is a quasi-variety.

Proof. Because of Prop. 8.11 we have to prove only one implication, namely that the class E^* of the models of any theory (Σ, E) that has reachable initial models, is a quasi-variety.

- We first show the preservation by submodels. Consider $N \hookrightarrow M$ a sub-model of a (Σ, E)-model M. We prove that $N \models_\Sigma E$.
  - Let $h\colon N_N \to M_N$ be the (Σ_N, E_N)-homomorphism $i^{-1}_{\Sigma,N}(N \hookrightarrow M)$.

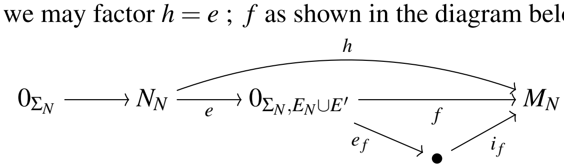


  - Let $E' = (\iota_\Sigma N)E$. Then

1 $M_N \models E'$ — Satisfaction Condition ($M_N\!\upharpoonright_{\iota_\Sigma N} = M$, definition of E')

2 $M_N \models E_N \cup E'$ — 1, $M_N \in |Mod(\Sigma_N, E_N)|$.

Thus we may factor $h = e\,;\,f$ as shown in the diagram below



where $0_{\Sigma_N, E_N \cup E'}$ denotes the initial reachable model of $(\Sigma_N, E_N \cup E')$. Let us also factor $f = e_f\ ;\ i_f$ in the inclusion system of $Mod\Sigma_N$ with e_f abstract surjection and i_f abstract inclusion.

- Then

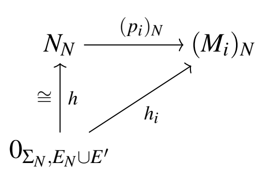
3	e abstract surjection	N_N, $0_{\Sigma_N, E_N \cup E'}$ reachable, Fact 8.10
4	$N \hookrightarrow M\ =\ e\restriction_{\iota_\Sigma N}\ ;\ e_f\restriction_{\iota_\Sigma N}\ ;\ i_f\restriction_{\iota_\Sigma N}$	definition of h, $h = e; e_f; i_f$
5	$e\restriction_{\iota_\Sigma N}$ isomorphism	$Mod\ \iota_\Sigma N$ preserves abstract surjections and inclusions, $N \hookrightarrow M$ abstract inclusion, epic inclusion systems
6	e isomorphism	$e = i^{-1}_{\Sigma,N}(e\restriction_{\iota_\Sigma N})$
7	$N_N \models E'$	4, $0_{\Sigma_N, E_N \cup E'} \models E'$
8	$N \models E$	$E' = (\iota_\Sigma N)E$, $N_N\restriction_{\iota_\Sigma N} = N$, Satisfaction Condition.

- For the preservation by direct products, consider $(p_i :\ N \to M_i)_{i \in I}$ a direct product of Σ-models such that $M_i \models E$ for each $i \in I$. We have to prove that $N \models E$.
  - For each $i \in I$ we let $(p_i)_N = i^{-1}_{\Sigma,N}(p_i :\ 1_N \to p_i)$.

9	$(p_i :\ 1_N \to p_i)_{i \in I}$ direct product in $N/Mod\Sigma \to Mod\Sigma$	$N/Mod\Sigma \to Mod\Sigma$ reflects direct products
10	$((p_i)_N = i^{-1}_{\Sigma,N} p_i)_{i \in I}$ direct product in $Mod(\Sigma_N, E_N)$	$i_{\Sigma,N} :\ Mod(\Sigma_N, E_N) \to N/Mod\Sigma$ isomorphism

  - Let $E' = (\iota_\Sigma N)E$ and $0_{\Sigma_N, E_N \cup E'}$ be a reachable initial model of $E_N \cup E'$. Then

11	$(M_i)_N \models E'$ for each $i \in I$	$(M_i)_N\restriction_{\iota_\Sigma N} = M_i$, definition of E', $M_i \models E$, Satisfaction Cond.
12	$(M_i)_N \models E_N \cup E'$	11, $(M_i)_N \in \lvert Mod(\Sigma_N, E_N)\rvert$
13	let $h_i :\ 0_{\Sigma_N, E_N \cup E'} \to (M_i)_N$ homomorphism	12, initiality of $0_{\Sigma_N, E_N \cup E'}$
14	let $h :\ 0_{\Sigma_N, E_N \cup E'} \to N_N$ s.th. $h; (p_i)_N = h_i$	13, $((p_i)_N)_{i \in I}$ direct product (10)



15	let $h' :\ N_N \to 0_{\Sigma_N, E_N \cup E'}$	initiality of N_N
16	$h\ ;\ h' = 1_{0_{\Sigma_N, E_N \cup E'}}$	initiality of $0_{\Sigma_N, E_N \cup E'}$
17	$h'\ ;\ h = 1_{N_N}$	initiality of N_N
18	$N_N \cong 0_{\Sigma_N, E_N \cup E'}$	16, 17
19	$N_N \models E'$	18, $0_{\Sigma_N, E_N \cup E'} \models E'$
20	$N \models E$	$N_N\restriction_{\iota_\Sigma N} = N$, $E' = (\iota_\Sigma N)E$, Satisfaction Condition.

□

Exercises

8.4. Any intersection of quasi-varieties is a quasi-variety. Any intersection of varieties is a variety.

8.5. Consider an institution I with small direct products of models and inclusion systems for each category of models. Let $\chi : \Sigma \to \Sigma'$ be a signature morphism and ρ' be a Σ'-sentence.

1. If χ is representable and ρ' is preserved by direct products then $(\forall\chi)\rho'$ is also preserved by direct products.
2. We say that χ *lifts inclusions* when for any inclusion of Σ-models $M \hookrightarrow N$ and M' any χ-expansion of M, there exists an inclusion $M' \hookrightarrow N'$ with $N'\upharpoonright_\chi = N$. We say that ρ' is *preserved by submodels* when for any inclusion of Σ'-models $M' \hookrightarrow N'$ if $N' \models \rho'$ then $M' \models \rho'$. If χ lifts inclusions and ρ' is preserved by submodels then $(\forall\chi)\rho'$ is also preserved by submodels.

Apply the results above to show that the models of the sentences of the form $(\forall\chi)H \Rightarrow C$ where C is a basic sentence preserved by submodels, H is accessible from basic sentences by conjunctions and disjunctions, and χ is representable, form a quasi-variety. Develop instances of this result in $\mathcal{FOL}$ and $\mathcal{PA}$.

8.6. In $\mathcal{MVL}^\sharp$ the class of models of any set of sentences of the form $((\forall X)H \Rightarrow C, x)$ (where H is any pre-sentence formed from (relational) atoms by $\wedge$, $\vee$, and $*$ and C is a single (relational) atom) forms a quasi-variety, and consequently has initial models. (*Hint:* Apply Ex. 8.5.)

8.7. Find a $\mathcal{FOL}$ theory that has a reachable initial model but whose class of models is not a quasi-variety.

8.4 Quasi-variety theorem

In Cor. 8.12 we have seen that the $\mathcal{FOL}$-models of infinitary Horn sentences form quasi-varieties. In this section, we will see that this holds more generally in abstract institutions. But the main purpose of this section is to address the reverse implication, to establish general conditions when quasi-varieties of models can be axiomatised by Horn theories. The full consequences of such a result can be understood in connection with the equivalence between initial semantics and quasi-varieties developed in Sect. 8.3. Put in simple terms, on the one hand, Horn theories guarantee initial semantics, and on the other hand, in general, initial semantics cannot go beyond Horn theories. In computing science, this resonates with the fact that the logics of both executable algebraic specifications and logic programming (both being based on initial semantics) are forms of Horn logic.

Both results in the section, namely that models of Horn theories form quasi-varieties and that quasi-varieties are axiomatizable by Horn theories are obtained based on the satisfaction by injectivity of Sect. 5.5. In both cases, we develop results at the very general level of abstract categories, in the style of Prop. 8.11, by simulating the satisfaction relation by categorical injectivity. These very general results constitute the backbone of the developments of this section as we can give them an institution theoretic form in which Horn theories appear. This Horn theoretic shape is obtained simply by invoking the relationship between satisfaction of Horn theories and categorical injectivity developed in Sect. 5.5.

The framework. The general concept of a Horn sentence as a sentence of the form $(\forall\chi)E \Rightarrow E'$ with χ being a representable signature morphism from a designated class $\mathcal{D}$ of signature morphisms, E being a set of epi basic sentences, and E' being a set of basic sentences, is too lax for the purpose of this section mainly because basic sentences capture significantly more than the atoms of the institutions (recall that existentially quantified atoms are also basic in $\mathcal{FOL}$ and other institutions). Although epi basic sentences might constitute a better abstract capture for the atoms of concrete institutions, we do not have any guarantee that in each situation each epi basic sentence is 'atomic'. The solution to this problem is to consider an abstract-designated sub-class of the class of general Horn sentences as a parameter for our framework. Therefore for this section, we introduce a framework consisting of the following additional data for an institution:

- a designated class $\mathcal{D}$ of representable signature morphisms,
- a system of diagrams ι for the institution such that each elementary extension $\iota_\Sigma M$ belongs to $\mathcal{D}$,
- a sub-functor Horn of *Sen*, such that each sentence of Horn is semantically equivalent to a $\mathcal{D}$-universal Horn sentence, and
- for each signature Σ, a designated co-well-powered inclusion system for the category $Mod\Sigma$ of the Σ-models.

We also assume that

- each category $Mod\Sigma$ has small direct products.

A typical example is to consider $\mathcal{D}$ the class of all $\mathcal{FOL}$-signature injective extensions with constants and for each $\mathcal{FOL}$-signature Σ the set HornΣ to be the set of all infinitary Horn sentences $(\forall X)H \Rightarrow C$ (with X being a set of variables, H [the conjunction of] a set of $\mathcal{FOL}$-atoms, and C a $\mathcal{FOL}$-atom). The finitary variant of this, i.e., when H is a finite conjunction of atoms, is also an example.

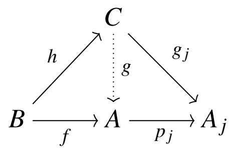
Models of Horn sentences form quasi-varieties

Proposition 8.15. *For each abstract surjection h of an inclusion system in a category with direct products, the class $Inj(h)$ of the objects that are injective with respect to h form a quasi-variety.*

Proof. Let $h: B \to C$.

- Consider a family of objects $(A_j)_{j\in J} \subseteq Inj(h)$ and let $(p_j: A \to A_j)_{j\in J}$ be their direct product. We prove that $A \in Inj(h)$. Let $f: B \to A$. Then

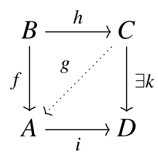
1	for each $j \in J$, there exists g_j such that $h; g_j = f; p_j$	$A_j \in Inj(h)$
2	there exists g s.th. for each $j \in J$, $g; p_j = g_j$	$(p_j)_{j\in J}$ direct product (existence property)
3	for each $j \in J$, $h\,;\,g\,;\,p_j = f\,;\,p_j$	1, 2
4	$h\,;\,g = f$	3, $(p_j)_{j\in J}$ direct product (uniqueness property).



- Now consider a subobject $i\colon\ A \hookrightarrow D$ of $D \in \mathit{Inj}(h)$. We prove that $A \in \mathit{Inj}(h)$ too. Let $f\colon\ B \to A$. Then

5 there exists k such that $h\ ;\ k = f\ ;\ i$ — $D \in \mathit{Inj}(h)$

6 there exists g such that $h\ ;\ g = f$ (and $g;i = h$) — 5, Diagonal-fill Lemma 4.16.



□

Theorem 8.16. *Consider any institution endowed with the structure specified in the beginning of the section and such that*

(QP1) *the abstract surjections are preserved by the model reducts corresponding to signature morphisms of $\mathcal{D}$, and*

(QP2) *for each* Horn*-sentence $(\forall\chi)E \Rightarrow E'$, for some basic models $M_E, M_{E\cup E'}$ for E and $E \cup E'$, respectively, the canonical model homomorphism $M_E \to M_{E\cup E'}$ is an abstract surjection.*

Then the models of any Horn*-sentence form a quasi-variety. Consequently the models of any* Horn*-theory form a quasi-variety.*

Proof. From Prop. 5.27 we know that for each Horn sentence $(\forall\chi)E \Rightarrow E'$ there exists a model homomorphism h such that for each model M,

$$M \models (\forall\chi)E \Rightarrow E' \ \text{ if and only if } \ M \models^{\mathrm{inj}} h \ \text{ (i.e., } M \text{ is injective with respect to } h).$$

Moreover, the above model homomorphism h is a χ-reduct of the canonical model homomorphism $M_E \to M_{E\cup E'}$, hence by *QP1-2* it is an abstract surjection. The conclusion follows by Prop. 8.15. □

Examples in $\mathcal{FOL}$. Let us apply Thm. 8.16 to $\mathcal{FOL}$. We have to fix two parameters, Horn and the inclusion systems for the categories of the models. Because of *QP2* the choice of these two parameters cannot be done independently. The easiest way to do this is first to fix the inclusion system and then to look at possibilities for Horn.

1. When we chose the closed inclusion system, *QP2* leaves us more freedom in choosing Horn because the abstract surjections are just the surjective homomorphisms. Then Horn can be many things, such as the standard Horn sentences of the form $(\forall X)H \Rightarrow C$ where H is a finite conjunction of atoms and C is a single atom or the set of the infinitary Horn sentences (H is allowed to be infinite), or we may even allow H to be a conjunction of existentially quantified atoms. Note that because of *QP2*, C cannot be an existentially quantified atom as the surjectivity of $M_H \to M_{H\cup\{C\}}$ may be lost.

2. When we chose the strong inclusion system, *QP2* constraints us more than in the other case because the abstract surjections, in addition to being surjective homomorphisms, should also be strong. This rules out C being a relational atom, it can only be an equational atom. For H we have all the possibilities like in the other case.

So we can formulate many instances of Thm. 8.16 in $\mathcal{FOL}$, Cor. 8.12 being just one of them.

Each quasi-variety is axiomatizable by a Horn theory

The axiomatizability of quasi-varieties by Horn theories require conditions that in the applications are more stringent than those required in Thm. 8.16. Like in the case of Thm. 8.16, this result is obtained on the basis of a very general category-theoretic correspondent in the style of Prop. 8.15. This follows now.

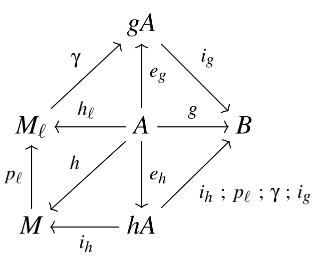
Proposition 8.17. *In any category with direct products and with a co-well-powered epic inclusion system, for each quasi-variety Q there exists a class E of abstract surjections such that $Q = Inj(E)$.*

Proof. Let us define $E = \{h \text{ abstract surjection } \mid Q \subseteq Inj(h)\}$. We notice immediately that $Q \subseteq Inj(E)$, therefore we have to prove only that $Inj(E) \subseteq Q$. Consider $A \in Inj(E)$. We prove that $A \in Q$.

- Because the inclusion system is co-well-powered we can choose a 'complete' *set* $(h_j : A \to M_j \in Q)_{j\in J}$ of quotient representatives of A in Q in the sense that for each quotient representative $h : A \to B \in Q$ there exists an isomorphism γ and some $j \in J$ such that $h;\gamma = h_j$. Then

1	$M \in Q$	$M_j \in Q$, Q closed under direct products
2	there exists unique $h : A \to M$ s.th. $h;p_j = h_j$, $j \in J$	$(p_j)_{j\in J}$ direct product
3	let $h = e_h ; i_h$ be the factoring of h through the inclusion system.	

- We prove that $Q \subseteq Inj(e_h)$.



(8.1)

Consider any $B \in Q$ and any $g\colon\ A \to B$. Let $g = e_g\ ;\ i_g$ be the factorisation of g in the inclusion system. Then

4	$gA \in Q$	$i_g\colon\ gA \to B$, B subobject, $B \in Q$, Q closed under subobjects
5	there exists $\ell \in J$, γ iso s.th. $e_g = h_\ell;\gamma$	4, $(h_j)_{j\in J}$ complete set of quotient repr.
6	$e_h\ ;\ (i_h;p_\ell;\gamma;i_g) = g$	chase diagram (8.1)
7	$B \in Inj(e_h)$	6.

- We continue with the proof of $A \in Q$.

8	$e_h \in E$	$Q \subseteq Inj(e_h)$, definition of E
9	$A \models^{\mathrm{inj}} e_h$	8, $A \in Inj(E)$
10	there exists m s.th. $e_h\ ;\ m = 1_A$	apply 9 to $1_A\colon\ A \to A$
11	e_h isomorphism	10, e_h epi (epic inclusion system)
12	$hA \in Q$	$M \in Q$ (1), Q closed under subobjects
13	$A \in Q$	11, 12.

□

Theorem 8.18 (Quasi-variety)**.** *Consider any institution endowed with the structure specified in the beginning of the section and such that*

(QA1) *the inclusion systems of the model categories are epic,*

(QA2) *each abstract surjection (of models) is* ι*-conservative, and*

(QA3) *for any abstract surjection (of models)* $h\colon\ M \to N$*, the 'internal' sentence* $(\forall \iota_\Sigma M)E_M \Rightarrow (\iota_\Sigma h)^{-1}E_N^{**}$ *is semantically equivalent to a set of* Horn*-sentences,*

any quasi-variety is the class of models of a set of Horn*-sentences.*

Proof. From Prop. 5.29 we know that for each ι-conservative model homomorphism h,

$$M \models^{\mathrm{inj}} h \ \text{ if and only if } \ M \models (\forall \iota_\Sigma M)E_M \Rightarrow (\iota_\Sigma h)^{-1}E_N^{**}.$$

From Prop. 8.17 and *QA1-3* we obtain the conclusion. □

Now we can put together both implications given by the Thm.s 8.16 and 8.18 and formulate an equivalence relationship between quasi-varieties and Horn theories.

Corollary 8.19. *In any institution institution endowed with the structure specified at the beginning of the section and satisfying* QP1-2 *and* QA1-3 *a class of models of a signature is a quasi-variety if and only if it is the class of models of a set of* Horn-*sentences.*

While the conditions *QA1-2* do not narrow the applicability of Quasi-variety Thm. 8.18 and its Cor. 8.19, the conjunction between *QA3* and *QP2* may eliminate some apparent possible applications as illustrated by the following example.

Quasi-varieties in $\mathcal{FOL}$. In $\mathcal{FOL}$, the conjunction between *QA3* and *QP2* eliminates the strong inclusion systems for the categories of models. The condition *QP2* restricts E' to a set of equations in the Horn sentences $(\forall\chi)E \Rightarrow E'$ which prevents the condition *QA3* from holding because diagrams contain also relational atoms (see also Lemma 8.20 below). Hence the only possible choice remains that of the closed inclusion systems. Note that the condition *QA3* holds by the semantical equivalence given by the result below (its proof is left to the reader).

Lemma 8.20 (*QA3* in $\mathcal{FOL}$)**.** *Let Σ be a signature in $\mathcal{FOL}$, which is considered with its the standard system of diagrams (see Sect. 4.4), and $h\colon\ M \to N$ be a surjective Σ-model homomorphism. Then*

$$(\forall\iota_\Sigma M)E_M \Rightarrow (\iota_\Sigma h)^{-1}E_N^{**} \quad \models\!\!\mid \quad \{(\forall\iota_\Sigma M)E_M \Rightarrow \rho \mid N_N \models (\iota_\Sigma h)\rho,\ \rho\ \ atom\}.$$

Therefore the $\mathcal{FOL}$ instance of Cor. 8.19 is as follows.

Corollary 8.21. *For any $\mathcal{FOL}$ signature a class of models is a quasi-variety for the closed inclusion system if and only if it is the class of models of a set of infinitary Horn sentences.*

The results of Cor. 8.19 and of Thm. 8.14 can be put together to identify the subinstitutions of a given institution that enjoys the property that all its theories admit initial reachable models. For example in the case of the infinitary extension of $\mathcal{FOL}$ (possibly infinite quantifiers and conjunctions) these results say that one cannot go beyond the infinitary Horn sentences.

Exercises

8.8. Prove Lemma 8.20.

8.9. Axiomatizability for quasi-varieties of partial algebras

1. As an instance of Cor. 8.19, a class of partial algebras is

axiomatizable by	iff it is closed under
QE_2-sentences	products and (plain) subalgebras
QE-sentences	products and closed subalgebras

(*Hint:* Use Ex. 4.68.)

2. A result similar to 1. for QE_1 fails on the condition *QP2*.
3. In $\mathcal{PA}$ each morphism between theories of universal quasi-existence equations is liberal.

8.5 Birkhoff variety theorem

On the one hand, if we consider a set of unconditional Horn sentences, (i.e. the set H of the hypotheses is empty, which is the same with considering it as absolutely true) then its quasi-variety of models has an additional property: it is also closed under 'homomorphic images' or, otherwise said, under 'quotients'. Such quasi-varieties are called varieties. On the other hand, a variety of models admits an axiomatization by unconditional Horn sentences. This is the essence of Birkhoff Variety Theorem, which originally has been developed for $\mathcal{EQL}$.

In this section, we develop an institution-independent version of this result as a refinement of the results on quasi-varieties and Horn axiomatizability of Sect. 8.4. The 'homomorphic images' are handled at the institution-theoretic level by abstract surjections like in the definition of varieties given in Sect. 8.3. Concerning sentences we impose the unconditionality restriction on the subfunctor Horn. Therefore we consider

- a sentence subfunctor $\mathsf{UA}: \ \mathit{Sig} \to \mathbb{S}et$ such that each UA-sentence is semantically equivalent to a sentence of the form $(\forall\chi)E'$ with $\chi \in \mathcal{D}$ and E' being a set of basic sentences.

A typical example for UA is given by the universally quantified $\mathcal{FOL}$-atoms.

Models of 'universal atoms' form varieties

Proposition 8.22. *Consider any institution that in addition to all conditions of Thm. 8.16 also satisfies that*

(VP) *for any* $(\chi: \ \Sigma \to \Sigma') \in \mathcal{D}$*, any abstract surjection* $h: \ M \to N$ *in* $\mathit{Mod}\Sigma$*, any* χ*-expansion* N' *of* N*, there exists a* χ*-expansion* $h': \ M' \to N'$ *of* h*.*

Then the models of any UA*-sentence form a variety.*

Proof. • The models of any UA-sentence form a quasi-variety because the sub-functor UA is a Horn sub-functor.

- Let $h: \ M \to N$ be an abstract surjection and $M \models (\forall\chi)E'$ where $\chi \in \mathcal{D}$ and E' is basic. For any χ-expansion N' of N, by (VP) we get a χ-expansion $h': \ M' \to N'$ of h. Then

1	$M' \models E'$	$M'\upharpoonright_\chi = M$, $M \models (\forall\chi)E'$
2	for $M_{E'}$ basic model for E', there exists a homomorphism $M_{E'} \to M'$	1, E' basic
3	there exists homomorphism $M_{E'} \to N'$	2, $h': \ M' \to N'$
4	$N' \models E'$	3, E' basic
5	$N \models (\forall\chi)E'$	4, N' arbitrary χ-expansion of N.

□

The condition (VP) is easy to check in the applications as shown by the following example.

Examples in $\mathcal{FOL}$. Let us consider any of the closed or strong inclusion systems for the categories of $\mathcal{FOL}$ models, and $\mathcal{D}$ the class of the injective signature extensions with constants. For both inclusion systems considered, the abstract surjections are surjective as functions. For each $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$, each surjective Σ-model homomorphism $h : M \to N$, for any χ-expansion N' of N, each constant x in $\Sigma' - \chi(\Sigma)$, for M'_x let us pick any element of $h^{-1}N'_x$. This lifts h to a Σ'-homomorphism $M' \to N'$. Note how the surjectivity of h is crucial for this lifting.

Based on the situation of quasi-varieties of models for Horn-sentences in $\mathcal{FOL}$ (analysed and discussed in Sect. 8.4), we can formulate the following:

Corollary 8.23. *For any $\mathcal{FOL}$ signature the models of any set of universally quantified atoms (equations) form a variety for the closed (strong) inclusion systems in the categories of models.*

Each variety is axiomatizable by a theory of 'universal atoms'

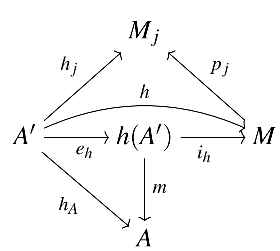
The development of the axiomatizability of varieties by unconditional Horn theories follows the same scheme as the axiomatization of quasi-varieties by Horn theories of Sect. 8.4. The core of this result is developed at the highest level of generality, that of abstract categories, and then this gets instantiated to institutions with adequate additional structure and conditions to obtain a more explicit form of the result.

Given a class $\mathcal{K} \subseteq |\mathbb{C}|$ of objects in a category $\mathbb{C}$ with an inclusion system, f is a *$\mathcal{K}$-surjection* when it is an abstract surjection and $dom(f) \in \mathcal{K}$.

Proposition 8.24. *In a category $\mathbb{C}$ with direct products and a co-well-powered inclusion system, let $\mathcal{K}$ be a class of objects such that for each object A of the category there exists an abstract surjection $A' \to A$ with $A' \in \mathcal{K}$. Then for each variety $V \subseteq |\mathbb{C}|$ there exists a class E of $\mathcal{K}$-surjections such that $V = Inj(E)$.*

Proof. Let us define $E = \{e\ \mathcal{K}\text{-surjection} \mid V \subseteq Inj(e)\}$. We notice immediately that $V \subseteq Inj(E)$, therefore we have to prove only that $Inj(E) \subseteq V$. Consider $A \in Inj(E)$. We will prove that $A \in V$.

- There exists an object $A' \in \mathcal{K}$ and an abstract surjection h_A such that $h_A : A' \to A$. Similarly to the argument in the proof of Prop. 8.17 we get $h : A' \to M \in V$.
- Also like in Prop. 8.17 we factor $h = e_h \,; \imath_h$ through the inclusion system and can prove that $e_h \in E$.



- Then

1	there exists m such that $h_A = e_h \,;\, m$	$e_h \in E, A \in Inj(E)$
2	m abstract surjection	e_h, h_A abstract surjections
3	$h(A') \in V$	$M \in V$, $i_h : A' \to M$ subobject, V variety
4	$A \in V$	3, m abstract surjection, V variety.

□

Theorem 8.25 (Birkhoff variety)**.** *In any institution institution endowed with the structure specified at the beginning of Sect. 8.4 and satisfying* QA2 *(of Thm. 8.18) and such that*

(VA1) *for each model M, $i_{\iota_\Sigma M} : Mod\Sigma_M \to M_{\iota_\Sigma M}/Mod\Sigma$ maps the initial (Σ_M, E_M)-model M_M to an abstract surjection $M_{\iota_\Sigma M} \to M$, and*

(VA2) *for any abstract surjection (of models) $h : M \to N$, the 'internal' sentence $(\forall \iota_\Sigma M)(\iota_\Sigma h)^{-1} E_N^{**}$ is semantically equivalent to a set of* UA*-sentences.*

Then any variety is the class of models of a set of UA*-sentences.*

Proof. For a given signature Σ in the role of $\mathcal{K} \subseteq |Mod\Sigma|$ let us consider the class of all representations of the signature morphisms $\Sigma \to \Sigma'$ which belong to $\mathcal{D}$, i.e.,

$$\mathcal{K} = \{M_\chi \mid (\chi : \Sigma \to \Sigma') \in \mathcal{D}\}.$$

- The condition *VA1* allows for the application of Prop. 8.24. Therefore for each variety V there exists a class E of abstract surjections with domains in $\mathcal{K}$ such that $V = Inj(E)$.
- Let $(h : M_\chi \to N) \in E$. We have that:

1	h ι-conservative	h $\mathcal{K}$-surjection, *QA2*
2	$(\iota_\Sigma h)^{-1} E_N^{**}$ basic with $N_N \upharpoonright_{\iota_\Sigma h}$ basic model	1, proof of Prop. 5.29
3	$N_N \upharpoonright_{\iota_\Sigma h} \upharpoonright_{\iota_\Sigma M_\chi} = N$	'functoriality' of ι.

From 2 and 3 we get easily that:

$$M \models^{\mathrm{inj}} h \quad \text{if and only if} \quad M \models_\Sigma (\forall \iota_\Sigma M_\chi)(\iota_\Sigma h)^{-1} E_N^{**}.$$

Now by *VA2* we obtain the conclusion of the theorem. □

Now we can put together both implications given by Prop. 8.22 and Thm. 8.25.

Corollary 8.26. *In any institution institution endowed with the structure specified at the beginning of Sect. 8.4 and satisfying* QP1-2*,* QA2*,* VP*, and* VA1-2 *a class of models of a signature is a variety if and only if it is the class of models of a set* UA*-sentences.*

Conditions *VA1-2* can be checked rather easily in concrete applications as suggested by the following example.

Varieties in $\mathcal{FOL}$. For each Σ-model M, $i_{\iota_\Sigma M} M_M : M_{\iota_\Sigma M} \to M$ is just the Σ-homomorphism $0_{\Sigma_M}\restriction_{\iota_\Sigma M} \to M$ that maps each element of M to itself. This is surjective but not strong, hence it is an abstract surjection only for the closed inclusion system for models. This eliminates the $\mathcal{FOL}$ variant corresponding to the strong inclusion systems from the possible instances of Cor. 8.26.

Condition *VA2* is fulfilled in $\mathcal{FOL}$ with UA being the universally quantified atoms because of the semantic equivalence below (its rather simple proof is left as an exercise to the reader).

Lemma 8.27. *Let Σ be a signature in $\mathcal{FOL}$, which is considered with the standard system of diagrams (see Sect. 4.4), and $h : M \to N$ be a surjective Σ-model homomorphism. Then*

$$(\forall \iota_\Sigma M)(\iota_\Sigma h)^{-1} E_N^{**} \models\!\!\mid \{(\forall \iota_\Sigma M)\rho \mid N_N \models (\iota_\Sigma h)\rho,\ \rho \ \mathit{atom}\}.$$

Therefore we can now formulate the following:

Corollary 8.28. *For any $\mathcal{FOL}$ signature a class of models is a variety for the closed inclusion system if and only if it is the class of models of a set of universally quantified atoms.*

Exercises

8.10. Prove Lemma 8.27.

8.11. Axiomatizability of varieties of partial algebras
As an instance of the general Birkhoff Variety Theorem 8.25, we establish that for any $\mathcal{PA}$ signature each class of models that is closed under products, closed submodels, and epi homomorphic images (see also Ex. 4.68) is the class of models of a set of universally quantified existence equations. However, the corresponding preservation result fails because not every universally quantified existence equation is preserved by any epi homomorphism. At the general level, this failure is reflected as a failure of the condition *(VP)* for the epi homomorphisms of partial algebras.

8.6 General Birkhoff axiomatizability

In $\mathcal{EQL}$ Cor. 8.28 says that any variety is (an) elementary (class of algebras). This can be presented in a different way in which the closure of any class of models $\mathbb{M}$ is given in terms of applying semantic operators on $\mathbb{M}$. So in the case of a class $\mathbb{M}$ of Σ-models in $\mathcal{EQL}$ we have that

$$\mathbb{M}^{**} = HSP\mathbb{M}. \tag{8.2}$$

where $P\mathbb{M}$ / $S\mathbb{M}$ / $H\mathbb{M}$ means taking all direct products / sub-models / homomorphic images of models from $\mathbb{M}$. On the one hand, it is quite clear that equation (8.2) represents a strengthening of the axiomatizability of varieties from Cor. 8.28 because from (8.2) we deduce immediately that if $\mathbb{M}$ is a variety then $\mathbb{M} = \mathbb{M}^{**}$, so $\mathbb{M}$ is axiomatizable by $\mathbb{M}^*$. On the other hand to derive from (8.2) the other side of Cor. 8.28, namely that the class E^* of the models of an equational theory E is a variety, is equally trivial: by letting

$\mathbb{M} = E^*$ we have $\mathbb{M} = \mathbb{M}^{**} = HSP\mathbb{M}$, which implies $H\mathbb{M} \subseteq \mathbb{M}$, $S\mathbb{M} \subseteq \mathbb{M}$, $P\mathbb{M} \subseteq \mathbb{M}$. When looking into the proof of Prop. 8.24 we can see exactly this order when establishing that an object A belongs to the variety we first consider a product, then a subobject of that product, and finally a homomorphic image of that subobject. Without equation (8.2) it is not obvious why $HSP\mathbb{M}$ should be a variety. An explanation for this is that there are some commutativity-like relations between the three semantic operators. These constitute the additional technical step for moving from the axiomatizability already obtained to their 'HSP' versions.

In this final section of the chapter, we will refine the axiomatizability results previously developed to the 'HSP' style of equation (8.2) and then formulate a general concept of axiomatizability that captures uniformly all these results and much more. This will be used later on in the book in the context of abstract institution-independent treatments of other model-theoretic topics, when axiomatizability properties constitute a cause for other model theoretic properties.

Application of relations. Given a binary relation $R \subseteq A \times B$, for each $A' \subseteq A$ let

$$R(A') = \{b \mid \langle a, b\rangle \in R, a \in A'\}.$$

Let us also recall

- that the composition of binary relations $R \subseteq A \times B$ and $R' \subseteq B \times C$ is a relation $R;R' \subseteq A \times C$ defined by

$$R;R' = \{\langle a, c\rangle \mid \text{ there exists } b \text{ such that } \langle a, b\rangle \in R \text{ and } \langle b, c\rangle \in R'\}$$

  and

- that the inverse R^{-1} of a binary relation $R \subseteq A \times B$ is a relation $R^{-1} \subseteq B \times A$ defined by

$$R^{-1} = \{\langle b, a\rangle \mid \langle a, b\rangle \in R\}.$$

Relations induced by classes of arrows. Any class of arrows H in a category $\mathbb{C}$ determines a (class) relation $\stackrel{H}{\to} \subseteq |\mathbb{C}| \times |\mathbb{C}|$ by

$$a \stackrel{H}{\to} b \text{ if there exists an arrow } h\colon a \to b \in H.$$

The inverse $(\stackrel{H}{\to})^{-1}$ is denoted by $\stackrel{H}{\leftarrow}$.

Axiomatizability revisited

Now we refine in the 'HSP style', one by one, the main four axiomatizability results already developed in this chapter. We will do this in a different order from the order they have originally been developed. In some cases, a few more axioms are needed, but in the applications they represent rather common sense properties. The need for these new axioms is not surprising if we consider that the new axiomatizability results come in a stronger format than their original correspondents. But in all cases, the former are obtained from the latter, so this is the sense in which we talk about a refinement of the original axiomatizability results.

For any class $\mathbb{M}$ of objects in a category (usually the category of the models of a signature in an institution) let $P\ \mathbb{M}$ ($Up\ \mathbb{M}$) denote the class of all objects that are direct products (ultraproducts) of objects from $\mathbb{M}$.

Quasi-variety theorem revisited. The refinement of the conclusion of Thm. 8.18 requires the following commutativity-like property.

Proposition 8.29. *Consider a category with small direct products and an inclusion system such that the class of inclusions I is weakly stable under isomorphisms. Then for any class $\mathbb{M}$ of objects*

$$P(\overset{I}{\leftarrow}\mathbb{M}) \ \subseteq\ \overset{I}{\leftarrow}(P\mathbb{M}).$$

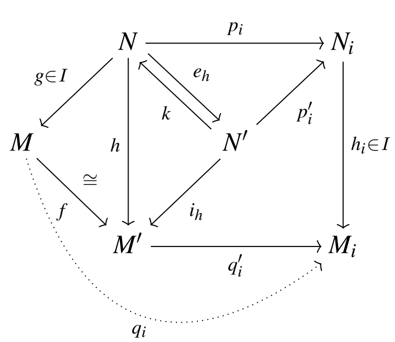
Proof. Let $(h_i: N_i \to M_i)_{i\in I} \subseteq I$ such that $(M_i)_{i\in I} \subseteq \mathbb{M}$ and $(p_i: N \to N_i)_{i\in I}$ be any direct product. We will find a direct product $(q_i: M \to M_i)_{i\in I}$ such that $N \subseteq M$.

1 Let $(q'_i: M' \to M_i)_{i\in I}$ be any direct product.

2 There exists an unique $h: M \to M'$ such that $h\ ;\ q'_i = p_i\ ;\ h_i$ 1.

Let $h = e_h\ ;\ i_h$ be the factorisation of h through the inclusion system. Then

3 for each $i \in I$ there exists $p'_i: N' \to M_i$ s.th. $e_h;p'_i = p_i$, $i_h;q'_i = p'_i;h_i$ — h_i abstract inclusion, e_h abstract surjection, Diagonal-fill Lemma 4.16.



We show that

4 $(p'_i: N' \to N_i)_{i\in I}$ is a direct product.

Let $k: N' \to N$ be the unique homomorphism such that $k;p_i = p'_i$ (since $(p_i)_{i\in I}$ is direct product). Then

5 $e_h\ ;\ k\ ;\ p_i = e_h\ ;\ p'_i = p_i$ — $k;p_i = p'_i$, definition of p'_i

6 $e_h \,;\, k = 1_N$ — 5, $(p_i)_{i\in I}$ direct product
7 $e_h \,;\, k \,;\, e_h = e_h$ — 6
8 $k \,;\, e_h = 1_{N'}$ — 7, e_h epi (epic inclusion system)
9 e_h isomorphism — 6, 8.

Since the abstract inclusions are weakly stable under isomorphisms there exists an inclusion $g\colon\ N \to M$ and an isomorphism $f\colon\ M \to M'$ such that $h = g \,;\, f$. Since $(q'_i)_{i\in I}$ is a direct product and f is isomorphism we also have that $(q_i = f;q'_i\colon\ M \to M_i)_{i\in I}$ is a direct product. □

The new condition involved, namely that the abstract inclusions are weakly stable under isomorphisms, is rather trivial in the concrete examples. In common categories of models this is very much like in $\mathbb{S}et$ considered with the standard inclusion system.

Theorem 8.30 (Quasi-variety)**.** *Consider an institution that satisfies the conditions of Theorems 8.16 and 8.18 and such that*

(QA4) *the class of model inclusions I are weakly stable under isomorphisms.*

For any class of Σ-models $\mathbb{M}$,

$$(\mathbb{M}^* \cap \mathsf{Horn}\Sigma)^* \;=\; \stackrel{I}{\leftarrow}(P\mathbb{M}).$$

Proof. The proof relies on the results of Theorems 8.16 and 8.18, on the result of Prop. 8.29, and on the fact that $PP = P$, which is straightforward to check in any category.

- First we prove that $\stackrel{I}{\leftarrow}(P\mathbb{M})$ is the least quasi-variety that contains $\mathbb{M}$.

1 $P(\stackrel{I}{\leftarrow}(P\mathbb{M})) \;\subseteq\; \stackrel{I}{\leftarrow}(PP\mathbb{M}) \;=\; \stackrel{I}{\leftarrow}(P\mathbb{M})$ — *QA4*, Prop. 8.29, $PP = P$
2 $\stackrel{I}{\leftarrow}\stackrel{I}{\leftarrow}(P\mathbb{M}) \;=\; \stackrel{I}{\leftarrow}(P\mathbb{M})$ — 1, $\stackrel{I}{\to};\stackrel{I}{\to}=\stackrel{I}{\to}$ (I sub-category).

1 and 2 show that $\stackrel{I}{\leftarrow}(P\mathbb{M})$ is a quasi-variety. It obviously contains $\mathbb{M}$. To show that it is the least one containing $\mathbb{M}$ let us consider any quasi-variety Q such that $\mathbb{M} \subseteq Q$. We have that:

3 $P\mathbb{M} \;\subseteq\; P(Q) \;=\; Q$ — $\mathbb{M} \subseteq Q$, P monotone, Q quasi-variety
4 $\stackrel{I}{\leftarrow}(P\mathbb{M}) \;\subseteq \stackrel{I}{\leftarrow} Q \;=\; Q$ — 3, $\stackrel{I}{\leftarrow}$ monotone, Q quasi-variety.

- The final part of the proof goes as follows:

5 $\stackrel{I}{\leftarrow}(P\mathbb{M}) \;\subseteq (\mathbb{M}^* \cap \mathsf{Horn}\Sigma)^*$ — $(\mathbb{M}^* \cap \mathsf{Horn}\Sigma)^*$ q.-variety (Thm. 8.16), $\mathbb{M} \subseteq (\mathbb{M}^* \cap \mathsf{Horn}\Sigma)^*$, 4
6 let $E \subseteq \mathsf{Horn}\Sigma$ such that $\stackrel{I}{\leftarrow}(P\mathbb{M}) = E^*$ — $\stackrel{I}{\leftarrow}(P\mathbb{M})$ quasi-variety, Thm. 8.18
7 $\mathbb{M} \subseteq E^*$ — 6, $\mathbb{M} \subseteq \stackrel{I}{\leftarrow}(P\mathbb{M})$

8 $E \subseteq \mathbb{M}^*$ — 7, Galois connection property

9 $E \subseteq \mathbb{M}^* \cap \mathsf{Horn}\Sigma$ — 8, $E \subseteq \mathsf{Horn}\Sigma$ (from the definition of E)

10 $(\mathbb{M}^* \cap \mathsf{Horn}\Sigma)^* \subseteq E^* \;=\; \overset{I}{\leftarrow}(P\mathbb{M})$ — Galois connection property, 6

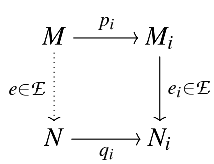
11 $(\mathbb{M}^* \cap \mathsf{Horn}\Sigma)^* \;=\; \overset{I}{\leftarrow}(P\mathbb{M})$ — 5, 10.

□

Birkhoff variety theorem revisited. We continue our refinement of the axiomatizability results with the Variety Theorem.

Theorem 8.31 (Birkhoff variety)**.** *Consider any institution endowed with the structure specified at the beginning of Sect. 8.4, that satisfies the axioms* QP1-2*,* QA1-4*,* VP*,* VA1-2*, and the following additional axiom also:*

(VA3) *for any signature Σ, in $Mod\Sigma$ the abstract surjections are stable with respect to direct products in the sense that if $(e_i : M_i \to N_i)_{i\in I}$ is any family of abstract surjections then for any direct products $(p_i : M \to M_i)_{i\in I}$ and $(q_i : N \to N_i)_{i\in I}$ the unique homomorphism $e : M \to N$ such that $e ; q_i = p_i ; e_i$, $i \in I$, is an abstract surjection too.*



Then for class $\mathbb{M}$ of Σ-models

$$(\mathbb{M}^* \cap \mathsf{UA}\Sigma)^* \;=\; \overset{\mathcal{E}}{\to}(\overset{I}{\leftarrow}(P\mathbb{M}))$$

(where $\langle I, \mathcal{E}\rangle$ is the inclusion system of $Mod\Sigma$).

Proof. • First we prove the following commutativity-like property of the semantic operators 'homomorphic image' and 'sub-model'.

1 For any Σ-model M, $\overset{I}{\leftarrow}(\overset{\mathcal{E}}{\to} M) \subseteq \overset{\mathcal{E}}{\to}(\overset{I}{\leftarrow} M)$.

Consider $(e : M \to A) \in \mathcal{E}$, $(i : B \to A) \in I$. We prove that $B \in \overset{\mathcal{E}}{\to}(\overset{I}{\leftarrow} M)$.

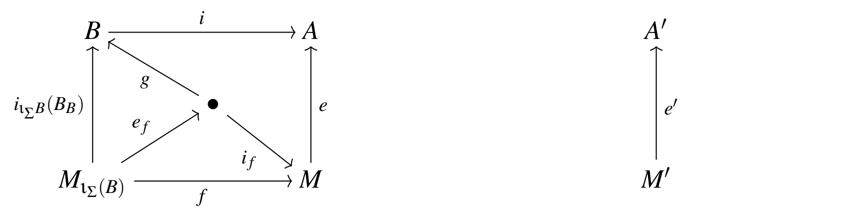
2 $\iota_\Sigma B$ representable — $\iota_\Sigma B \in \mathcal{D}$, morphisms of $\mathcal{D}$ are representable

3 let $A' = i^{-1}_{\iota_\Sigma B}(\, i_{\iota_\Sigma B}(B_B) \,;\, i\,)$ — 2

4 let $e' : M' \to A'$ be $\iota_\Sigma B$-expansion of e — $e \in \mathcal{E}$, *VP*

5 let $f = i_{\iota_\Sigma B} M'(: M_{\iota_\Sigma B} \to M)$

6 $f\ ;\ e = i_{\iota_\Sigma B}(B_B)\ ;\ i$ $\qquad e = i_{\iota_\Sigma B}e' :\ i_{\iota_\Sigma B}M' \to i_{\iota_\Sigma B}A'$



Let $f = e_f\ ;\ i_f$ be the factorisation of f through the inclusion system $\langle I, \mathcal{E}\rangle$.

7 there exists g such that $e_f; g = i_{\iota_\Sigma B}(B_B)$, $g; i = i_f; e$ $\qquad$ 6, Diagonal-fill Lemma 4.16

8 $g \in \mathcal{E}$ $\qquad$ 7, $e_f \in \mathcal{E}$, $i_{\iota_\Sigma B}(B_B) \in \mathcal{E}$ (cf. *VA1*)

Since $i_f \in I$ and $g \in \mathcal{E}$ we obtain that $B \in \stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow} M)$.

- Now we prove that $\stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow} P\mathbb{M})$ is a variety:

9 $\stackrel{\mathcal{E}}{\to}(\stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow} P\mathbb{M})) = \stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow} P\mathbb{M})$ $\qquad$ $\mathcal{E}$ sub-category

10 $\stackrel{I}{\leftarrow}(\stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow}(P\mathbb{M}))) \subseteq \stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow}(\stackrel{I}{\leftarrow}(P\mathbb{M})))$ $\qquad$ 1

$= \stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow}(P\mathbb{M}))$ $\qquad$ I subcategory

11 $P(\stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow}(P\mathbb{M}))) \subseteq \stackrel{\mathcal{E}}{\to}(P(\stackrel{I}{\leftarrow}(P\mathbb{M})))$ $\qquad$ *VA3*

$\subseteq \stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow}(PP\mathbb{M}))$ $\qquad$ Prop. 8.29, $\stackrel{\mathcal{E}}{\to}$ monotone

$= \stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow}(P\mathbb{M}))$ $\qquad$ $PP = P$.

- The final part of this proof replicates the ideas of the final part of the proof of Thm. 8.30 and that $\stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow} P\mathbb{M})$ is the *least* variety that contains $\mathbb{M}$, that $\mathbb{M}^* \cap \mathsf{UA}\Sigma \models E$, where E is the axiomatization of $\stackrel{\mathcal{E}}{\to}(\stackrel{I}{\leftarrow} P\mathbb{M})$ as given by Thm. 8.25.

□

This refined version of the Birkhoff Variety Theorem introduces the new axiom *VA3*. This condition is rather trivial in concrete applications. For example in the case of the $\mathcal{FOL}$ models, *VA3* holds for the closed inclusion system, and luckily this is the inclusion system for which the $\mathcal{FOL}$ Cor. 8.28. It can be established by the simple fact that products of surjective functions are still surjective.

Axiomatizability by ultraproducts revisited. With Thm. 8.32 below we refine the axiomatizability results of Thm. 8.5 and its Cor. 8.7 in the '*HSP*' style. For any class $\mathbb{M}$ of models let Ur can be the *ultraradical relation* on models defined by $\langle M, N\rangle \in Ur$ if and only if N is an ultrapower of M.

Theorem 8.32. *Consider an institution with negations and conjunctions such that sentences are preserved by ultraproducts. Then for each class* $\mathbb{M}$ *of* Σ*-models:*

- $\mathbb{M}^{**} \;=\; \equiv(Up\,\mathbb{M})$*, and*
- $\mathbb{M}^{**} \;=\; Ur^{-1}(Up\,\mathbb{M})$ *when in addition the institution has the Keisler-Shelah property and* Up *is idempotent (i.e.,* $Up;Up = Up$*).*

Proof. • The first part follows immediately with an inspection of the proof of Thm. 8.5.

- For the second part, it is therefore enough to show that $Up;Ur^{-1} = Up;\equiv$. This goes as follows:

1	$Ur^{-1} \;\subseteq\; \equiv$	each model is elementary equivalent to any of its ultrapowers
2	$Up;Ur^{-1} \;\subseteq\; Up;\equiv$	2
3	$\equiv \;\subseteq\; Up;Ur^{-1}$	Keisler-Shelah property
4	$Up;\equiv \;\subseteq\; Up;Up;Ur^{-1} \;=\; Up;Ur^{-1}$	3, $Up;Up = Up$.

□

To establish that Up is idempotent at the general categorical level is not easy, but with somehow less effort it can be established in concrete institutions such as $\mathcal{FOL}$. In both cases, the main idea is as follows. Let $(I_j)_{j\in J}$ be a family of sets and I be a disjoint union of it. If F is an ultrafilter over J and $(F_j)_{j\in J}$ is a family of ultrafilters, each F_j being an ultrafilter over I_j, then

$$\hat{F} = \{X \subseteq I \mid \{j \in J \mid X \cap I_j \in F_j\} \in F\}$$

is an ultrafilter too. Then given F_j-ultraproducts of families $(M_{j,i})_{i\in I_j}$ of models, for each $j \in J$, then any of their F-ultraproducts is an $\hat{F}$-ultraproduct of $(M_{j,i})_{j\in J, i\in I_j}$.

Axiomatizability by universal sentences revisited. We refine the conclusion of Cor. 8.8 in the '*HSP*' spirit.

Theorem 8.33. *Under the framework and the hypotheses of Cor. 8.8 we also assume that the institution is a Łoś-institution and that*

(UAX1) *each elementary extension invents strongly and completely ultraproducts,*

(UAX2) *the ultraproduct construction is idempotent (i.e.,* $Up;Up = Up$*), and*

(UAX3) *the* Sen^0*-submodels are preserved by expansions along elementary extensions.*

Then for any class of Σ*-models* $\mathbb{M}$

$$(\mathbb{M}^* \cap \neg Sen^0\Sigma)^* \;=\; \overset{Sen^0}{\longleftarrow}(Up\,\mathbb{M}).$$

Proof. The proof plans to show that $\overset{Sen^0}{\longleftarrow}(Up\ \mathbb{M})$ is closed under Sen^0-submodels and ultraproducts and then finalize the proof by using Cor. 8.8.

- The closure of $\overset{Sen^0}{\longleftarrow}(Up\ \mathbb{M})$ under $\overset{Sen^0}{\longleftarrow}$ follows from the transitivity of $\overset{Sen^0}{\longrightarrow}$. Let $f: M \to N$ such that $M_M[Sen^0]N_f$ and $g: N \to P$ such that $N_N[Sen^0]P_g$. We show that $M_M[Sen^0]P_{f;g}$.

1	$N_N\upharpoonright_{\iota_\Sigma f} = (i^{-1}_{\Sigma,N}1_N)\upharpoonright_{\iota_\Sigma f} = i^{-1}_{\Sigma,M} f = N_f$	naturality of i, definition of N_f
2	$P_g\upharpoonright_{\iota_\Sigma f} = (i^{-1}_{\Sigma,N} g)\upharpoonright_{\iota_\Sigma f} = i^{-1}_{\Sigma,M}(f;g) = P_{f;g}$	naturality of i, definition of $P_{f;g}$.

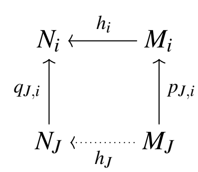
Let us consider any $\rho \in (M_M)^* \cap Sen^0\Sigma_M$. Then

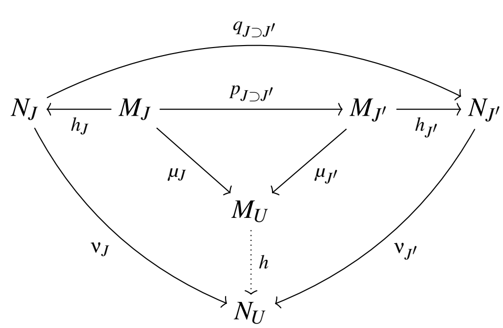
3	$N_f \models \rho$	$(M_M)^* \cap Sen^0\Sigma_M \subseteq (N_f)^* \cap Sen^0\Sigma_M$, $\rho \in (M_M)^* \cap Sen^0\Sigma_M$
4	$N_N \models (\iota_\Sigma f)\rho$	1, 3, Satisfaction Condition
5	$(\iota_\Sigma f)\rho \in Sen^0\Sigma_N$	naturality of sub-functor $Sen^0 \subseteq Sen$
6	$(\iota_\Sigma f)\rho \in (N_N)^* \cap Sen^0\Sigma_N$	4, 5
7	$P_g \models (\iota_\Sigma f)\rho$	6, $N_N[Sen^0]P_g$
8	$P_{f;g} \models \rho$	2, 7, Satisfaction Condition

- Now we show that $\overset{Sen^0}{\longleftarrow}(Up\ \mathbb{M})$ is closed under ultraproducts. For this, we first prove that for each class $\mathbb{N}$ of Σ-models we have that

9 $\quad Up(\overset{Sen^0}{\longleftarrow}\mathbb{N}) \ \subseteq\ \overset{Sen^0}{\longleftarrow}(Up\ \mathbb{N}).$

Let us consider $(h_i: M_i \to N_i \in \mathbb{N})_{i\in I}$ such that $(M_i)_{M_i}[Sen^0](N_i)_{h_i}$, $i \in I$, and ultraproducts $(\mu_J: M_J \to M_U)_{J\in U}$, $(\nu_J: N_J \to N_U)_{J\in U}$. Let $h_J: M_J \to N_J$ be the unique homomorphism such that $h_J;q_{J,i} = p_{J,i};h_i$ for each $i \in J$ (p and q being families of projections of the respective direct products).





Let $h: M_U \to N_U$ be the unique homomorphism such that $h_J;\nu_J = \mu_J;h$ for each $J \in U$. We show that $(M_U)_{M_U}[Sen^0](N_U)_h$ meaning that $M_U \xrightarrow{Sen^0} N_U$ which gives that $M_U \in \overset{Sen^0}{\longleftarrow} (Up\ \mathbb{N})$.

- By applying *UAX1* to $(M_U)_{M_u}$ there exists $\iota_\Sigma M_U$-expansions M'_i of M_i and ultraproduct $(\mu'_J : M'_J \to M'_U = (M_U)_{M_U})_{J \in U}$ such that $\mu' \restriction_{\iota_\Sigma M_U} = \mu$.
- Since $\iota_\Sigma M_U$ is representable, each M'_i determines an unique expansion $h'_i : M'_i \to N'_i$ of h_i. By Propositions 6.7 and 6.9 we can expand the ultraproduct ν to an ultraproduct $(\nu'_J : N'_J \to N'_U)_{J \in U}$ in two stages, first expand all h_J to $h'_J : M'_J \to N'_J$, and then ν to ν'.
- Let $h' : (M_U)_{M_U} = M'_U \to N'_U$ be the unique homomorphism such that $\mu'_J;h' = h'_J;\nu'_J$, which is also the unique expansion of h to a homomorphism from M'_U. Since the unique homomorphism $(M_U)_{M_U} \to (N_U)_h$ is an expansion of h, it follows that $N'_U = (N_U)_h$.
- Let $\rho \in ((M_U)_{M_U})^* \cap Sen^0 \Sigma_{M_U}$. Then

10 there exists $J \in U$ such that $M'_i \models \rho$ for each $i \in J$ — ρ preserved by ultrafactors

11 $N'_i \models \rho$ for each $i \in J$ — $h'_i : M'_i \to N'_i$ Sen^0-submodel ($h'_i \restriction_{\iota_\Sigma M_U} = h_i$, *UAX3*

12 $(N_U)_h = N'_U \models \rho$ — ρ preserved by ultraproducts.

This completes the proof of 9. Now we finalise the proof that $\overset{Sen^0}{\longleftarrow} (Up\ \mathbb{M})$ is closed under ultraproducts.

13 $Up(\overset{Sen^0}{\longleftarrow} (Up\ \mathbb{M})) \subseteq \overset{Sen^0}{\longleftarrow} (Up(Up\ \mathbb{M}))$ — 9 for $\mathbb{N} = Up\mathbb{M}$

14 $\subseteq \overset{Sen^0}{\longleftarrow} (Up\ \mathbb{M})$ — 13, $Up;Up = Up$ (*UAX2*).

- We have thus proved that $\overset{Sen^0}{\longleftarrow} (Up\ \mathbb{M})$ is closed under Sen^0-models and ultraproducts, and in fact it is the least class of models with this property that contains $\mathbb{M}$. By applying the same line of reasoning as in the final parts of the proof of Theorems 8.30 and 8.31, which in the current case includes reliance on the result of Cor. 8.8, we get the conclusion of the theorem.

□

The following concrete instance of Thm. 8.33 illustrates how its specific conditions can be established in concrete situations. In $\mathcal{FOL}$, let S_c be the class of the closed injective model homomorphisms and Univ be the functor of the universal sentences.

Corollary 8.34. *For any $\mathcal{FOL}$ signature Σ and any class of Σ-models $\mathbb{M}$,*

$$(\mathbb{M}^* \cap \mathsf{Univ}\Sigma)^* = \overset{S_c}{\leftarrow} (Up\ \mathbb{M})).$$

Proof. In Thm. 8.33 we take Sen^0 to be the subfunctor Exist of the existential sentences such as in Prop. 8.1 and Corollaries 8.4 and 8.9. How the axiom *UAX2* holds in $\mathcal{FOL}$ we have discussed above; this is by far the most difficult check of the validity of the specific conditions of Thm. 8.33 in $\mathcal{FOL}$as the other two axioms hold as follows:

UAX1: Elementary extensions invent strongly and completely ultraproducts because as injective signature extensions with constants they meet the conditions of Prop. 6.11.

UAX3: By Prop. 8.1 the Sen^0-submodels are precisely the closed injective homomorphisms which are preserved by expansions along the injective signature extensions with constants.

□

Birkhoff institutions

The way we have developed and presented the axiomatizability results in this section follows a certain pattern. One starts with an arbitrary class of models $\mathbb{M}$. On the one hand one considers the models of the sentences of a certain kind which are satisfied by all models in $\mathbb{M}$, and on the other hand one takes the closure of $\mathbb{M}$ first under a class of filtered products, and afterwards under certain relations defined in terms of certain classes of model homomorphisms. These two operations give the same result; this is the respective axiomatizability result. (In the literature the latter semantic closure operators are called *axiomatizable hulls*.)

The definition. The pattern for axiomatizability results discussed above is captured formally by the concept of Birkhoff institution. For any class of filters $\mathcal{F}$ and any class of Σ-models, by $\mathcal{F}\mathbb{M}$ we denote the class of all $\mathcal{F}$-products of models from $\mathbb{M}$.
Then $(Sig, Sen, Mod, \models, \mathcal{F}, \mathcal{B})$ is a *Birkhoff institution* when

- $\mathcal{F}$ is a class of filters with $\{\{*\}\} \in \mathcal{F}$, and
- $(Sig, Sen, Mod, \models)$ is an institution such that for each signature $\Sigma \in |Sig|$ the category $Mod\Sigma$ has $\mathcal{F}$-products,
- for each signature Σ, $\mathcal{B}_\Sigma \subseteq |Mod\Sigma| \times |Mod\Sigma|$ is a binary reflexive relation which is closed under isomorphisms, i.e., $(\mathcal{B}_\Sigma; \cong_\Sigma) = \mathcal{B}_\Sigma = (\cong_\Sigma; \mathcal{B}_\Sigma)$,

such that for each class $\mathbb{M}$ of Σ-models

$$\mathbb{M}^{**} = \mathcal{B}_\Sigma^{-1}(\mathcal{F}\mathbb{M}).$$

Examples. Based on the results we have already developed we can now present a list of Birkhoff institutions obtained around $\mathcal{FOL}$ by varying the style of the sentences. The second part of the list below contains some examples of Birkhoff institutions not developed in this book, but which are known in the literature in terms of the corresponding axiomatizability result.

institution	$\mathcal{B}$	$\mathcal{F}$	*source*
$\mathcal{FOL}$	$\equiv$	all ultrafilters	Thm. 8.32
$\mathcal{FOL}$	*ultraradical relation*	all ultrafilters	Thm. 8.32
$\mathcal{PL}$	$=$	all ultrafilters	Thm. 8.32
$\mathcal{UNIV}$	$\stackrel{S_c}{\to}$	all ultrafilters	Cor. 8.34
$\mathcal{HCL}_{\infty,\omega}$	$\stackrel{S_c}{\to}$	$\{\{I\} \mid I \text{ set}\}$	Thm. 8.30
universal $\mathcal{FOL}$-atoms	$\stackrel{H_r}{\leftarrow};\stackrel{S_c}{\to}$	$\{\{I\} \mid I \text{ set}\}$	Thm. 8.25
$\mathcal{EQL}$	$\stackrel{H_r}{\leftarrow};\stackrel{S_w}{\to}$	$\{\{I\} \mid I \text{ set}\}$	Thm. 8.30
universal $\mathcal{FOL}_{\infty,\omega}$ sentences	$\stackrel{S_c}{\to}$	$\{\{\{*\}\}\}$	[9]
$\mathcal{HCL}$	$\stackrel{S_c}{\to}$	all filters	[9]
$\forall\vee$ (universal disjunctions of atoms)	$\stackrel{H_s}{\leftarrow};\stackrel{S_c}{\to}$	all ultrafilters	[9]
$\forall\vee_\infty$ (univ. infinitary disj. of atoms)	$\stackrel{H_s}{\leftarrow};\stackrel{S_c}{\to}$	$\{\{\{*\}\}\}$	[9]
$\forall\exists$ (universal-existential sentences)	*sandwiches* ([42])	all ultrafilters	[9]

where H_r denotes the class of surjective, H_s the class of strong surjective, S_w the class of injective, and S_c the class of closed injective model homomorphisms.

Exercises

8.12. Prove the idempotency of the ultraproduct construction in $\mathcal{PL}$.

8.13. Birkhoff institutions of partial algebras
The following institutions of partial algebras arise as Birkhoff institutions according to the following table:

institution	$\mathcal{B}$	$\mathcal{F}$
$\mathcal{UNIV}(\mathcal{PA})$	$\stackrel{S_c}{\to}$	all ultrafilters
$\mathcal{QE}_2(\mathcal{PA})$	$\stackrel{S_w}{\to}$	$\{\{I\} \mid I \text{ set}\}$
$\mathcal{QE}(\mathcal{PA})$	$\stackrel{S_c}{\to}$	$\{\{I\} \mid I \text{ set}\}$

where S_w and S_c are the classes of plain, respectively closed, injective homomorphisms and where $\mathcal{UNIV}(\mathcal{PA})$ is the institution of the 'universal' sentences in $\mathcal{PA}$ (see Ex. 8.2).

Notes. Thm. 8.5 and Cor. 8.6 are institution-independent generalizations of basic axiomatizability results in first-order logic of [109] (see also [42]). Our general preservation-by-saturation Thm. 8.2 generalizes and extends its first-order logic Cor. 8.4 which can be found in [42]. Its axiomatizability consequence Cor. 8.9 can also be found in [42] while Cor. 8.8 constitutes its institution-independent generalization. The ultraradicals have been introduced and used in [205].

Similar quasi-variety concepts to ours have been formulated and results obtained within the framework of factorization systems (see [227, 228] or [9] for a very general approach), however, the inclusion systems framework leads to greater simplicity. Thm. 8.14 generalizes a well-known result from universal algebra [134] and conventional model theory of first-order logic [169]. A similar institution-independent result has been obtained by Tarlecki [227] within the framework of the so-called 'abstract algebraic institutions'. However, the concept of abstract algebraic institution provides a set of conditions much more complex than our framework. Within the same setting, [228] develops an institution-independent approach to the quasi-variety theorem related to ours, however Birkhoff Variety Thm. 8.25 seems to have no previous institution-independent variant.

Both quasi-variety and Birkhoff variety theorems have rather old roots in universal algebra. The former had been discovered by Mal'cev [169] while the latter by Birkhoff back in 1935 (see [27]). Lemmas 8.17 and 8.24 are inclusion system versions of well-known Birkhoff-like axiomatizability results for satisfaction by injectivity originally developed within the framework of factorization systems [197, 10]. They appeared in their current form as axiomatizability results for the so-called 'inclusive equational logic' of [212].

For a proof of the idempotency of the ultraproduct construction in $\mathcal{FOL}$ we may consult [12] and for the general categorical one we may look into [7].

Birkhoff institutions were introduced in [66]. A more complete list of Birkhoff sub-institutions of first-order logic can be obtained by using results from [9]. Examples of Birkhoff institutions in the context of less conventional logics arise in the context of Birkhoff-style axiomatizability results for these logics. For example, a large list of Birkhoff institutions based on partial algebra can also be obtained from [9]. Moreover, the very general axiomatizability results of [9] can be applied to obtaining Birkhoff institutions out of recent algebraic specification logics.

Chapter 9

Interpolation

Interpolation is one of the most important topics of logic and model theory. It has been studied extensively and in-depth in various logical contexts. Its manifold applications have been explored both in logic and in computing science.

In the first section of this chapter, we present several different perspectives on the concept of interpolation. A true understanding of such central concept in logic and model theory and of its applications may involve a structural categorical view on the one hand, a logical meaning on the other hand, and an understanding of the relationship between them. It also requires an understanding of the exact relationship between its traditional forms and its generalised abstract forms. Once we have covered these aspects, we will devote the rest of the chapter on methods for obtaining interpolation properties.

We develop two direct methods for obtaining interpolation results, one of them based on the Birkhoff-style axiomatizability properties of institutions, and the other one based on Robinson consistency. Although these two methods have quite complementary application domains, interpolation in $\mathcal{FOL}$ arises as an application of both of them, with the former method involving the 'heavy artillery' of the Keisler-Shelah property (Cor. 7.25). For the interpolation caused by axiomatizability, as a technical device, we use a semantic interpretation of interpolation. Apart from bringing uniformity to interpolation-by-axiomatizability, this has also other applications, such as in institutions supporting higher-order quantifications.

A third method to establish interpolation, which is presented here, is an indirect one, which 'borrows' interpolation along the institution comorphisms.

Another topic of this chapter refers to an extension of the Craig interpolation concept to the so-called 'Craig-Robinson interpolation' which is the variant of interpolation appropriate for several applications such as definability and semantics of structured specifications.

9.1 What is interpolation?

The structural view of interpolation is based on mappings between theories. This requires a concept that is dual to that of theory morphism which at this point in our discussion is already familiar to us.

Anti-morphisms of theories. In Sec. 4.1 we have introduced 'morphisms of theories'. Recall that given theories (Σ, E) and (Σ', E') a morphism $\varphi\colon\ (\Sigma, E) \to (\Sigma', E')$ is just a morphism between the underlying signatures, i.e. $\varphi\colon\ \Sigma \to \Sigma'$, such that $E' \models \varphi E$. This concept comes from model-theoretic computing science, especially algebraic specification and other logic-based declarative paradigms, where theories represent program or specification modules and morphisms of theories represent various connections between them that support their systematic aggregation into bigger modules. The most common such connection is that of an 'import', but there are others too. In all situations the target theory (i.e. (Σ', E')) is 'bigger' than the source theory (i.e. (Σ, E)). This is what $E' \models \varphi E$ says, another way to write this being $(\varphi E)^{**} \subseteq E'^{**}$. However, it is mathematically legitimate to reverse this, in other words to have the target theory 'smaller' than the source theory. This would be expressed as $\varphi E \models E'$. Let us call this an *anti-morphism of theories*. Anti-morphisms of theories share similar properties with the morphisms of theories, for instance, they form a category under the composition defined at the level of their underlying signature morphisms. But a morphism and an anti-morphism cannot be composed. To distinguish between them let us adopt the following notation: $(\varphi, \models\!\!\!\!\dashv)$ for morphisms and $(\varphi, \models)$ for anti-morphisms.

Pseudo-commutative morphism-anti-morphism (m-a-m) squares. These are squares like the left-hand side one below:

$$
\begin{array}{ccc}
(\Sigma, E) & \xrightarrow{(\varphi_1, =\!\mid)} & (\Sigma_1, E_1) \\
{\scriptstyle (\varphi_2, \models)} \downarrow & & \downarrow {\scriptstyle (\theta_1, \models)} \\
(\Sigma_2, E_2) & \xrightarrow[(\theta_2, =\!\mid)]{} & (\Sigma', E')
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\
{\scriptstyle \varphi_2} \downarrow & & \downarrow {\scriptstyle \theta_1} \\
\Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma'
\end{array}
\qquad (9.1)
$$

where

- the horizontal (vertical) arrows represent morphisms (anti-morphisms) of theories, and
- the underlying signature morphisms form a commutative square (depicted as the right-hand side square in the above figure).

Note that the idea of the commutativity of an m-a-m square does not make any sense as such a square contains both morphisms and anti-morphisms of theories and the mappings of the same kind are disconnected. The attribute 'pseudo-commutative' refers to the fact that the square formed by the underlying signature morphisms commutes.

Interpolants. Let us consider the following two questions:

1. Given a span $(\varphi_1, \models)$, $(\varphi_2, \models)$, can we complete it to a pseudo-commutative m-a-m square like in Fig. 9.1?
2. Given a sink $(\theta_1, \models)$, $(\theta_2, \models)$, can we complete it to a pseudo-commutative m-a-m square like in Fig. 9.1?

These are dual questions, however in terms of their answers the difference between them cannot be any bigger. While the answer to the former question is always an unconditional 'yes', the justification being very simple, the latter question represents the *interpolation problem*, which admits manifold solutions, all of them very contextual and difficult. We will solve the former question immediately, by Prop. 9.1 below, and then in the rest of this chapter, we will address the latter question in extenso.

Proposition 9.1. *For any m-a-m span* $(\varphi_1, \models)$*,* $(\varphi_2, \models)$*, any commutative square of signature morphisms like in Fig. 9.1 determines at least one pseudo-commutative m-a-m square like in Fig. 9.1.*

Proof. Let us define $E' = \theta_k(\varphi_k E)$ (by the commutativity of the square of signature morphisms we have that $\theta_1(\varphi_1 E) = \theta_2(\varphi_2 E)$). Then

- $(\theta_1, \models) : (\Sigma_1, E_1) \to (\Sigma', E')$ is an anti-morphism from the morphism property of $(\varphi_1, \models)$ and by an application of the 'translation' property of the semantic consequence for θ_1.
- $(\theta_2, \models) : (\Sigma_2, E_2) \to (\Sigma', E')$ is a morphism from the anti-morphism property of $(\varphi_2, \models$ $)$ is an anti-morphism and by an application of the 'translation' property for θ_2.

□

Concerning the latter question let us make the following remarks.

- By the 'transitivity' property of semantic consequence, a m-a-m sink $(\theta_1, \models)$, $(\theta_2, \models)$ is essentially the same with a sink of signature morphisms θ_1, θ_2 such that $\theta_1 E_1 \models \theta_2 E_2$. In this case, E' can be any set of sentences 'in-between' $\theta_1 E_1$ and $\theta_2 E_2$, i.e. $\theta_1 E_1 \models E' \models \theta_2 E_2$. As terminology, E_1 is called the *premise of the interpolation*, while E_2 is called its *conclusion*.
- The completion of an m-a-m sink as above has two aspects: first a cone φ_1, φ_2 for the sink θ_1, θ_2 of signature morphisms, and then a set of sentences E such that $(\varphi_1, \models)$, $(\varphi_2, \models)$ is a m-a-m span. The set E is called an *interpolant* for E_1 and E_2.
- Traditionally the interpolation problem is concerned with the second aspect only, by assuming already a cone φ_1, φ_2. This is true both in the case of the very classical logic contexts ($\mathcal{PL}$, $\mathcal{FOL}$), of less classical ones (e.g., modal logics), and even the institution-theoretic ones. Later on in this section, we will have a more extensive discussion on this. However, we think that the more ambitious idea of considering (Σ, E) as being the interpolant (which means including the signature Σ as part of the concept of interpolant) deserves further exploration.

Now we are ready to formulate the main interpolation concept in our book. In any institution, a commuting square of signature morphisms like in the Fig. 9.1 is a *Craig Interpolation square* (abbreviated *Ci square*) when any sink $(\theta_1, \models)$, $(\theta_2, \dashv)$ can be completed to a pseudo-commutative m-a-m square like in Fig. 9.1. The familiar form of this definition is in its explicit form: for any $E_1 \subseteq Sen\Sigma_1$, $E_2 \subseteq Sen\Sigma_2$ such that $\theta_1 E_1 \models \theta_2 E_2$ there exists $E \subseteq Sen\Sigma$ such that $E_1 \models \varphi_1 E$ and $\varphi_2 E \models E_2$.

Finiteness aspects. In traditional interpolation studies as well as in computing science applications the theories involved in interpolation problems are usually finite. Concerning that, we can make the following remarks. Consider a commutative square of signature morphisms like the above.

1. If it supports the Ci property for the finite / singleton conclusions (i.e. E_2 finite / singleton) then it is a Ci square. If E_2 is infinite then we can take the union of all the interpolants obtained from the interpolation problems with E_1 as premise and each sentence of E_2 as a conclusion.
2. Let us assume that the institution is compact. If the conclusion E_2 of an interpolation problem is finite then by compactness any interpolant E can be reduced to a finite one.

Down to earth. Let us look now into a concrete example of interpolation and see how it fits our abstract interpolation concepts. In $\mathcal{FOL}$ consider a single-sorted signature with a binary operation symbol $\_\star\_$ and constants a, a', b, b', c. This signature is the Σ' from our above definition of interpolation squares. Consider the following semantic consequence

$$(a = a' \ \wedge\ a \star b = c)\ \models\ (b = b'\ \Rightarrow\ a' \star b' = c). \tag{9.2}$$

Based on this consequence we can formulate an (institution theoretic) interpolation problem. We will use the notations familiar from the discussion above. The consequence (9.3) can be represented by some m-a-m sink $(\theta_1, \models)$, $(\theta_2, \dashv)$ where Σ_1 and Σ_2 are just Σ' from which we remove b' and a, respectively and θ_1, θ_2 are the resulting signature inclusions. E_1 is the premise and E_2 the conclusion of the consequence and E' can be either of them or even another theory that satisfies the requirements, since given a consequence like (9.3), what E' exactly is does not constitute an issue.

A 'smart' way to justify the semantic consequence (9.3), which also reveals its main point, is to factor it as follows:

$$(a = a' \ \wedge\ a \star b = c)\ \models\ a' \star b = c\ \models\ (b = b'\ \Rightarrow\ a' \star b' = c).$$

This factoring represents a completion of the above sink to a pseudo-commutative m-a-m square, where the m-a-m span $(\varphi_1, \dashv)$, $(\varphi_2, \models)$ is given by $\Sigma = \Sigma_1 \cap \Sigma_2$, φ_1, φ_2 being the obvious inclusions, and the interpolant E being the equality $a' \star b = c$. This square of signature inclusions is a Ci square, but why and how this happens we will see later on in this chapter. From this information, we can understand that the existence of interpolants for this problem was inevitable. Moreover, from our discussion on finiteness aspects, because $\mathcal{FOL}$ is compact and also has conjunctions any interpolant for this problem could be presented in a single-sentence form.

The traditional versus the institution-theoretic view of interpolation. The traditional view of interpolation is illustrated well by the concrete example above. Firstly, it is only about intersection-union squares of signatures like below.

$$\begin{array}{ccc} \Sigma_1 \cap \Sigma_2 & \xrightarrow[\subseteq]{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2} \downarrow {\scriptstyle \subseteq} & & {\scriptstyle \subseteq} \downarrow {\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{\subseteq} & \Sigma_1 \cup \Sigma_2 \end{array}$$

The idea is that an interpolant is always made of symbols that are shared by the premise and the conclusion of interpolation. Since in the traditional contexts we do not consider signature morphisms other than inclusions, the idea of shared symbols leads to $\Sigma = \Sigma_1 \cap \Sigma_2$ as this is the maximal signature such that $\Sigma \subseteq \Sigma_1, \Sigma_2$. This maximality guarantees that we do not miss interpolants just because of not having enough symbols available.

Then it is only about single sentences.

From our definition of interpolation, it is quite clear that in institution theory there is a revision of these two aspects. These have to do with pragmatics both at the level of the theory and of the applications. The traditional concept of interpolation is dependent on the classical concrete context in which it had originally been developed, that tacitly enjoys some very specific properties. This is just one reason among others why it lacks the sort of generality that is required by modern logical contexts, especially those related to applications in computing science. Let us discuss in more detail the motivations behind these revisions.

- There are a couple of motivations for generalising from intersection-union squares of signatures to arbitrary commutative squares. One is abstraction, at the abstract level bothering with concepts such as inclusion, union, and intersection constitute an unnecessary technical complication that has nothing to do with interpolation as such. Of course, inclusion systems are an ideal tool for dealing at the abstract level with such concepts, but this would still be a technical complication. The second motivation has to do with some applications of interpolation to algebraic specification where some of the signature morphisms involved in the interpolation squares may be non-injective. We will see details about this in Chap. 15. The latter motivation is of course stronger than the former.

- In the traditional context of interpolation, if instead of single sentences we refer to finite sets of sentences then we get the same thing. But this is so only because $\mathcal{FOL}$ has conjunctions. In institutions without conjunctions the traditional single-sentence variant of interpolation can be unnecessarily restrictive when compared to the one based on sets of sentences, and in some cases it makes almost no sense in the applications. For instance in $\mathcal{EQL}$, in the single-sentence variant the interpolation problems fail in general, while in the set-of-sentences formulation they admit interpolants in general. And we should know that $\mathcal{EQL}$ is an important institution as its good computational properties recommend it as foundation for some important specification and programming paradigms. The same situation happens with $\mathcal{HCL}$, the institution underlying logic programming. And after all, what counts in logic and its applications are theories rather than individual sentences. Therefore the institution theoretic view of interpolation adopts a form based on sets of sentences.

$(\mathcal{L},\mathcal{R})$**-interpolation.** The commutativity of a square of signature morphisms is necessary for defining interpolation but in general, is too loose for supporting interpolation properties. If we look at the traditional context, there is a tightness aspect to the intersection-union squares. $\Sigma_1 \cap \Sigma_2$ being the maximal signature Σ such that $\Sigma \subseteq \Sigma_1, \Sigma_2$ takes the sharing to its most permissive level. At the other end of the interpolation square, the union $\Sigma_1 \cup \Sigma_2$ is the minimal signature Σ' that can accommodate both the premise and the conclusion of an interpolation problem. At the general categorical level 'intersection' means 'pullback' while 'union' means 'pushout'. This suggests that we should consider interpolation for those commutative squares of signature morphisms that are both pullback and pushout squares. However the reality of the developments in institution-theoretic interpolation is a bit different, the pullback condition being not only unnecessary but even a hindrance in the applications. Otherwise said the sharing between Σ_1 and Σ_2 should be thought in a broader sense than that of a mere intersection. On the other hand, the pushout condition is crucial. There are several reasons for these as follows.

- When building an algebraic specification or a declarative program usually Σ comes before Σ_1 and Σ_2, and Σ' comes at the end as a kind of parameterized 'union' of Σ_1 and Σ_2, the parameter of the 'union' is Σ. This 'union' is achieved by a pushout construction. Moreover, the sharing is not always a span of inclusions, it can be a span of other types of signature morphisms also. On the other hand, a pullback represents a reverse order and has no meaning in the world of software module aggregation. But what does this have to do with interpolation? Such commutative squares of signature morphisms that arise from software modules composition in logic-based contexts should enjoy interpolation properties so that the respective module compositions have good properties, that for instance enable modular theorem proving and formal verifications. We will see exactly what this means in Chap. 15.
- Then, and not unrelated to the previous item, it is common for pushout squares of signature morphisms to enjoy model amalgamation, which plays an important role in some methods of establishing establishing interpolation properties, especially when relying upon axiomatizability results.

Let us say this with clarity: all the extensions of the traditional concept of interpolation that take place in institution theory do cover fully the traditional concept of interpolation, so these should be regarded as a mere enrichment rather than something else. An enrichment that broadens remarkably the application domain of interpolation.

In many institutions only *some* pushout squares of signature morphisms have the Ci property. For example, while in $\mathcal{FOL}^1$ (the unsorted version of $\mathcal{FOL}$) *all* pushout squares have the Ci property, this is not the case in $\mathcal{FOL}$. Also, in $\mathcal{EQL}$ and $\mathcal{HCL}$, not all pushout squares have the Ci property. It is often convenient to capture such classes of Ci squares by restricting independently φ_1 and φ_2 to belong to certain classes of signature morphisms.

Therefore, for any classes of signature morphisms $\mathcal{L}, \mathcal{R}$, we say that the institution has the *Craig* $(\mathcal{L}, \mathcal{R})$*-Interpolation property* if each pushout square of signature morphism of the form

$$\begin{array}{ccc} \bullet & \xrightarrow{\mathcal{L}} & \bullet \\ {\scriptstyle \mathcal{R}}\big\downarrow & & \big\downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

is a Ci square.

The list below anticipates some of the concrete $(\mathcal{L}, \mathcal{R})$-interpolation properties obtained in this chapter. But before presenting this list let us establish the following notation for $\mathcal{FOL}$ signature morphisms.

(xyz)**-morphisms of signatures.** Let us define the following syntactic properties for signature morphisms. A $\mathcal{FOL}$ signature morphism φ is an *(xyz)-morphism*, with $x, z \in \{i, s, b, *\}$ and $y \in \{i, i', s, b, e, *\}$ when the sort component φ^{st} has the property x, the operation component φ^{op} has the property y, and the relation component φ^{rl} has the property z. In the case of the families of functions φ^{op} and φ^{rl} the properties refer to their components. The meanings of these symbols are as follows:

- The symbols i, s, b, e stand for 'injective', 'surjective', 'bijective', 'injective and encapsulated', respectively. The symbol $*$ stands for 'any'.
- That $\varphi^{\mathrm{op}}_{w\to s}$ is encapsulated means that no 'new' operation symbol, i.e., outside the range of φ, is allowed to have the sort within the range of φ. In other words, if $\varphi\colon (S, F, P) \to (S', F', P')$ and $\sigma' \in F'_{w'\to s'}$ with $s' \in \varphi S$ then there exists $\sigma \in F_{w\to s}$ such that $\varphi\sigma = \sigma'$.
- The symbol i' stands for 'injective' plus that φ does not introduce any new operation whose sort is empty. The emptiness of a sort is established in the source signature if it comes from there.

For example, an $(ss*)$-morphism of signatures is surjective on the sorts and on the operations, while a (bis)-morphism of signatures is bijective on the sorts, is injective on the operations, and is surjective on the relations.

This notational convention can be extended to other institutions too, such as for example $\mathcal{PA}$, $\mathcal{EQL}$ or $\mathcal{FOL}^1$. In the case of $\mathcal{EQL}$, because we do not have relation symbols, the last component is missing. The same applies to $\mathcal{FOL}^1$, in this case, the first component (i.e., the sort component) is missing.

The list. Below is the above-mentioned list of $(\mathcal{L}, \mathcal{R})$-interpolation properties:

institution	$\mathcal{L}$	$\mathcal{R}$	*reference*
$\mathcal{FOL}^1$	$**$	$**$	Cor. 9.11 or 9.22
$\mathcal{FOL}$	$i**$	$***$	Cor. 9.17 or 9.22
	$***$	$i**$	Cor. 9.11 or 9.22
$\mathcal{EQL}$	$**$	ii'	Cor. 9.9
	ie	$**$	Cor. 9.14
$\mathcal{HCL}$	$***$	$ii'i$	Cor. 9.9
	$ie*$	$***$	Cor. 9.14
$\mathcal{SOL}$	$***$	iii	Cor. 9.6

Exercises

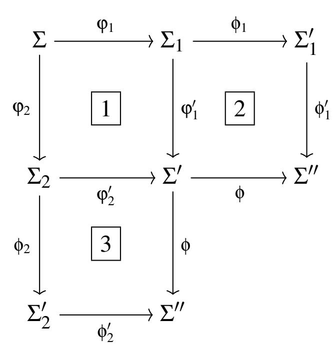
9.1. In $\mathcal{FOL}$ consider single-sorted signatures Σ_1 and Σ_2 such that Σ_1 has two constants a, b, a unary operation symbol f, and a unary relation symbol q, while Σ_2 has one constant c.

1. Prove that

$$q(fa) \wedge \neg q(fb) \models_{\Sigma_1 \cup \Sigma_2} (\exists v) v \neq c. \tag{9.3}$$

2. In $\Sigma_1 \cap \Sigma_2$, find a interpolant for the consequence (9.3).

9.2. Composition of interpolation squares
Ci squares can be composed both 'horizontally' and 'vertically': in any institution, consider the commuting squares of signature morphisms



Then

1. $\boxed{12}$ ($\boxed{13}$) is a Ci square if $\boxed{1}$ and $\boxed{2}$ ($\boxed{3}$) are Ci squares.
2. $\boxed{1}$ is a Ci square if $\boxed{12}$ ($\boxed{13}$) is a Ci square and ϕ_1 (ϕ_2) is conservative.

(*Hint:* Use the m-a-m square definition of interpolation.)

9.2 Semantic interpolation

By using the Galois connection between sets of sentences and classes of models given by the satisfaction relation, we may shift the interpolation concept from sets of sentences to classes of models. This has very little logical significance, but as we will see later on in the chapter, it can be technically very useful. The semantic interpretation of interpolation is based on the following observations for any commuting square of signature morphisms in an arbitrary institution.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array} \tag{9.4}$$

If $E_1 \subseteq Sen\Sigma_1$, $E_2 \subseteq Sen\Sigma_2$, and $E \subseteq Sen\Sigma$ then we can rewrite:

- $\theta_1 E_1 \models_{\Sigma'} \theta_2 E_2$ as $(Mod\theta_1)^{-1}E_1^* \subseteq (Mod\theta_2)^{-1}E_2^*$,
- $E_1 \models_{\Sigma_1} \varphi_1 E$ as $E_1^* \subseteq (Mod\varphi_1)^{-1}E^*$, and
- $\varphi_2 E \models_{\Sigma_2} E_2$ as $(Mod\varphi_2)^{-1}E^* \subseteq E_2^*$.

Now let us abstract E_1^* to any class $\mathbb{M}_1 \subseteq |Mod\Sigma_1|$, E_2^* to any class $\mathbb{M}_2 \subseteq |Mod\Sigma_2|$, and E to any class $\mathbb{M} \subseteq |Mod\Sigma|$. Then the interpolation situation defined by E_1, E_2 and E gets translated to the following: for any classes of models $\mathbb{M}_1 \subseteq |Mod\Sigma_1|$, $\mathbb{M}_2 \subseteq |Mod\Sigma_2|$ such that

$$(Mod\theta_1)^{-1}\mathbb{M}_1 \ \subseteq \ (Mod\theta_2)^{-1}\mathbb{M}_2$$

there exists a class of models $\mathbb{M} \subseteq |Mod\Sigma|$ such that

$$\mathbb{M}_1 \subseteq (Mod\varphi_1)^{-1}\mathbb{M} \ \text{ and } \ (Mod\varphi_2)^{-1}\mathbb{M} \subseteq \mathbb{M}_2. \tag{9.5}$$

$\mathbb{M}$ is called a *semantic interpolant* for $\mathbb{M}_1$ and $\mathbb{M}_2$. Note however that if we want to solve an interpolation problem by interpreting it as semantic interpolation as above, then we still have to do another step, namely to get (a syntactic interpolant) E from $\mathbb{M}$. This can be done only if $\mathbb{M}$ is elementary, otherwise said if it is axiomatizable. We may therefore state the following Principle of Semantic Interpolation:

> *The existence of a (syntactic) interpolant for E_1 and E_2 is equivalent to the existence of a semantic interpolant $\mathbb{M}$ for E_1^* and E_2^* such that $\mathbb{M}$ is elementary, in this case the syntactic interpolant being $\mathbb{M}^*$.*

Existence of semantic interpolants. The following simple result shows that the existence of semantic interpolants is easy and subject only to a mild condition commonly satisfied in the applications. So the real difficulty of interpolation, when approached by the Principle of Semantic Interpolation, is to get the semantic interpolant axiomatized.

Proposition 9.2. *In any institution, for any weak model amalgamation square like (9.4), if $\mathbb{M}_k \subseteq |Mod\Sigma_k|$, $k = 1,2$, such that*

$$(Mod\theta_1)^{-1}\mathbb{M}_1 \ \subseteq \ (Mod\theta_2)^{-1}\mathbb{M}_2$$

then $\mathbb{M}_1$ and $\mathbb{M}_2$ have $\mathbb{M} = \mathbb{M}_1\restriction_{\varphi_1}$ as a semantic interpolant.

Proof. • That $\mathbb{M}_1 \subseteq (Mod\varphi_1)^{-1}\mathbb{M}$ follows immediately from the definition of $\mathbb{M}$.

- For showing that $(Mod\varphi_2)^{-1}\mathbb{M} \subseteq \mathbb{M}_2$ we consider any $M_2 \in (Mod\varphi_2)^{-1}\mathbb{M}$ and prove that $M_2 \in \mathbb{M}_2$. This goes as follows:

1	$M_2\upharpoonright_{\varphi_2} \in \mathbb{M}$	$M_2 \in (Mod\varphi_2)^{-1}\mathbb{M}$
2	there exists $M_1 \in \mathbb{M}_1$ such that $M_1\upharpoonright_{\varphi_1} = M_2\upharpoonright_{\varphi_2}$	1, definition of $\mathbb{M}$
3	there exists $M' \in \lvert Mod\Sigma'\rvert$ s.th. $M'\upharpoonright_{\theta_k} = M_k$, $k = 1,2$	2, model amalgamation hypothesis
4	$M' \in (Mod\theta_1)^{-1}\mathbb{M}_1$	$M_1 \in \mathbb{M}_1$ (2), $M'\upharpoonright_{\theta_1} = M_1$ (3)
5	$M' \in (Mod\theta_2)^{-1}\mathbb{M}_2$	4, $(Mod\theta_1)^{-1}\mathbb{M}_1 \subseteq (Mod\theta_2)^{-1}\mathbb{M}_2$
6	$M_2 \in \mathbb{M}_2$	5, $M'\upharpoonright_{\theta_2} = M_2$.

□

Semantic operators. Fixed points of semantics operators will assist us in the search for axiomatizable semantic interpolants. With examples of semantics operators we have already met in Chap. 8 (on preservation and axiomatizability) but we have not yet given them a definition. Given a signature Σ, a *semantic* Σ*-operator* is just a mapping of Σ-classes of Σ-models $\mathcal{U}_\Sigma : \mathcal{P}\lvert Mod\Sigma\rvert \to \mathcal{P}\lvert Mod\Sigma\rvert$. It is a *semantic closure operator* when it has the following additional properties:

– $\mathbb{M} \subseteq \mathcal{U}_\Sigma\mathbb{M}$	reflexivity
– $\mathbb{M} \subseteq \mathbb{M}'$ implies $\mathcal{U}_\Sigma\mathbb{M} \subseteq \mathcal{U}_\Sigma\mathbb{M}'$	monotonicity
– $\mathcal{U}_\Sigma(\mathcal{U}_\Sigma\mathbb{M}) = \mathcal{U}_\Sigma\mathbb{M}$	idempotency
– if $\mathbb{M}$ is closed under isomorphisms, then $\mathcal{U}_\Sigma\mathbb{M}$ is also closed under isomorphisms	closure under isomorphisms.

Some examples of semantic closure operators are as follows.

- The trivial operators: the identity operators and the maximal operators.
- The *isomorphic closure* operator Iso defined by Iso $\mathbb{M} = \{M \mid M \cong N \text{ for some } N \in \mathbb{M}\}$.
- The *elementary closure* operator $(-)^{**}$ mapping each class of models $\mathbb{M}$ to $\mathbb{M}^{**}$.
- In the context of the developments in Chap. 8, when a class of models $\mathbb{M}$ gets mapped to $\overset{I}{\leftarrow}\mathbb{M}$, $\overset{\mathcal{E}}{\to}\mathbb{M}$, $P\mathbb{M}$, $Up\mathbb{M}$ (when Up is idempotent, such as in $\mathcal{FOL}$), $\overset{\mathcal{E}}{\to}(\overset{I}{\leftarrow}\mathbb{M})$, $\overset{Sen^0}{\longleftarrow}(Up\mathbb{M})$ (when Up is idempotent), $\overset{I}{\leftarrow}(P\mathbb{M})$, $\overset{\mathcal{E}}{\to}(P\mathbb{M})$, $\overset{\mathcal{E}}{\to}(\overset{I}{\to}(P\mathbb{M}))$, all these are closure operators under the conditions of the respective results where they appear.

Fixed points of semantic operators. A class $\mathbb{M}$ of Σ-models is a *fixed point* for a semantic operator $\mathcal{U}_\Sigma$ when $\mathcal{U}_\Sigma \mathbb{M} = \mathrm{Iso}\mathbb{M}$. Let $FX\ \mathcal{U}_\Sigma$ be the class of all fixed points of $\mathcal{U}_\Sigma$. Fixed points $\mathbb{M}$ of semantic operators may guarantee the closure of $\mathbb{M}$ under certain semantic operators, a closure that may determine the axiomatizability of $\mathbb{M}$. For instance if we take $\mathcal{U}\,\mathbb{M} = \overset{I}{\leftarrow}(P\mathbb{M})$ then the closure of $\mathbb{M}$ under Iso and $\mathcal{U}$ means that $\mathbb{M}$ is closed under $\overset{I}{\leftarrow}$ and under direct products (of course by and under the conditions of Prop. 8.29). Via Thm. 8.18 this leads to the axiomatizability of $\mathbb{M}$.

The following is a rather abstract generic result which gives a set of heavily technical sufficient conditions for the existence of a semantic interpolant that in the applications can be easily shown to be elementary as a consequence of being a fixed point of semantic operators that are linked to axiomatizability properties. By giving various meanings to the parameters $\mathcal{U}$ and $\mathcal{V}$ we will be able to apply this result to obtain various general proper interpolation properties.

Theorem 9.3. *In any institution, consider any weak model amalgamation square of signature morphisms*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

and pairs of semantic operators $\mathcal{U} = \langle \mathcal{U}_\Sigma, \mathcal{U}_{\Sigma_1} \rangle$ and $\mathcal{V} = \langle \mathcal{V}_\Sigma, \mathcal{V}_{\Sigma_2} \rangle$ such that

1. $\mathcal{U}_\Sigma ; \mathcal{V}_\Sigma ; \mathcal{U}_\Sigma = \mathcal{U}_\Sigma ; \mathcal{V}_\Sigma$,
2. $\mathcal{V}$ *are closure operators,*
3. $Mod\varphi_1$ *preserves fixed points of $\mathcal{U}$ (i.e., $(FX\ \mathcal{U}_{\Sigma_1})\restriction_{\varphi_1} \subseteq FX\ \mathcal{U}_\Sigma$)),*
4. $(Mod\varphi_1);\mathrm{Iso};\mathcal{V}_\Sigma = \mathrm{Iso};(Mod\varphi_1);\mathcal{V}_\Sigma$, *and*
5. $\mathcal{V}_\Sigma;(Mod\varphi_2)^{-1} \subseteq (Mod\varphi_2)^{-1};\mathcal{V}_{\Sigma_2}$.

Then all classes of models $\mathbb{M}_1 \in FX\ \mathcal{U}_{\Sigma_1}$ and $\mathbb{M}_2 \in FX\ \mathcal{V}_{\Sigma_2}$ which are closed under isomorphisms and such that $(Mod\theta_1)^{-1}\mathbb{M}_1 \subseteq (Mod\theta_2)^{-1}\mathbb{M}_2$ have a semantic interpolant $\mathbb{M}$ in $(FX\ \mathcal{U}_\Sigma) \cap (FX\ \mathcal{V}_\Sigma)$ which is closed under isomorphisms.

Proof. The semantic interpolant $\mathbb{M}$ is defined as $\mathcal{V}_\Sigma(\mathbb{M}_1\restriction_{\varphi_1})$. We prove the properties of $\mathbb{M}$ in reverse order of their importance.

- We first show that $\mathbb{M}$ is closed under isomorphisms. We have that:

1	$\mathcal{V}_\Sigma\ \mathbb{M}_1\restriction_\varphi = \mathcal{V}_\Sigma\ (\mathrm{Iso}\mathbb{M}_1)\restriction_{\varphi_1}$	$\mathbb{M}_1$ closed under isomorphisms
2	$= \mathcal{V}_\Sigma\ \mathrm{Iso}(\mathbb{M}_1\restriction_{\varphi_1})$	model reducts, as functors, preserve isomorphisms.

  Since $\mathrm{Iso}(\mathbb{M}_1\restriction_{\varphi_1})$ is closed under isomorphisms, by the closure under isomorphisms property of $\mathcal{V}_\Sigma$ (as semantic closure operator), from (2), it follows that $\mathcal{V}_\Sigma\ \mathbb{M}_1\restriction_\varphi$ is closed under isomorphisms too.

- Now we show that $\mathbb{M} \in FX\ \mathcal{U}_\Sigma$ as follows:

3	$\mathbb{M}_1\restriction_{\varphi_1} \in FX\ \mathcal{U}_\Sigma$	$\mathbb{M}_1 \in FX\ \mathcal{U}_{\Sigma_1}$, $Mod\varphi_1$ preserves fixed points
4	$\mathcal{U}_\Sigma\mathbb{M} = \mathcal{U}_\Sigma(\mathcal{V}_\Sigma\ \mathbb{M}_1\restriction_{\varphi_1}) = \mathcal{U}_\Sigma(\mathcal{V}_\Sigma\ \mathrm{Iso}(\mathbb{M}_1\restriction_{\varphi_1}))$	definition of $\mathbb{M}$, 2
5	$= \mathcal{U}_\Sigma(V_\Sigma(\mathcal{U}_\Sigma\ \mathbb{M}_1\restriction_{\varphi_1})) = V_\Sigma(\mathcal{U}_\Sigma\ \mathbb{M}_1\restriction_{\varphi_1})$	3, condition 1. of the theorem
6	$= \mathcal{V}_\Sigma\ \mathrm{Iso}(\mathbb{M}_1\restriction_{\varphi_1}) = \mathcal{V}_\Sigma\ (\mathrm{Iso}\mathbb{M}_1)\restriction_{\varphi_1}$	3, 2
7	$= \mathcal{V}_\Sigma\ \mathbb{M}_1\restriction_{\varphi_1} = \mathbb{M} = \mathrm{Iso}\mathbb{M}$	$\mathbb{M}_1$ closed under isomorphisms, definition of $\mathbb{M}$, $\mathbb{M}$ closed under isomorphisms.

- Now we show that $\mathbb{M} \in FX\ \mathcal{V}_\Sigma$ as follows:

8	$\mathcal{V}_\Sigma\mathbb{M} = \mathcal{V}_\Sigma^2\ \mathbb{M}_1\restriction_{\varphi_1} = \mathcal{V}_\Sigma\ \mathbb{M}_1\restriction_{\varphi_1}$	definition of $\mathbb{M}$, $\mathcal{V}_\Sigma$ idempotent (closed operator)
9	$= \mathbb{M} = \mathrm{Iso}\mathbb{M}$	definition of $\mathbb{M}$, $\mathbb{M}$ closed under isomorphisms.

- Finally, we show that $\mathbb{M}$ is a semantic interpolant for $\mathbb{M}_1$ and $\mathbb{M}_2$ (properties 11 and 15 below).

10	$\mathbb{M}_1\restriction_{\varphi_1} \subseteq \mathcal{V}_\Sigma\ \mathbb{M}_1\restriction_{\varphi_1} = \mathbb{M}$	reflexivity of $\mathcal{V}_\Sigma$ (closure operator), definition of $\mathbb{M}$
11	$\mathbb{M}_1 \subseteq (Mod\varphi_1)^{-1}\mathbb{M}$	equivalent way to write 10
12	$(Mod\varphi_2)^{-1}\mathbb{M} = (Mod\varphi_2)^{-1}(\mathcal{V}_\Sigma\ \mathbb{M}_1\restriction_{\varphi_1})$	definition of $\mathbb{M}$
13	$\subseteq \mathcal{V}_{\Sigma_2}((Mod\varphi_2)^{-1}\mathbb{M}_1\restriction_{\varphi_1})$	condition 5.
14	$\subseteq \mathcal{V}_{\Sigma_2}\mathbb{M}_2$	Prop. 9.2, monotonicity of $\mathcal{V}_{\Sigma_2}$ (closure operator property)
15	$= \mathrm{Iso}\mathbb{M}_2 = \mathbb{M}_2$	$\mathbb{M}_2 \in FX\ \mathcal{V}_{\Sigma_2}$ (condition 3.), $\mathbb{M}$ closed under isomorphisms.

□

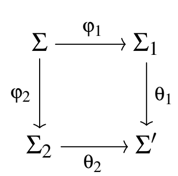
Let us remark that:

Fact 9.4. *In the context of Thm. 9.3 the relation* $\mathbb{M} \in (FX\ \mathcal{U}_\Sigma) \cap (FX\ \mathcal{V}_\Sigma)$ *can be equivalently expressed as* $\mathbb{M} \in FX(\mathcal{V}_\Sigma; \mathcal{U}_\Sigma)$.

Proof. By using that $\mathbb{M}$ is closed under isomorphisms and the first two conditions of Thm. 9.3. □

Higher-order interpolation. An immediate application of the general semantic interpolation Thm. 9.3 is the following general result.

Corollary 9.5. *In any institution with universal* $\mathcal{R}$*-quantification for a class* $\mathcal{R}$ *of signature morphisms, any weak model amalgamation square*



for which $\varphi_2 \in \mathcal{R}$ *is a Craig interpolation square.*

Proof. In Thm. 9.3 let us take

- $\mathcal{U}$ to be identities, and
- $\mathcal{V}$ to be elementary closures, i.e., $\mathcal{V}\,\mathbb{M} = \mathbb{M}^{**}$.

Since for this setting of $\mathcal{U}$ and $\mathcal{V}$ the first four conditions of Thm. 9.3 are rather trivial, let us focus on the last condition, that $(Mod\varphi_2)^{-1}\mathbb{N}^{**} \subseteq ((Mod\varphi_2)^{-1}\mathbb{N})^{**}$ for each $\mathbb{N} \subseteq |Mod\Sigma|$. Consider a Σ_2-model N_2 such that $N_2\upharpoonright_{\varphi_2} \in \mathbb{N}^{**}$ and let $\rho_2 \in ((Mod\varphi_2)^{-1}\mathbb{N})^{*}$. We need to show that $N_2 \models_{\Sigma_2} \rho_2$. Let ρ_2' be a universal φ_2-quantification of ρ_2. Then:

1	$\rho_2' \in \mathbb{N}^{**}$	$\rho_2 \in ((Mod\varphi_2)^{-1}\mathbb{N})^{*}$, ρ_2' universal φ_2-quantification of ρ_2
2	$N_2\upharpoonright_{\varphi_2} \models \rho_2'$	1, $N_2\upharpoonright_{\varphi_2} \in \mathbb{N}^{**}$
3	$N_2 \models \rho_2$	2, ρ_2' universal φ_2-quantification of ρ_2.

By the conclusion of Thm. 9.3 we get a semantic interpolant $\mathbb{M}$, closed under isomorphisms, and such that $\mathbb{M}^{**} = \mathrm{Iso}\mathbb{M}$ (as a fixed point for $\mathcal{V}$), which means $\mathbb{M}^{**} = \mathbb{M}$. The conclusion of this corollary follows now by the Principle of Semantic Interpolation. □

The following interpolation properties are instances of Cor. 9.5. Recall that $\mathcal{SOL}$ is the 'second-order' extension of $\mathcal{FOL}$ admitting quantifiers over *any* injective signature extensions with a finite number of symbols and that a signature morphism in $\mathcal{FOL}$ has non-empty sorts if there exists at least one term of each sort.

Corollary 9.6. *The institutions $\mathcal{FOL}$, $\mathcal{HCL}$, $\mathcal{EQL}$, $\mathcal{SOL}$ have Craig $(Sig, \mathcal{R})$-interpolation where $\mathcal{R}$*

- *is the class of all injective signature extensions with constants $\varphi \colon \Sigma \to \Sigma'$ such that Σ has non-empty sorts, in the case of $\mathcal{FOL}$, $\mathcal{HCL}$ and $\mathcal{EQL}$, and*
- *is the class of (iii)-morphisms of signatures $\varphi \colon \Sigma \to \Sigma'$ such that both Σ and Σ' have non-empty sorts, in the case of $\mathcal{SOL}$.*

Proof. In order to apply Cor. 9.5 we have to establish that the considered institutions admit universal $\mathcal{R}$-quantification. In any of the considered institutions let $\varphi \colon \Sigma \to \Sigma'$ be a signature morphism in $\mathcal{R}$ and let ρ' be a Σ'-sentence. We have to show that there exists a Σ-sentence ρ such that

$$\text{for each } \Sigma\text{-model } M,\ M \in \rho^{*} \text{ if and only if } (Mod\varphi)^{-1}M \subseteq \rho'^{*}. \tag{9.6}$$

Of course, when φ is finitary, i.e., extends with a finite number of symbols, this holds because all the considered institutions having explicit finitary universal $\mathcal{R}$-quantifications we can just consider ρ to be a universal φ-quantification of ρ'. So the issue is relevant only when φ extends with an infinite number of symbols.

- Because ρ' has only a finite number of symbols and Σ' has non-empty sorts, there exists a sub-signature $\Sigma_0 \subseteq \Sigma'$ such that Σ_0 is finite and has non-empty sorts, and there exists a Σ_0-sentence ρ_0' such that $\rho' = \chi'\rho_0'$ where χ' denotes the signature inclusion $\Sigma_0 \subseteq \Sigma'$.

- Let us consider the following square of signature inclusions.

$$\begin{array}{ccc} \Sigma \cap \Sigma_0 & \xrightarrow[\subseteq]{\varphi_0} & \Sigma_0 \\ \chi \downarrow \subseteq & & \subseteq \downarrow \chi' \\ \Sigma & \xrightarrow[\varphi]{\subseteq} & \Sigma' \end{array} \tag{9.7}$$

  This is a weak model amalgamation square because:

  – the intersection-union square determined by Σ and Σ_0 is a pushout square,

  – hence it is a model amalgamation square because all considered institutions are semi-exact, and moreover

  – it is a weak model amalgamation square because the inclusion $\Sigma \cup \Sigma_0 \subseteq \Sigma'$ has the model expansion property because both Σ and Σ_0 have non-empty sorts (and consequently their union too) and by Fact 5.6.

- Because all considered institutions have finitary $\mathcal{R}$-quantifications, there exists a $\Sigma \cap \Sigma_0$-sentence ρ_0 which is an universal φ_0-quantification of ρ'_0. We define $\rho = \chi\rho_0$ and prove (9.6). Consider any Σ-model M. From the weak model amalgamation property of the square (9.7) we have:

  1 $(Mod\varphi_0)^{-1}((Mod\chi)M) = (Mod\chi')((Mod\varphi)^{-1}M)$.

  Then (9.6) is obtained by the following succession of equivalent statements:

– $M \models \rho$	
– $M \models \chi\rho_0$	$\rho = \chi\rho_0$
– $M\upharpoonright_\chi \models \rho_0$	Satisfaction Condition
– $(Mod\varphi_0)^{-1}(M\upharpoonright_\chi) \subseteq (\rho'_0)^*$	ρ_0 universal φ_0-quantification of ρ'_0
– $(Mod\chi')((Mod\varphi)^{-1}M) \subseteq (\rho'_0)^*$	1
– $(Mod\varphi)^{-1}M \subseteq \rho'^*$	$\rho' = \chi'\rho'_0$, Satisfaction Condition.

□

The interpolation properties for $\mathcal{FOL}$, $\mathcal{EQL}$, $\mathcal{HCL}$ given by Cor. 9.6 are rather weak because in all these cases $\mathcal{R}$ is quite narrow. Later in the section, we will prove much stronger interpolation results for these institutions. On the other hand, the interpolation property for $\mathcal{SOL}$ given by Cor. 9.6 is rather substantial. This difference is caused by the possibility of higher-order quantifications in $\mathcal{SOL}$ which is missing in $\mathcal{FOL}$, $\mathcal{EQL}$ or $\mathcal{HCL}$.

Exercises

9.3. Interpolation in $\mathcal{HNK}$
The institution of higher order logic with Henkin semantics ($\mathcal{HNK}$) has Craig $(Sig^{\mathcal{HNK}}, (bi))$-interpolation. (*Hint:* From Cor. 9.5.)

9.3 Interpolation by axiomatizability

In this section, we derive a couple of general interpolation results for Birkhoff institutions from the abstract semantic interpolation Thm. 9.3. We also apply them to actual institutions and thus obtain a series of concrete interpolation results. For this, we need the following concept of lifting relations.

Lifting relations. Let $\varphi\colon\ \Sigma \to \Sigma'$ be a signature morphism and $\mathcal{R} = \langle \mathcal{R}_\Sigma, \mathcal{R}_{\Sigma'} \rangle$ with $\mathcal{R}_\Sigma \subseteq |Mod\Sigma| \times |Mod\Sigma|$ and $\mathcal{R}_{\Sigma'} \subseteq |Mod\Sigma'| \times |Mod\Sigma'|$ be a pair of binary relations. We say that φ *lifts* $\mathcal{R}$ if and only if for each $M' \in |Mod\Sigma'|$ and $N \in |Mod\Sigma|$ if $\langle M' \upharpoonright_\varphi, N \rangle \in \mathcal{R}_\Sigma$, then there exists $N' \in |Mod\Sigma'|$ such that $N' \upharpoonright_\varphi = N$ and $\langle M', N' \rangle \in \mathcal{R}_{\Sigma'}$.

$$
\begin{array}{ccccc}
\Sigma & \quad & Mod\Sigma & \quad & M'\upharpoonright_\varphi \xrightarrow{\ \mathcal{R}_\Sigma\ } N = N'\upharpoonright_\varphi \\
\varphi \big\downarrow & & \big\uparrow Mod\varphi & & \\
\Sigma' & & Mod\Sigma' & & M' \xrightarrow[\ \mathcal{R}_{\Sigma'}\ ]{} (\exists) N'
\end{array}
$$

This situation can be expressed more compactly by the following inequality:

$$(Mod\varphi); \mathcal{R}_\Sigma(\_) \ \subseteq\ \mathcal{R}_{\Sigma'}(\_); (Mod\varphi).$$

Can you relate this to the 5th condition of the semantic interpolation Theorem 9.3?

The ‘right’ interpolation theorem

The first interpolation theorem derived from Birkhoff axiomatizability properties, which is presented below, relies upon the properties of the morphisms on the ‘right-hand side’ of the interpolation squares.

Theorem 9.7. *Consider a Birkhoff institution* $(Sig, Sen, Mod, \models, \mathcal{F}, \mathcal{B})$ *and a weak model amalgamation square of signature morphisms:*

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\ \varphi_1\ } & \Sigma_1 \\
\varphi_2 \big\downarrow & & \big\downarrow \theta_1 \\
\Sigma_2 & \xrightarrow[\ \theta_2\ ]{} & \Sigma'
\end{array}
$$

such that

1. *$Mod\varphi_1$ preserves $\mathcal{F}$-products, and*

2. φ_2 *lifts* $\mathcal{B}$

Then this is a Craig Interpolation square.

Proof. We apply Thm. 9.3 by setting the semantic operators $\mathcal{U}$ and $\mathcal{V}$ as follows (we omit the signature subscripts from the notation of the operators):

- $\mathcal{U}\,\mathbb{M} = \mathcal{F}\,\mathbb{M}$, and
- $\mathcal{V}\,\mathbb{M} = (\mathcal{B}^{-1})^{+}\mathbb{M}$, where $(\mathcal{B}^{-1})^{+}$ is the transitive closure of $\mathcal{B}^{-1}$.

The hypotheses of Thm. 9.3 can be checked as follows:

1. This hypothesis is $\mathcal{U}_\Sigma;V_\Sigma;\mathcal{U}_\Sigma = \mathcal{U}_\Sigma;\mathcal{V}_\Sigma$.

   - On the one hand,

     1 $\mathcal{V}(\mathcal{U}\mathbb{M}) \subseteq \mathcal{U}(\mathcal{V}(\mathcal{U}\mathbb{M}))$ — $\mathbb{N} \subseteq \mathcal{F}\,\mathbb{N}$ (since $\{\{*\}\} \in \mathcal{F}$, a hypothesis of Birkhoff institutions).

   - On the other hand,

     2 $\mathcal{U}(\mathcal{V}(\mathcal{U}\mathbb{M})) \subseteq \mathcal{V}(\mathcal{U}(\mathcal{V}(\mathcal{U}\mathbb{M})))$ — $\mathbb{N} \subseteq \mathcal{V}\mathbb{N}$ ($\mathcal{B}$ reflexive)

     Now let us prove by induction on $n \in \omega$ that for each $\mathbb{N} \subseteq |Mod\Sigma|$

     3 $\mathcal{B}^{-n}(\mathcal{F}\,\mathbb{N}) = \mathbb{N}^{**}$

     – The base case, $n = 1$, follows by the definition of Birkhoff institutions.

     – For the induction step we do the following reasoning:

     4 $\mathcal{B}^{-(n+1)}(\mathcal{F}\,\mathbb{N}) = \mathcal{B}^{-1}(\mathcal{B}^{-n}(\mathcal{F}\,\mathbb{N})) = \mathcal{B}^{-1}\,\mathbb{N}^{**}$ — induction step

     5 $\subseteq \mathcal{B}^{-1}(\mathcal{F}\,\mathbb{N}^{**})$ — $\{\{*\}\} \in \mathcal{F}$

     6 $= \mathbb{N}^{****} = \mathbb{N}^{**}$ — Birkhoff institution, $(\_)^{**}$ idempotent (closure operator).

     Also

     7 $\mathbb{N}^{**} = \mathcal{B}^{-n}(\mathcal{F}\,\mathbb{N}$ — induction hypothesis

     8 $\subseteq \mathcal{B}^{-(n+1)}(\mathcal{F}\,\mathbb{N})$ — $\mathcal{B}$ reflexive.

     Thus 6 and 8 prove the induction step. Hence

     9 $\mathcal{V}(\mathcal{U}\mathbb{N}) = (\mathcal{B}^{-1})^{+}(\mathcal{F}\,\mathbb{M}) = \bigcup_{n\in\omega}\mathcal{B}^{-n}(\mathcal{F}\,\mathbb{N}) = \mathbb{N}^{**}$ — 3

     10 $\mathcal{V}(\mathcal{U}(\mathcal{V}(\mathcal{U}\mathbb{N}))) = \mathcal{V}(\mathcal{U}N^{**}) = \mathbb{N}^{****}$ — 9 applied twice

     11 $= \mathbb{N}^{**} = \mathcal{V}(\mathcal{U}\mathbb{N})$ — $(\_)^{**}$ idempotent (closure operator), 9

     12 $\mathcal{U}(\mathcal{V}(\mathcal{U}\mathbb{M})) \subseteq \mathcal{V}(\mathcal{U}\mathbb{M})$ — 2, 11

     Therefore condition 1. of Thm. 9.3 is fulfilled by 1 and 12.

2. $\mathcal{V}$ are closure operators the reflexivity of $\mathcal{B}$, by the transitivity of $(\mathcal{B}^{-1})^{+}$, and because $\mathcal{B}$ is closed under isomorphism.

3. Consider any $\mathbb{M}_1 \in FX\ \mathcal{U}_{\Sigma_1}$. We have to prove that $\mathbb{M}_1\!\upharpoonright_{\varphi_1} \in FX\ \mathcal{U}_{\Sigma}$. This goes as follows:

13	$\mathcal{F}(\mathbb{M}_1\!\upharpoonright_{\varphi_1}) = \mathrm{Iso}((\mathcal{F}\mathbb{M}_1)\!\upharpoonright_{\varphi_1})$	$Mod\varphi_1$ preserves $\mathcal{F}$-products
14	$= \mathrm{Iso}((\mathrm{Iso}\mathbb{M}_1)\!\upharpoonright_{\varphi_1})$	$\mathbb{M}_1 \in FX\ \mathcal{U}_{\Sigma_1}$
15	$= \mathrm{Iso}(\mathrm{Iso}(\mathbb{M}_1\!\upharpoonright_{\varphi_1})) = \mathrm{Iso}(\mathbb{M}_1\!\upharpoonright_{\varphi_1})$	$Mod\varphi_1$ preserves isomorphisms.

4. The condition to be proved is

$$(Mod\varphi_1)\ ;\ \mathrm{Iso}\ ;\ \mathcal{V}_{\Sigma} = \mathrm{Iso}\ ;\ (Mod\varphi_1)\ ;\ \mathcal{V}_{\Sigma}.$$

Let $\mathbb{N} \subseteq |Mod\Sigma_1|$. Then, on the one hand

16	$\mathcal{B}^{-1}(\mathrm{Iso}\ \mathbb{N}\!\upharpoonright_{\varphi_1}) = \mathcal{B}^{-1}\ \mathbb{N}\!\upharpoonright_{\varphi_1}$	$\mathcal{B}$ closed under isomorphisms
17	$\subseteq \mathcal{B}^{-1}\ (\mathrm{Iso}\mathbb{N})\!\upharpoonright_{\varphi_1}$	reflexivity of Iso, monotonicity of $\mathcal{B}^{-1}(\_)$

and on the other hand

18	$\mathcal{B}^{-1}\ (\mathrm{Iso}\mathbb{N})\!\upharpoonright_{\varphi_1} \subseteq \mathcal{B}^{-1}\ \mathrm{Iso}(\mathbb{N}\!\upharpoonright_{\varphi_1})$	$\mathcal{B}^{-1}(\_)$ monotone, $Mod\varphi_1$ preserves isomorphisms.

5. That $\mathcal{V}_{\Sigma}\ ;\ (Mod\varphi_2)^{-1} \subseteq (Mod\varphi_2)^{-1}\ ;\ \mathcal{V}_{\Sigma_2}$ means that φ_2 lifts $\mathcal{B}^+$ (the transitive closure of $\mathcal{B}$) which holds by the hypothesis that φ_2 lifts $\mathcal{B}$.

Now consider a set E_1 of Σ_1-sentences and E_2 a set of Σ_2-sentences such that $\theta_1 E_1 \models \theta_2 E_2$. By setting $\mathbb{M}_1 = E_1^*$ and $\mathbb{M}_2 = E_2^*$ in the statement of Thm. 9.3, according to its conclusion we obtain a semantic interpolant closed under isomorphisms $\mathbb{M} \subseteq |Mod\Sigma|$ and such that $\mathbb{M} \in (FX\ \mathcal{U}_{\Sigma}) \cap (FX\ \mathcal{V}_{\Sigma})$. This implies $\mathbb{M} = \mathcal{B}^{-1}(\mathcal{F}\mathbb{M})$ which by the Birkhoff institution property means $\mathbb{M}^{**} = \mathbb{M}$. Thus $E = \mathbb{M}^*$ is an interpolant. □

Apart from the fundamental axiomatizability framework of a Birkhoff institution, from the hypotheses of Thm. 9.7 only the lifting condition sets substantial limits to its applicability. Note that this condition is a reflection of condition 5. of the generic Thm. 9.3 to a more concrete framework. The other conditions can usually be handled as follows:

- Regarding the model amalgamation hypothesis, for interpolation squares we usually look among pushout squares of signature morphisms. Thus we can do with the basic assumption that the institution has weak model amalgamation.
- It is common that in institutions in which the signatures contain only symbols with finite arities, the filtered products of models are preserved by the model reducts corresponding to *any* signature morphism. For the case of $\mathcal{FOL}$ and related institutions this has been shown in Sect. 6.2.

The lifting condition. We now focus on the condition underlying Thm. 9.7, that φ_2 lifts $\mathcal{B}$. Below we give an emblematic example of how this condition can be solved at the level of concrete institutions.

Towards the end of Sect. 8.6 we have introduced some classes of $\mathcal{FOL}$ model homomorphisms. Let us recall them in the form of the following table.

class name	injective / surjective	model theoretic property
H_r	surjective	
H_s	surjective	strong
S_w	injective	
S_c	injective	closed

Proposition 9.8. *In $\mathcal{FOL}$, each $(ii'i)$-morphism of signatures lifts $\mathcal{B}$ for each $\mathcal{B} \in \{\stackrel{S_w}{\to}, \stackrel{S_c}{\to}, \stackrel{H_r}{\leftarrow}, \stackrel{H_s}{\leftarrow}\}$. Consequently, each $(ii'i)$-morphism of signatures lifts $\stackrel{H}{\leftarrow};\stackrel{S}{\to}$ for each $H \in \{H_r, H_s\}$ and each $S \in \{S_w, S_c\}$.*

Proof. What follows is very much a proof by construction. Let $\varphi\colon\ \Sigma \to \Sigma'$ be a $(ii'i)$-morphism of $\mathcal{FOL}$-signatures, with $\Sigma = (S,F,P)$ and $\Sigma' = (S',F',P')$ being their actual forms. We treat the H and the S cases simultaneously. Consider a Σ-model homomorphism h as follows: $h\colon\ N \to M'\upharpoonright_\varphi$ when $h \in H$ and $h\colon\ M'\upharpoonright_\varphi \to N$ when $h \in S$. In the H case we lift h to a Σ'-homomorphism $h'\colon\ N' \to M'$ while in the S case to a Σ'-homomorphism $h'\colon\ M' \to N'$. Moreover, concerning membership to one of the four classes of homomorphisms, we aim that h' must be of the same kind as h.

- For each symbol z in Σ we define $N'_{\varphi z} = N_z$. This is equivalent to $N'\upharpoonright_\varphi = N$. The definition holds because φ is injective ((iii)-morphism). Also, h' is defined by

  $$h'_{s'} = \begin{cases} h_s & s' = \varphi s \\ 1_{M'_{s'}}, & s' \notin \varphi S. \end{cases}$$

  This implies that $N'_{s'} = M'_{s'}$ whenever s' is outside the range of φ. Note that lifting of h to h' maintains the injectivity / surjectivity of h. It remains to define N' on the operation and relation symbols outside the range of φ, and this constitutes the challenging part of this proof.

- At the level of the operations, let us consider $\sigma' \in F'_{w\to s}$ but not within the range of φ.
  - In the H case, for each appropriate sequence x of arguments, by using the surjectivity of h'_s, we let $N'_{\sigma'}x \in h'^{-1}_s(M'_{\sigma'}(h'_w x))$.
  - In the S case we let

    $$N'_{\sigma'}x = \begin{cases} h'_s(M'_{\sigma'}(h'^{-1}_w x)), & x \in h'_w M'_w \\ \text{any element of } N'_s, & \text{otherwise.} \end{cases}$$

    The correctness of this definition can be justified as follows. In the former case, it follows by the injectivity of h'. In the latter case, it is about making sure that

$N'_s \neq \emptyset$. This follows from the definition of $(ii'i)$. If s is not within the range of φ then $N'_s = M'_s \neq \emptyset$ because $\sigma' \in F_{w \to s}$ is a 'new' operation symbol. If s is within the range of φ then $N'_s = N_{\varphi^{-1}s} \neq \emptyset$ by the same argument like in the previous case.

- At the level of the relations, for each $\pi' \in P'_w$ but outside the range of φ, we let $N'_{\pi'} = h'^{-1}_w M'_{\pi'}$ in the H case and $N'_{\pi'} = h'_w M'_{\pi'}$ in the S case.

We can see that under this lifting, $N' \upharpoonright_\varphi = N$, that h' is a Σ'-homomorphism and is exactly of the same kind as h concerning their membership to one of the four classes of homomorphisms. □

Some concrete interpolation consequences of Thm. 9.7. Based upon some of the axiomatizability results listed at the end of Sect. 8.6, by the interpolation Thm. 9.7 and the lifting Prop. 9.8 we have the following interpolation results:

Corollary 9.9. *The institutions $\mathcal{UNIV}$, of the universal $\mathcal{FOL}_{\infty,\omega}$-sentences, $\mathcal{HCL}$, $\mathcal{HCL}_{\infty,\omega}$, of universal $\mathcal{FOL}$-atoms, $\mathcal{EQL}$, $\forall\vee$, $\forall\vee_\infty$ have Craig $(Sig,(ii'i))$-interpolation.*

The counterexample below shows that the injectivity condition on the signature morphisms from $\mathcal{R}$ is necessary too, as in its absence interpolation may fail.

Failure of interpolation because of non-injectivity. In $\mathcal{EQL}$ consider the pushout square of signature morphisms

$$\begin{array}{ccc}
\Sigma = \{f,g\} & \xrightarrow[\subseteq]{\varphi_1} & \Sigma_1 = \{f,g,h\} \\
{\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\
\Sigma_2 = \{k\} & \xrightarrow[\theta_2]{\subseteq} & \Sigma' = \{k,h\}
\end{array}$$

such that all the signatures involved contain only one sort s and one constant a (not shown in the diagram) and only unary operations as shown in the diagram. Let

- $E_1 = \{(\forall x)gx = h(fx),\ (\forall x)f(gx) = h(gx)\}$ and
- $E_2 = \{(\forall x)k(kx) = kx\}$.

It is easy to see that $\theta_1 E_1 \models_{\Sigma'} \theta_2 E_2$ (check it!). We show that the interpolation problem defined by this consequence does not have a solution. By *Reductio ad Absurdum* let us suppose that there exists an interpolant E.

- E may contain only reflexive equations $(\forall x)t = t'$. We prove this by considering the following Σ_1-model M_1.
  - $(M_1)_s = \{t(h^n a) \mid t\ (\Sigma + x)\text{-term},\ n \in \omega\}$ where $h^0 a = a$, $h^{n+1}a) = h(h^n a)$, and $t(h^n a)$ is the Σ_1-term obtained by substituting variable x by $h^n a$.
  - $(M_1)_f t = ft$, $(M_1)_g t = gt$ for each $t \in (M_1)_s$.

–
$$(M_1)_h t = \begin{cases} gt_0, & t = ft_0 \\ f(gt_0), & t = gt_0 \\ h^{n+1}a, & t = h^n a. \end{cases}$$

Obviously, $M_1 \models E_1$. Since $E_1 \models \varphi_1 E$ (E interpolant) it follows that $M_1\upharpoonright_{\varphi_1} \models E$. If E contains an equation $(\forall x)t = t'$ such that t and t' are different terms, then $M_1\upharpoonright_{\varphi_1} \models t(a) = t'(a)$. However this is not possible as $t(a)$ and $t'(a)$, being just their own interpretation in M_1, are different elements in M_1 (and in $M_1\upharpoonright_{\varphi_1}$ too, as $M_1\upharpoonright_{\varphi_1}$ just forgets the interpretation of h).

- Hence E is a trivial theory, which implies that $\varphi_2 E$ is a trivial theory. Then $\varphi_2 E \not\models E_2$.

As a side remark aimed at people who are familiar with algebraic specification and term rewriting we may note the following. M_1 is an initial model of E_1 as it is the model of the normal forms of E_1 turned into a rewriting system $\{h(fx) \to gx,\ h(gx) \to f(gx)\}$. This is terminating because each application of one of the two rules decreases the number of the hs in a term. It is also confluent because it is non-lapsing and orthogonal (left-linear and non-overlapping). For the connaisseurs, in this case, these properties are straightforward to check.

Interpolation by the Keisler-Shelah property. In situations when $\mathcal{B}$ is rather weakly defined, the lifting condition can be rather difficult to establish. The cost is thus shifted from the axiomatizability property to the lifting condition on φ_2. A typical example is given by $\mathcal{FOL}$, regarded as a Birkhoff institution with $\mathcal{B}$ being the elementary equivalence relation $\equiv$, and $\mathcal{F}$ the class of all ultrafilters (cf. Thm. 8.32). A solution to this problem is given by the Keisler-Shelah property (cf. Cor. 7.25) via Thm. 8.32 which says that a class of $\mathcal{FOL}$-models is elementary if and only if it is closed under ultraproducts and ultraradicals. This provides a characterization of elementary equivalence $\equiv$ strong enough to support an easy applicability of the interpolation Thm. 9.7. But before doing this let us consider the following simple fact. We invite the reader to be convinced of this fact by himself.

Fact 9.10. *A $\mathcal{FOL}$ signature morphism lifts isomorphisms if and only if it is $(i**)$.*

Corollary 9.11. *$\mathcal{FOL}$ has Craig $(Sig^{\mathcal{FOL}}, (i**))$-interpolation.*

Proof. We use the Birkhoff axiomatizability characterization of elementary classes in $\mathcal{FOL}$,

$$\mathbb{M}^{**} = Ur^{-1}(Up\mathbb{M})$$

given by Thm. 8.32. Let us show that each $\mathcal{FOL}$ signature morphism φ which lifts isomorphisms, also lifts the ultraradical relation Ur. If we did this then the conclusion followed by the virtue of Fact 9.10. Here we go:

1 $(M'\upharpoonright_{\varphi})_U = N$ for some ultrafilter U

2 $(M'_U)\upharpoonright_{\varphi} \cong (M'\upharpoonright_{\varphi})_U = N$ — $\mathcal{FOL}$-signature morphisms preserve filtered products (Sec. 6.2), 1

3	$M'_U \cong N'$ for some N' such that $N' \upharpoonright_{\varphi} = N$	2, φ lifts isomorphisms
4	$M' (Ur) N'$	3.

□

The 'left' interpolation theorem

The second general interpolation theorem that relies on Birkhoff axiomatizability is presented below. It shifts the reliance upon the lifting property of the signature morphisms from those on the 'right-hand side' to those on the 'left-hand side' of the interpolation squares of signature morphisms. One consequence of this is that the lifting condition on $\mathcal{B}$ rather becomes a lifting condition on its inverse $\mathcal{B}^{-1}$.

Theorem 9.12. *Consider a Birkhoff institution $(Sig, Sen, Mod, \models, \mathcal{F}, \mathcal{B})$ and a weak model amalgamation square of signature morphisms:*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

such that

1. *$Mod\varphi_1$ preserves $\mathcal{F}$-products, and*
2. *φ_1 lifts $\mathcal{B}^{-1}$ and isomorphisms.*

Then this is a Craig Interpolation square.

Proof. We apply the abstract semantic interpolation Thm. 9.3 by setting the semantic operators $\mathcal{U}$ and $\mathcal{V}$ as follows:

- $\mathcal{U}$ are the elementary closure operators, i.e., $\mathcal{U}\mathbb{M} = \mathbb{M}^{**}$, and
- $\mathcal{V}$ are the identities operators.

Because the hypotheses 1,2 and 5 of Thm. 9.3 are trivial to check, we focus on the remaining two ones.

3. Let $\mathbb{M}_1 \in FX\ \mathcal{U}_{\Sigma_1}$ which means that $\mathbb{M}_1^{**} = \mathrm{Iso}\mathbb{M}_1$. We have to show that $(\mathbb{M}_1 \upharpoonright_{\varphi_1})^{**} = \mathrm{Iso}(\mathbb{M}_1 \upharpoonright_{\varphi_1})$. We have the following sequence of relations:

$$\begin{aligned}
(\mathbb{M}_1 \upharpoonright_{\varphi_1})^{**} &= \mathcal{B}_\Sigma^{-1}(\mathcal{F}(\mathbb{M}_1 \upharpoonright_{\varphi_1})) && \text{Birkhoff institution} \\
&= \mathcal{B}_\Sigma^{-1}(\mathcal{F}\,\mathrm{Iso}(\mathbb{M}_1 \upharpoonright_{\varphi_1})) && \text{filtered products are defined up to isomorphisms} \\
&= \mathcal{B}_\Sigma^{-1}(\mathcal{F}((\mathrm{Iso}\mathbb{M}_1) \upharpoonright_{\varphi_1})) && \varphi_1 \text{ lifts isomorphisms} \\
&= \mathcal{B}_\Sigma^{-1}(\mathcal{F}(\mathbb{M}_1^{**} \upharpoonright_{\varphi_1})) && \mathbb{M}_1 \in FX\ \mathcal{U}_{\Sigma_1} \\
&= \mathcal{B}_\Sigma^{-1}(\mathrm{Iso}((\mathcal{F}\mathbb{M}_1^{**}) \upharpoonright_{\varphi_1})) && Mod(\varphi_1) \text{ preserves } \mathcal{F}\text{-products}
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{B}_{\Sigma}^{-1}((\mathcal{F}\mathbb{M}_1^{**})\restriction_{\varphi_1}) && \mathcal{B} \text{ closed under isomorphisms}\\
&\subseteq \mathcal{B}_{\Sigma}^{-1}((\mathcal{B}_{\Sigma_1}^{-1}(\mathcal{F}\mathbb{M}_1^{**}))\restriction_{\varphi_1}) && \mathcal{B}\text{'s reflexive}\\
&= \mathcal{B}_{\Sigma}^{-1}(\mathbb{M}_1^{****}\restriction_{\varphi_1}) && \text{Birkhoff institution}\\
&= \mathcal{B}_{\Sigma}^{-1}(\mathbb{M}_1^{**}\restriction_{\varphi_1}) && (\_)^{**} \text{ closure operator}\\
&\subseteq (\mathcal{B}_{\Sigma_1}^{-1}(\mathbb{M}_1^{**}))\restriction_{\varphi_1} && \varphi_1 \text{ lifts } \mathcal{B}^{-1}\\
&\subseteq (\mathcal{B}_{\Sigma_1}^{-1}(\mathcal{F}\mathbb{M}_1^{**}))\restriction_{\varphi_1} && \mathbb{N} \subseteq \mathcal{F}\mathbb{N} \text{ because } \{\{*\}\} \in \mathcal{F}\\
&= \mathbb{M}_1^{****}\restriction_{\varphi_1} && \text{Birkhoff institution}\\
&= \mathbb{M}_1^{**}\restriction_{\varphi_1} && (\_)^{**} \text{ closure operator}\\
&= (\mathrm{Iso}\mathbb{M}_1)\restriction_{\varphi_1} && \mathbb{M}_1 \in FX\,\mathcal{U}_{\Sigma_1}\\
&\subseteq \mathrm{Iso}(\mathbb{M}_1\restriction_{\varphi_1}) && Mod\varphi_1 \text{ preserves isomorphisms.}
\end{aligned}
$$

Since we work only with institutions closed under isomorphisms, we also have that $\mathrm{Iso}(\mathbb{M}_1\restriction_{\varphi_1}) \subseteq (\mathbb{M}_1\restriction_{\varphi_1})^{**}$, hence $(\mathbb{M}_1\restriction_{\varphi_1})^{**} = \mathrm{Iso}(\mathbb{M}_1\restriction_{\varphi_1})$.

4. This condition holds because $(Mod\varphi_1);\mathrm{Iso}_{\Sigma} = \mathrm{Iso}_{\Sigma_1};(Mod\varphi_1)$ which is another way of expressing that φ_1 lifts isomorphisms.

The conclusion of Thm. 9.3 tells us that for any sets E_1 of Σ_1-sentences and E_2 of Σ_2-sentences such that $\theta_1 E_1 \models \theta_2 E_2$ there exists a semantic interpolant $\mathbb{M}$ closed under isomorphisms and such that $\mathbb{M}^{**} = \mathrm{Iso}\mathbb{M}$. By the closure of $\mathbb{M}$ under isomorphisms this means that $\mathbb{M}^{**} = \mathbb{M}$, hence $\mathbb{M}$ is elementary. By the Principle of Semantic Interpolation, $E = \mathbb{M}^*$ is an interpolant for E_1 and E_2. □

For obtaining concrete instances of the general 'left' interpolation Thm. 9.12 we follow the same path as in the previous 'right' interpolation Thm. 9.7. Therefore we have to establish classes of signature morphisms that lift various concrete relations that can be used in building various operators $\mathcal{B}$.

The lifting condition. The following result establishes lifting of the inverses of the relations considered by Prop. 9.8.

Proposition 9.13. *In $\mathcal{FOL}$, each $(ie*)$-morphism of signatures lifts $\mathcal{B}^{-1}$ for each $\mathcal{B} \in \{\stackrel{S_w}{\to}, \stackrel{S_c}{\to}, \stackrel{H_s}{\leftarrow}\}$ and each (iei)-morphism lifts $\stackrel{H_r}{\to}$. Consequently, for each $S \in \{S_w, S_c\}$, each $(ie*)$-morphism of signatures lifts $\stackrel{S}{\leftarrow};\stackrel{H_s}{\to}$ and each (iei)-morphism lifts $\stackrel{S}{\leftarrow};\stackrel{H_r}{\to}$.*

Proof. We will develop a proof by construction following similar steps like in the proof of Prop. 9.8. Let $\varphi\colon\ \Sigma \to \Sigma'$ be an $(ie*)$-morphism of $\mathcal{FOL}$-signatures, with $\Sigma = (S,F,P)$ and $\Sigma' = (S',F',P')$ being their actual forms. Like in the proof of Prop. 9.8 we treat the H and the S cases simultaneously. Consider a Σ-model homomorphism h as follows: $h\colon\ N \to M'\restriction_{\varphi}$ when $h \in S$ and $h\colon\ M'\restriction_{\varphi} \to N$ when $h \in H$. Note that this is inverse to how we considered h in the proof of Prop. 9.8; this is so because now we lift $\mathcal{B}^{-1}$ rather than $\mathcal{B}$.

- The definition of N' for the symbols φz, z in Σ, and of h' is almost the same like in the proof of Prop. 9.8, the aim being to have $N'\upharpoonright_\varphi = N$ and $h'\upharpoonright_\varphi = h$. The only difference is that in the H case, when $s' \notin \varphi S$, we let

$$N'_{s'} = \begin{cases} \{*\} \text{ (a singleton set)}, & M'_{s'} \neq \emptyset \\ \emptyset, & \text{otherwise.} \end{cases}$$

  In the S case it is like in the proof of Prop. 9.8, i.e. $h'_{s'} = 1_{M'_{s'}}$.

- For the operations $\sigma' \in F'_{w \to s}$ but not within the range of φ:
  - In the S case we define $N'_{\sigma'}x = h'^{-1}_s(M'_{\sigma'}(h'_w x))$. The encapsulation condition guarantees that $N'_{\sigma'}x$ exists when $s \in \varphi S$. When $s \notin \varphi S$, $N'_{\sigma'}x = M'_{\sigma'}(h'_w x)$ since h'_s is identity. The injectivity of h' implies the uniqueness of $N'_{\sigma'}x$.
  - In the H case
$$N'_{\sigma'}x = \begin{cases} N_\sigma x, & \sigma' = \varphi\sigma \\ *, & \sigma' \notin \varphi F. \end{cases}$$
    The encapsulation condition guarantees that we can define $N'_{\sigma'}$ in a correct way and such that h' satisfies the homomorphism condition on the operations. In its absence , for $s \in \varphi S$ the homomorphism condition $h'_s(M'_{\sigma'}x) = N'_{\sigma'}(h'_w x)$ could not be satisfied in situations when $h'(M'_{\sigma'}x) \neq h'_s(M'_{\sigma'}y)$ but $h'_w x = h'_w y$. On the other hand, in the presence of the encapsulation condition this is not an issue because σ' is an operation originating from Σ and we can say that all things regarding σ' happen in fact at the level of Σ. This includes the definition of $N'_{\sigma'}$ and the homomorphism property.

- For the relation symbols $\pi' \in P'_w$ outside the range of φ we define $N'_{\pi'} = h'^{-1}_w M'_{\pi'}$ in the S case and $N'_{\pi'} = h'_w M'_w$ in the H case.

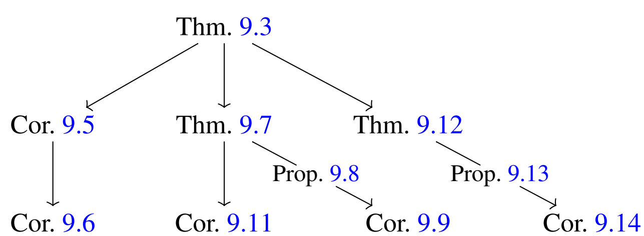
□

Some concrete interpolation consequences of Thm. 9.12. Based upon some of the axiomatizability results listed at the end of Sect. 8.6, by the interpolation Thm. 9.12 and the lifting Prop. 9.13, and also because each $(i**)$-morphism of signatures lifts isomorphisms of $\mathcal{FOL}$-models (cf. Fact 9.10) we have the following concrete interpolation results:

Corollary 9.14. *The institutions below have Craig $(\mathcal{L}, Sig)$-interpolation as indicated in the table below:*

institution	$\mathcal{L}$
$\mathcal{UNIV}$ universal $\mathcal{FOL}_{\infty,\omega}$-sentences $\mathcal{HCL}$, $\mathcal{HCL}_{\infty,\omega}$, $\forall\vee$, and $\forall\vee_\infty$	*ie∗*
universal $\mathcal{FOL}$-atoms	*iei*
$\mathcal{EQL}$	*ie*

Three abstraction levels of interpolation in retrospective. All interpolation results in this section have been derived from the highly abstract semantic interpolation result of Thm. 9.3. This process involved the development of interpolation results on three levels of abstraction, in a top-down movement from abstract to concrete. The top level is that of the abstract semantic interpolation. Below it there is the level of general proper interpolation. At the bottom level sit the interpolation results in concrete institutions. This development process is emblematic for the institution-theoretic development of model theory in general and we can visualise it in the form of the following tree.



Exercises

9.4. Interpolation in $\mathcal{PL}$
In propositional logic ($\mathcal{PL}$) each pushout square of signatures is a Ci square. (*Hint:* $\mathcal{PL}$ is a Birkhoff institution with $\mathcal{F}$ the class of all ultrafilters and $\mathcal{B}$ the identity relation.)

9.5. Interpolation for partial algebra
By the general axiomatizability results of this section, through the axiomatizability results for partial algebras of the Exercises 8.3, 8.9 and 8.11, formulate and prove interpolation results for partial algebras.

9.6. Given a weakly semi-exact institution I, let $\mathcal{C}$ be the class of the signature morphisms φ that admit the model expansion property and for which $Sen^I(\varphi)$ is surjective. Then I has both the Craig $(\mathcal{C}, Sig^I)$ and $(Sig^I, \mathcal{C})$-interpolation properties.

9.4 Interpolation by consistency

In the early age of model theory, first-order Craig interpolation was obtained in several different ways. One prominent way was to derive it from a consistency property of first-order logic, known as 'Robinson consistency'. In this section we generalise this method to the institution-independent level. First we develop the definition of institution-independent Robinson consistency and establish its mutual causality relationship with Craig interpolation. Then we develop a general Robinson consistency theorem at the level of abstract institutions. This relies on some properties of the respective institution that are essentially those that give $\mathcal{FOL}$ its character, such as admitting all Boolean connectives and a kind of quantification that in the applications amounts to first-order quantification. This means that the range of the applicability of the results of this section is confined to $\mathcal{FOL}$-like institutions, a situation very different from the developments of Sec. 9.3. Moreover, this difference is not only about applicability. While the results of Sec. 9.3 have only little correspondence to developments outside institution theory, what will follow in this section represent just a high generalisation of well known results from first-order model theory.

Robinson consistency. Recall that a Σ-theory E in an arbitrary institution is *consistent* if $E^* \neq \emptyset$. A commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is a *Robinson Consistency square* (abbreviated *RC* square) if and only if all theories $E_i \subseteq Sen\Sigma_i$, $i \in \{1,2\}$, with 'inter-consistent reducts', i.e., $(Sen\varphi_1)^{-1}E_1^{**} \cup (Sen\varphi_2)^{-1}E_2^{**}$ is consistent, have 'inter-consistent Σ'-translations', i.e., $\theta_1 E_1 \cup \theta_2 E_2$ is consistent.

Note that Robinson consistency has substance in institutions where consistency is not a trivial property in the sense of being a property of each theory. For instance in $\mathcal{HCL}$ and $\mathcal{EQL}$, according to Cor. 4.28, any theory is consistent.

Quasi-compactness. The method to obtain interpolation that is put forward in this section relies on compactness in several diferent ways. This is unlike the interpolation results previously developed in this chapter. However in all results of this section (and also of Sec. 10) compactness can be technically replaced with the existence of infinite conjunctions. Note that in general these two properties are mutually exclusive. We can take advantage of this technical situation and widen significantly the applicability of the results to institutions admitting infinite conjunctions but lacking compactness. We say that an institution is *quasi-compact* when it is compact or has infinite conjunctions.

Robinson consistency versus Craig interpolation

The method to obtain Craig interpolation by Robinson consistency relies on a result about the latter implying the former.

Theorem 9.15. *In any quasi-compact institution with negations and conjunctions, each RC square is a Ci square.*

Proof. For each sentence e let $\neg e$ denote any negation of e. For each Γ finite set of sentences let $\wedge\Gamma$ denote any conjunction of the sentences in Γ. Consider $E_i \subseteq Sen\Sigma_i$, $i = 1,2$, such that $\theta_1 E_1 \models \theta_2 E_2$. In this proof we will use tacitly several general properties of the semantic consequence relation $\models$, such as those from Prop. 3.7 or others that can be derived from those. Also, we will rely on the general preservation of the semantics of negation and conjunctions by translation along signature morphisms, properties that follow easily from the Satisfaction Condition. For each $e_2 \in E_2$ we have that:

1 $\theta_1 E_1 \cup \theta_2 \neg e_2$ inconsistent $\qquad \theta_1 E_1 \models \theta_2 e_2,\ \neg\theta_2 e_2 \models\!\!| \ \theta_2 \neg e_2$

2	$(Sen\varphi_1)^{-1}E_1^{**} \,\cup\, (Sen\varphi_2)^{-1}(\neg e_2)^{**}$ inconsistent	1, RC property
3	there exists $\Gamma_1(e_2) \subseteq (Sen\varphi_1)^{-1}E_1^{**}$, $\Gamma_2(e_2) \subseteq (Sen\varphi_2)^{-1}(\neg e_2)^{**}$ *finite* such that $\Gamma_1(e_2) \cup \Gamma_2(e_2)$ inconsistent	2, quasi-compactness
4	$\Gamma_1(e_2) \models \neg \wedge \Gamma_2(e_2)$	3
5	$E_1 \models \varphi_1\ \Gamma_1(e_2)$	$\Gamma_1(e_2) \subseteq (Sen\varphi_1)^{-1}E_1^{**}$
6	$\neg e_2 \models \varphi_2\ \Gamma_2(e_2)$	$\Gamma_2(e_2) \subseteq (Sen\varphi_2)^{-1}(\neg e)_2^{**}$
7	$\varphi_2 \neg \wedge \Gamma_2(e_2) \models e_2$	6, $\varphi_2(\neg \wedge \Gamma_2(e_2)) \models\!\mid \neg \wedge \varphi_2 \Gamma_2(e_2)$
8	$\varphi_2\ \Gamma_1(e_2) \models e_2$	4, 7
9	$\varphi_2 \bigcup_{e_2 \in E_2} \Gamma_1(e_2) \ \models\ E_2$	8
10	$E_1 \models \varphi_1 \bigcup_{e_2 \in E_2} \Gamma_1(e_2)$	5.

The relations 9 and 10 show that $\bigcup_{e_2 \in E_2} \Gamma_1$ is an interpolant for E_1 and E_2. □

Under the same conditions as those of Thm. 9.15, its reverse holds too. From the perspective of obtaining interpolation, this does not have any value. However besides of its theoretical value, very importantly, it can be used to achieve symmetry of interpolation. We will see how this works after the following theorem.

Theorem 9.16. *In any quasi-compact institution with negations and conjunctions, each Ci square is an RC square.*

Proof. For each sentence e let $\neg e$ denote any negation of e. For each Γ finite set of sentences let $\wedge\Gamma$ denote the conjunction of the sentences in Γ. We prove that for any theories E_1, E_2 such that $\theta_1 E_1 \cup \theta_2 E_2$ is inconsistent, $(Sen\varphi_1)^{-1}E_1^{**} \,\cup\, (Sen\varphi_2)^{-1}E_2^{**}$ is inconsistent too. We have that

1	$\exists$ *finite* $\Gamma_2 \subseteq E_2$ s.th. $\theta_1 E_1 \cup \theta_2 \Gamma_2$ inconsistent	$\theta_1 E_1 \cup \theta_2 E_2$ inconsistent, quasi-compactness
2	$\theta_1 E_1 \models \theta_2(\neg \wedge \Gamma_2)$	1, $\theta_2(\neg \wedge \Gamma_2) \models\!\mid \neg \wedge \theta_2 \Gamma_2$

Let E be an interpolant for E_1 and $\neg \wedge \Gamma_2$. By quasi-compactness we may assume that E is finite. Then

3	$E_1 \models \varphi_1 E$	E interpolant
4	$E \subseteq (Sen\varphi_1)^{-1}E_1^{**}$	3
5	$\varphi_2 E \models \neg \wedge \Gamma_2$	E interpolant
6	$\Gamma_2 \models \varphi_2(\neg \wedge E)$	5, $\varphi_2(\neg \wedge E) \models\!\mid \neg \wedge \varphi_2 E$
7	$E_2 \models \varphi_2(\neg \wedge E)$	6, $\Gamma_2 \subseteq E_2$
8	$\neg \wedge E \in (Sen\varphi_2)^{-1}E_2^{**}$	7.

From 4 and 8 we deduce that any model of $(Sen\varphi_1)^{-1}E_1^{**} \,\cup\, (Sen\varphi_2)^{-1}E_2^{**}$ satisfies both E and $\neg \wedge E$. Such model cannot exist, hence $(Sen\varphi_1)^{-1}E_1^{**} \,\cup\, (Sen\varphi_2)^{-1}E_2^{**}$ is inconsistent. □

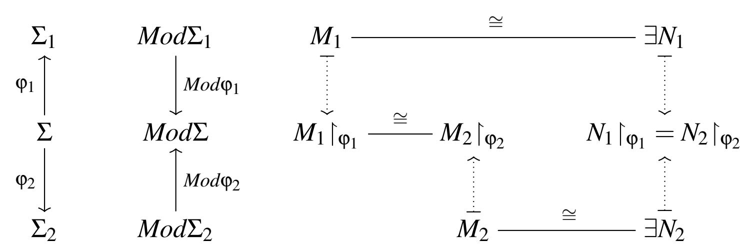
Ci is generally an asymmetric property with respect to the reflection in the mirror of the considered squares of signature morphisms (we have seen lot of that in Sec. 9.3), while RC is a symmetric property. The equivalence between Ci and RC given by Thm. 9.15 brings the symmetry of RC to Ci. This may allow for the extension of Ci properties of institutions. The following is an example of such an extension of the interpolation property of $\mathcal{FOL}$ formulated by Cor. 9.11.

Corollary 9.17. *$\mathcal{FOL}$ has both Craig $(Sig^{\mathcal{FOL}},(i{*}{*}))$ and $((i{*}{*}),Sig^{\mathcal{FOL}})$-interpolation.*

Robinson consistency theorem

The next step is to develop an institution-independent RC theorem. The following concept of lifting of isomorphisms will be needed.

Lifting isomorphisms. A span of signature morphisms $\Sigma_1 \xleftarrow{\varphi_1} \Sigma \xrightarrow{\varphi_2} \Sigma_2$ is said to *lift isomorphisms* if for any Σ_i-models M_i, $i = 1,2$, such that $M_1\upharpoonright_{\varphi_1} \cong M_2\upharpoonright_{\varphi_2}$ then there exists Σ_i-models N_i, $i = 1,2$, such that $M_i \cong N_i$, $i = 1,2$, and $N_1\upharpoonright_{\varphi_1} = N_2\upharpoonright_{\varphi_2}$.



At the beginning of Sec. 9.3 we already introduced a concept of lifting relations by single signature morphisms, which can be applied to the isomorphism relations $\cong$ between models too. On the one hand, lifting isomorphisms by single signature morphisms is a particular case of lifting isomorphisms by spans by taking the identity as the other morphism of the span. On the other hand, given a span, if any of the two constituent morphisms lifts isomorphisms then the span lifts isomorphisms. How this works can be seen in the proof of Prop. 9.18 below.

Proposition 9.18. *A span of signature morphisms $\Sigma_1 \xleftarrow{\varphi_1} \Sigma \xrightarrow{\varphi_2} \Sigma_2$ lifts isomorphisms if either φ_1 or φ_2 lifts isomorphisms.*

Proof. Consider Σ_i-models M_i, $i = 1,2$, such that $M_1\upharpoonright_{\varphi_1} \cong M_2\upharpoonright_{\varphi_2}$. We assuem that φ_1 lifts isomorphisms. Then there exists a Σ_1-model N_1 such that $N_1 \cong M_1$ and $N_1\upharpoonright_{\varphi_1} = M_2\upharpoonright_{\varphi_1}$. Now, in the definition of lifting isomorphisms by spans, we take $N_2 = M_2$. □

From Fact 9.10 and Prop. 9.18 we have the following concrete situation.

Corollary 9.19. *A span of $\mathcal{FOL}$ signature morphisms $\Sigma_1 \xleftarrow{\varphi_1} \Sigma \xrightarrow{\varphi_2} \Sigma_2$ lifts isomorphisms when either φ_1 or φ_2 is an $(i{*}{*})$-morphism.*

Theorem 9.20 (Robinson consistency)**.** *Consider any institution with diagrams ι such that*

1. *$M^* \subseteq N^*$ if there exists a model homomorphism $M \to N$,*
2. *it has pushouts of signatures and has weak model amalgamation,*
3. *it has universal χ-quantification for χ signature morphisms of the forms $\iota_\Sigma h$ and $\iota_\Sigma M$ for all Σ-model homomorphisms $h : M \to N$,*
4. *it has negations and conjunctions,*
5. *it has ω-co-limits*[1] *of models and moreover these are preserved by the model reduct functors, and*
6. *it is quasi-compact.*

Then any weak amalgamation square of signature morphisms like below

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

such that the span φ_1, φ_2 lifts isomorphisms, is a Robinson Consistency square.

Proof. Let $E_i \subseteq Sen\Sigma_i$, $i = 1,2$, be theories. Denote $\Gamma_i = (Sen\varphi_i)^{-1}E_i^{**}$; they are closed theories. Assume $\Gamma_1 \cup \Gamma_2$ is consistent. We have to prove that $\theta_1 E_1 \,\cup\, \theta_2 E_2$ is consistent too. It suffices to find Σ_i-models $N_i \models E_i$, $i = 1,2$, such that $N_1\!\upharpoonright_{\varphi_1} = N_2\!\upharpoonright_{\varphi_2}$, and then we apply weak amalgamation to find the desired Σ'-model N' of $\theta_1 E_1 \,\cup\, \theta_2 E_2$.

In brief, the way to achieve this is to get the N_i, $i = 1,2$, as liftings of reducts $M_1\!\upharpoonright_{\varphi_1} \cong M_2\!\upharpoonright_{\varphi_2}$ where M_i are obtained as co-limits of ω-chains in $Mod(\Sigma_i, E_i)$. The crucial isomorphism $M_1\!\upharpoonright_{\varphi_1} \cong M_2\!\upharpoonright_{\varphi_2}$ is obtained by constructing the two ω-chains in such a way that when reduced to $Mod\Sigma$ they are both final sub-diagrams of the same larger diagram. Let us provide detail to this process.

- Let $(f_i^n : A_i^n \to A_i^{n+1})_{n\in\omega}$, $i = 1,2$, denote the two ω-chains mentioned above. Their reducts to $Mod\Sigma$ are linked together by families of Σ-homomorphisms $(g^n)_{n\in\omega}$ and $(h^n)_{n\in\omega}$ as shown by the commutative diagram below:

$$\begin{array}{ccccccccc} A_1^0\!\upharpoonright_{\varphi_1} & & \xrightarrow{f_1^0\upharpoonright_{\varphi_1}} & & A_1^1\!\upharpoonright_{\varphi_1} & & \xrightarrow{f_1^1\upharpoonright_{\varphi_1}} & & A_1^2\!\upharpoonright_{\varphi_1} \quad \cdots \\ & {\scriptstyle g^0}\searrow & & \nearrow{\scriptstyle h^0} & & {\scriptstyle g^1}\searrow & & \nearrow{\scriptstyle h^1} & \quad {\scriptstyle g^2}\searrow \\ & & A_2^0\!\upharpoonright_{\varphi_2} & \xrightarrow[f_2^0\upharpoonright_{\varphi_2}]{} & & & A_2^1\!\upharpoonright_{\varphi_2} & \xrightarrow[f_2^1\upharpoonright_{\varphi_2}]{} & \quad A_2^2\!\upharpoonright_{\varphi_2} \cdots \end{array} \tag{9.8}$$

  We will also need that $A_1^0 \models E_1 \cup \varphi_1\Gamma_2$ and $A_2^0 \models E_2$. At this stage let us just assume the diagram (9.8) and continue with the details of the proof of the theorem.

[1] Here ω is the totally ordered set of the natural numbers.

- Because the model reduct functors preserve ω co-limits, the co-limits of $(f_i^n)_{n\in\omega}$, $i = 1,2$, in $Mod\Sigma_i$ (with vertices denoted as M_i) are mapped by $Mod\varphi_i$ to co-limits in $Mod\Sigma$.
- Since both $(f_i^n\restriction_{\varphi_i})_{n\in\omega}$, $i = 1,2$, are final sub-diagrams of (9.8), it follows (by Thm. 2.4) that $M_1\restriction_{\varphi_1} \cong M_2\restriction_{\varphi_2}$ (and isomorphic to the vertex of the co-limit of (9.8)).
- Because satisfaction is preserved along model homomorphisms (condition 1. of the theorem) and $A_i^0 \models E_i$, $i = 1,2$, we have that $M_i \models E_i$, $i = 1,2$.
- Because the span φ_1, φ_2 lifts isomorphisms there are N_i, $i = 1,2$, such that $N_i \cong M_i$ and $N_1\restriction_{\varphi_1} = N_2\restriction_{\varphi_2}$. Moreover $N_i \models E_i$ because $M_i \models E_i$ and isomorphisms preserve satisfaction.

The theorem is thus proved modulo the construction of the two ω-chains and of the families g and h such that the diagram (9.8) commutes. The remaining part of this proof is devoted entirely to the developments of this construction by an inductive process that consists of four steps as follows.

0.1 At the first base case we initialise the chain f_1 by taking $A_1^0 \in (E_1 \cup \varphi_1\Gamma_2)^*$. This is possible only if $E_1 \cup \varphi_1\Gamma_2$ is consistent. By *Reduction ad Absurdum* suppose that it is inconsistent. This implies that:

1	there exists $\Gamma_2' \subseteq \Gamma_2$ finite s.th. $E_1 \cup \varphi_1\Gamma_2'$ inconsistent	quasi-compactness
2	$E_1 \cup \varphi_1(\wedge\Gamma_2')$ inconsistent	institution has conjunctions
3	$E_1 \models \varphi_1(\neg\wedge\Gamma_2')$	institution has also negations, 2
4	$\neg\wedge\Gamma_2' \in \Gamma_1$	3, definition of Γ_1.

Since Γ_2 is a closed theory and $\Gamma_2' \subseteq \Gamma_2$, we have that $\wedge\Gamma_2' \in \Gamma_2$, which together with relation 4 contradicts that $\Gamma_1 \cup \Gamma_2$ is consistent. Therefore we can take $A_1^0 \in (E_1 \cup \varphi_1\Gamma_2)^*$.

0.2 The second base case represents the initialisation of the chain f_2 by setting a Σ_2-model $A_2^0 \in E_2^*$ and a Σ-homomorphism $g^0 : A_1^0\restriction_{\varphi_1} \to A_2^0\restriction_{\varphi_2}$. Let us abbreviate $A_1^0\restriction_{\varphi_1}$ by B^0. We consider a pushout square of signature morphisms as follows:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\iota_\Sigma B^0} & \Sigma_{B^0} \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle u} \\ \Sigma_2 & \xrightarrow[v]{} & \bullet \end{array} \tag{9.9}$$

If $uE_{B^0} \cup vE_2$ were consistent then we let B be one of its models. Then we can set $A_2^0 = B\restriction_v$ and $g^0 = i_{\Sigma,B^0}\ B\restriction_u$ (as by the Satisfaction Condition $B\restriction_u \models E_{B^0}$). Thus it remains to show the consistency of $uE_{B^0} \cup vE_2$. We do this by *Reductio ad Absurdum*, so we suppose it is inconsistent. We have that:

5 $\exists\, e \in Sen\Sigma_{B^0}$ s.th. $E_{B^0} \models e$, $vE_2 \models u(\neg e)$ — quasi-compactness, conjunctions, negations

Let $e' \in Sen\Sigma$ be any universal $\iota_\Sigma B^0$-quantification of $\neg e$. We prove that:

6 $E_2 \models \varphi_2 e'$.

Let $M_2 \in E_2^*$. By the Satisfaction Condition $M_2 \models \varphi_2 e'$ is equivalent to $M_2\!\upharpoonright_{\varphi_2} \models e'$. For *any* $\iota_\Sigma B^0$-expansion M^0 of $M_2\!\upharpoonright_{\varphi_2}$, by the weak model amalgamation of the square (9.9), there exists a model M such that $M\!\upharpoonright_u = M^0$ and $M\!\upharpoonright_v = M_2$. Then:

7 $M \models vE_2$ — $M\!\upharpoonright_v = M_2$, $M_2 \models E_2$, Satisfaction Condition

8 $M \models u(\neg e)$ — 7, $vE_2 \models u(\neg e)$ (5)

9 $M^0 \models \neg e$ — $M^0 = M\!\upharpoonright_u$, 8, Satisfaction Condition.

Hence $M_2\!\upharpoonright_{\varphi_2} \models e'$ and thus 6 is proved. We continue as follows.

10 $e' \in \Gamma_2$ — 6, $\Gamma_2 = (Sen\varphi_2)^{-1}E_2$

11 $B^0 \models e'$ — 10, $B^0 \models \Gamma_2$ ($A_1^0 \models \varphi_1\Gamma_2$, Satisfaction Condition)

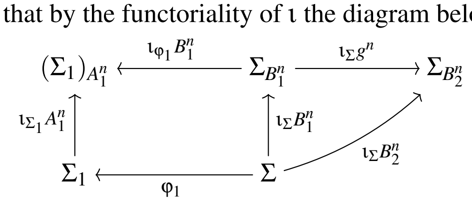
12 $(B^0)_{B^0} \models \neg e$ — 11, e' universal $\iota_\Sigma B^0$-quantification of $\neg e$

13 $(B^0)_{B^0} \models e$ — $E_{B^0} \models e$ (5), $(B^0)_{B^0} \models E_{B^0}$ (as initial model of the diagram of B^0).

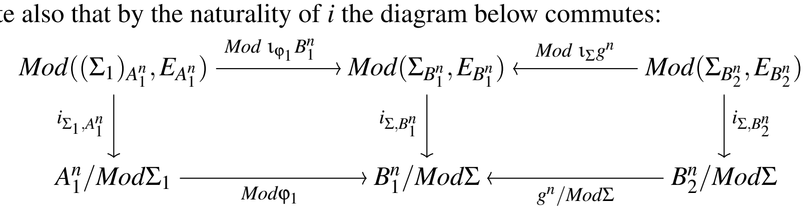
The relations 12 and 13 altogether represent a contradiction, hence $uE_{B^0} \cup vE_2$ is consistent. Consequently the model B exists, which gives A_2^0 and g^0.

n.1 Now we find f_1^n and h^n such that $f_1^n\!\upharpoonright_{\varphi_1} = g^n;h^n$. In order to simplify formulas we denote $A_i^n\!\upharpoonright_{\varphi_i}$ by B_i^n, $i = 1,2$. We first show that it suffices to find a $((\Sigma_1)_{A_1^n}, E_{A_1^n})$-model F_1^n and a $(\Sigma_{B_2^n}, E_{B_2^n})$-model H^n such that $F_1^n\!\upharpoonright_{\iota_{\varphi_1} B_1^n} = H^n\!\upharpoonright_{\iota_\Sigma g^n}$.

- By assuming that F_1^n and H^n exist, we define $f_1^n = i_{\Sigma_1, A_1^n} F_1^n$ and $h^n = i_{\Sigma, B_2^n} H^n$ and let us prove that $f_1^n\!\upharpoonright_{\varphi_1} = g^n;h^n$.
- Note that by the functoriality of ι the diagram below commutes:



- Note also that by the naturality of i the diagram below commutes:



– Therefore, by following the naturality of i in the diagram above

$$f_1^n\restriction_{\varphi_1} = (i_{\Sigma_1,A_1^n}F_1^n)\restriction_{\varphi_1} = i_{\Sigma,B_1^n}(F_1^n\restriction_{\iota_{\varphi_1}B_1^n}) \quad \text{definition of } f_1^n\text{, naturality of } i$$
$$= i_{\Sigma,B_1^n}(H^n\restriction_{\iota_\Sigma g^n}) \quad F_1^n\restriction_{\iota_{\varphi_1}B_1^n} = H^n\restriction_{\iota_\Sigma g^n}$$
$$= g^n;(i_{\Sigma,B_2^n}H^n) = g^n;h^n \quad \text{naturality of } i\text{, definition of } h^n.$$

It remained to get F_1^n and H^n. For this it is enough to consider a pushout like below

$$\begin{array}{ccc} \Sigma_{B_1^n} & \xrightarrow{\iota_{\varphi_1}B_1^n} & (\Sigma_1)_{A_1^n} \\ {\scriptstyle \iota_\Sigma g^n}\downarrow & & \downarrow{\scriptstyle u} \\ \Sigma_{B_2^n} & \xrightarrow[v]{} & \bullet \end{array}$$

and find a model for $uE_{A_1^n} \cup vE_{B_2^n}$. Its u-reduct will be F_1^n and its v-reduct will be H^n. We show the consistency of $uE_{A_1^n} \cup vE_{B_2^n}$ by *Reductio ad Absurdum*. Suppose it is inconsistent. Then

14 exists $e \in Sen\Sigma_{B_2^n}$ s.th. $E_{B_2^n} \models e$, $uE_{A_1^n} \models v(\neg e)$ — quasi-compactness, conjunctions, negations.

Let $e' \in Sen\Sigma_{B_1^n}$ be any universal $\iota_\Sigma g^n$-quantification of $\neg e$. Then

15 $E_{A_1^n} \models (\iota_{\varphi_1}B_1^n)e'$ — like with 6

16 $(A_1^n)_{A_1^n} \models (\iota_{\varphi_1}B_1^n)e'$ — 15, $(A_1^n)_{A_1^n}$ initial $((\Sigma_1)_{A_1^n}, E_{A_1^n})$-model

17 $(A_1^n)_{A_1^n}\restriction_{\iota_{\varphi_1}B_1^n} \cong (B_1^n)_{B_1^n}$ — naturality of i

18 $(B_1^n)_{B_1^n} \models e'$ — 16, 17, Satisfaction Condition

19 $(B_2^n)_{B_2^n} \models (\iota_\Sigma g^n)E_{B_1^n}$ — $\iota_\Sigma g^n : (\Sigma_{B_1^n}, E_{B_1^n}) \to (\Sigma_{B_2^n}, E_{B_2^n})$ theory morphism

20 $(B_2^n)_{B_2^n}\restriction_{\iota_\Sigma g^n} \models E_{B_1^n}$ — 19, Satisfaction Condition

21 $(B_2^n)_{B_2^n}\restriction_{\iota_\Sigma g^n} \models e'$ — 18, 20, e' preserved by the unique homomorphism $(B_1^n)_{B_1^n} \to (B_2^n)_{B_2^n}\restriction_{\iota_\Sigma g^n}$

22 $(B_2^n)_{B_2^n} \models \neg e$ — 21, e' $\iota_\Sigma g^n$-quantification of $\neg e$

23 $(B_2^n)_{B_2^n} \models e$ — $E_{B_2^n} \models e$ (14).

Since 22 and 23 represent a contradiction, it follows that $uE_{A_1^n} \cup vE_{B_2^n}$ is consistent, hence we can get F_1^n and H^n.

n.2 The last part of the proof consists of finding f_2^n and g^{n+1} such that $f_2^n\restriction_{\varphi_2} = h^n;g^{n+1}$. Since this is very similar to the proof of (n.1) we may skip it.

□

Obtaining concrete Robinson Consistency Theorems. We now see how the hypotheses of Thm. 9.20 can be established in concrete institutions.

1. Like in Thm. 7.11 on the existence of saturated models, in the applications the model homomorphisms of the institutions should be restricted to the elementary ones. Moreover, in this context, we also need diagrams. Cor. 5.36 provides a solution to this as it establishes institutions with diagrams for the elementary homomorphisms.

2. This is a common property of many concrete institutions of interest. However, here we have to consider that the model homomorphisms are elementary, which may pose some difficulties with the semi-exactness. But, luckily, here we need amalgamation only for the models and not for the homomorphisms.

3. In the applications this condition essentially requires that the institution has universal quantification for the class of the injective signature extensions with constants. This is justified by the fact that usually both elementary extensions $\iota_\Sigma M$ and the signature morphisms of the form $\iota_\Sigma h$ for $h:\ M \to N$ elementary Σ-homomorphism are signature extensions with constants. In the case of $\iota_\Sigma h$ this is so because in the applications h is injective as an elementary homomorphism in an institution with negations. For establishing the universal quantification for the injective signature extensions with an arbitrary number of constants, in institutions where quantification is defined only for injective signature extensions with a *finite* number of constants, one uses the same argument as in the proof of Cor. 9.6 which relies upon sentences being finitary.

4. This is perhaps the condition that constrains most the applicability domain of the theorem, but does not require any explanations concerning its realisation in the concrete institutions.

5. The existence of ω-co-limits for models is handled by instances of the general result on directed co-limits of elementary homomorphisms given by Cor. 7.4. Moreover, since the directed co-limits of elementary homomorphisms are obtained as co-limits using ordinary model homomorphisms (and afterwards the co-limiting co-cone is shown to consist of elementary homomorphisms), the preservation of ω-co-limits of elementary homomorphisms follows as a consequence of the arities of the symbols of the signatures being finite as in the typical example given by Prop. 6.8.

6. The final hypothesis of the theorem, the quasi-compactness, can be established in the applications by methods from Chap. 6 on ultraproducts, but also by other methods (not yet discussed) too, such as completeness. Though, as a side remark, we may notice that compactness, from all hypotheses of Thm. 9.20 being the most demanding one to establish, was not required by any of the previous interpolation results that were based on axiomatizability properties. Those results did not require semantic negations either. If we look at the proof of Thm. 9.20, we can see that the roles of the negations and of the compactness are very related to each other. Moreover, because of negations and conjunctions it does not matter which form of compactness we chose, model-theoretic or consequence-theoretic, as in this case these are equivalent forms.

Since the conditions of the general implication of Craig interpolation from Robinson consistency (Thm. 9.15) are part of the list of conditions of Thm. 9.20, we can formulate the expected interpolation consequence of the latter result.

Corollary 9.21. *In any institution satisfying the list of hypotheses of Thm. 9.20, any weak model amalgamation square of signature morphisms such that its span lifts isomorphisms, is a Craig interpolation square.*

Note how the lifting of isomorphisms is a recurrent condition in our interpolation results, in its single signature version it was required in Thm. 9.12.

We can now obtain again the $\mathcal{FOL}$ interpolation result of Cor. 9.17, but this time as an instance of Robinson consistency Thm. 9.20.

Corollary 9.22. *$\mathcal{FOL}$ has Craig $(Sig^{\mathcal{FOL}}, (i**))$ and $((i**), Sig^{\mathcal{FOL}})$-interpolation.*

But let us compare the mathematical costs of obtaining this result by the two methods. The Robinson Consistency method has a lower cost as the other method, based on the Keisler-Shelah theorem in $\mathcal{FOL}$, requires results about saturated models, also in relation with ultraproducts, some non-trivial set theory, the assumption of GCH. Some of these are results of complex developments beyond the effort required by the Robinson consistency method.

Failure of interpolation (and consequently of Robinson consistency) because of non-injectivity. The $\mathcal{FOL}$ interpolation result given by Cor. 9.17 (or 9.22) and consequently the Robinson consistency in $\mathcal{FOL}$, are sharp indeed in the sense that if none of the signature morphisms of a span $\Sigma_1 \xleftarrow{\varphi_1} \Sigma \xrightarrow{\varphi_2} \Sigma_2$ is $(i**)$ then the pushout of the span might fail to be a Ci square. The following gives an example for this situation. Consider the following pushout of $\mathcal{FOL}$ signatures containing only sorts and constants as shown in the diagram below:

$$\begin{array}{ccc}
\Sigma = \{a:\ s_1, b:\ s_2\} & \xrightarrow{\varphi_1} & \Sigma_1 = \{a:\ s\} \\
\varphi_2 \big\downarrow & & \big\downarrow \theta_1 \\
\Sigma_2 = \{a, b:\ s\} & \xrightarrow[\theta_2]{} & \Sigma' = \{a:\ s\}
\end{array}$$

Evidently, $\theta_1(a = a) \models \theta_2(a = b)$. However we will show that there is no interpolant E for this. By *Reductio ad Absurdum* suppose that there is one. Then

$$(a = a) \models \varphi_1 E \quad \text{and} \quad \varphi_2 E \models (a = b).$$

Consider the models

model X	signature of X	X_s	X_{s_1}	X_{s_2}	X_a	X_b
M	Σ	–	$\{A,B\}$	$\{A,B\}$	A	A
N	Σ	–	$\{A,B\}$	$\{A,B\}$	B	B
N_1	Σ_1	$\{A,B\}$	–	–	B	–
M_2	Σ_2	$\{A,B\}$	–	–	A	B

Note that $N = N_1\upharpoonright_{\varphi_1}$, $M = M_2\upharpoonright_{\varphi_2}$ and that $M \cong N$ by the isomorphism which is identity on s_2 and swaps the elements of s_1. Then

1	$N_1 \models (a = a) \models \varphi_1 E$	E interpolant
2	$N \models E$	1, $N = N_1\upharpoonright_{\varphi_1}$, Satisfaction Condition
3	$M \models E$	2, $M \cong N$
4	$M_2 \models \varphi_2 E$	3, $M = M_2\upharpoonright_{\varphi_2}$, Satisfaction Condition
5	$M_2 \models (a = b)$	4, $\varphi_2 E \models (a = b)$ (E interpolant).

Since 5 is false, E does not exist.

Interpolation in $\mathcal{FOL}$ versus interpolation in some of its sub-institutions. We have obtained a series of concrete interpolation results in $\mathcal{FOL}$ and in sub-institutions of $\mathcal{FOL}$ as instances of general institution-independent result. Now we are in the position to compare the interpolation properties in $\mathcal{FOL}$ on the one hand, and in sub-institutions such as $\mathcal{HCL}$, $\mathcal{EQL}$, $\mathcal{UNIV}$, etc. on the other hand.

- All interpolation properties developed in the above mentioned institutions require that at least one morphism of the span φ_1, φ_2 of the respective interpolation square, is injective on the sorts.
- However, in the sub-institutions the injectivity requirement is more stringent than in $\mathcal{FOL}$ as it extends also operation and relation symbols. Moreover, in the case of the operation symbols, besides mere injectivity other technical conditions are necessary, such as encapsulation or non-empty sorts. Thus we can say that $\mathcal{FOL}$ has stronger interpolation properties than the mentioned sub-institutions.
- Even if the conditions of interpolation in $\mathcal{HCL}$, $\mathcal{EQL}$, or $\mathcal{UNIV}$, etc., are more stringent than in $\mathcal{FOL}$, we will see later on in the chapter on applications to specification that they fit well with what is required by the modularisation technologies.

Exercises

9.7. [3] **Elementary amalgamation squares**
A commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is an *elementary amalgamation square* if for each Σ_1-model M_1 and each Σ_2-model M_2 such that $M_1\upharpoonright_{\varphi_1} \equiv M_2\upharpoonright_{\varphi_2}$ there exists a unique Σ'-model M' such that $M'\upharpoonright_{\theta_1} \equiv M_1$ and $M'\upharpoonright_{\theta_2} \equiv M_2$. In any institution with negation, a commuting square of signature morphisms is an elementary amalgamation square if and only if it is a Robinson consistency square.

9.8. In any institution with diagrams such that each pushout of elementary extensions is a Robinson consistency square, any two elementary equivalent models can be "embedded" into a common model in the sense that for each $M_1 \equiv M_2$ there exists homomorphisms $M_1 \xrightarrow{h_1} M \xleftarrow{h_2} M_2$. (*Hint:* Consider the pushout of the span of elementary extensions along the models M_1 and M_2, and consider the theories $(M_1)_{M_1}{}^*$ and $(M_2)_{M_2}{}^*$.)

9.9. Interpolation in $\mathcal{FOL}_{\infty,\omega}$
$\mathcal{FOL}_{\infty,\omega}$ has Craig $(Sig, (i**))$ and $((i**), Sig)$-interpolation. (*Hint:* Use Robinson consistency Thm. 9.20.)

9.10. Robinson consistency in $\mathcal{PA}$
Develop the Robinson consistency result for $\mathcal{PA}$ as an instance of Thm. 9.20. Derive a corresponding interpolation result for $\mathcal{PA}$.

9.5 Craig-Robinson interpolation

In this section we will introduce a more refined form of interpolation than Craig interpolation. Although it is rather marginal in the mainstream logic, for some computing science applications it does appear as the appropriate form of interpolation. Moreover, as we will see, the same is true for some model theory as such applications.

The Craig interpolation property can be strengthened by adding to the 'primary' premises E_1 a set Γ_2 (of Σ_2-sentences) as 'secondary' premises. In any institution we say that a commuting square of signature morphisms like below

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is a *Craig-Robinson Interpolation square* (abbreviated *CRi* square) when for each set E_1 of Σ_1-sentences and each sets E_2 and Γ_2 of Σ_2-sentences, if $\theta_1 E_1 \cup \theta_2 \Gamma_2 \models_{\Sigma'} \theta_2 E_2$, then there exists a set E of Σ-sentences such that $E_1 \models_{\Sigma_1} \varphi_1 E$ and $\Gamma_2 \cup \varphi_2 E \models_{\Sigma_2} E_2$. Also the $\langle \mathcal{L}, \mathcal{R} \rangle$-interpolation concept can be extended in a straightforward way from Craig interpolation to Craig-Robinson interpolation.

Craig-Robinson versus Craig interpolation. By taking Γ_2 to be the empty set $\emptyset$ we can see that

Fact 9.23. *Any CRi square is also a Ci square.*

The opposite implication does not hold in general. The following gives a sufficient condition when Ci and CRi are equivalent interpolation concepts.

Proposition 9.24. *In any institution that has implications and is quasi-compact, a commuting square of signature morphisms is a CRi square if and only if it is a Ci square.*

Proof. We focus only on the implication not covered by Fact 9.23, that Ci implies CRi. Consider $E_1 \subseteq Sen\Sigma_1$ and $E_2, \Gamma_2 \subseteq Sen\Sigma_2$ such that $\theta_1 E_1 \cup \theta_2 \Gamma_2 \models \theta_2 E_2$.

- First we notice that without loss of generality we may assume that E_2 consists of only one sentence e, i.e., $E_2 = \{e\}$. Indeed, if we assumed that CRi property holds for each $e \in E_2$, let E_e be the interpolant corresponding to each $e \in E_2$. Then $\bigcup_{e \in E_2} E_e$ is an interpolant corresponding to E_2.
- Because we may assume that $E_2 = \{e\}$, then by the quasi-compactness assumption, we may further assume without loss of generality that E_1 and Γ_2 are finite.
- Let $\Gamma_2 \Rightarrow e$ denote $\gamma_1 \Rightarrow (\cdots \Rightarrow (\gamma_n \Rightarrow e))$ where $\Gamma_2 = \{\gamma_1, \ldots, \gamma_n\}$. Then

1	$\theta_1 E_1 \models \theta_2(\Gamma_2 \Rightarrow e)$	$\theta_1 E_1 \cup \theta_2 \Gamma_2 \models \theta_2 e$, induction on the size of Γ_2
2	there exists $E \subseteq Sen\Sigma$ s.th. $E_1 \models \varphi_1 E$, $\varphi_2 E \models (\Gamma_2 \Rightarrow e)$	1, Ci property
3	$E_1 \models \varphi_1 E,\ \varphi_2 E \cup \Gamma_2 \models e$	2, induction on the size of Γ_2.

□

Prop. 9.24 gives the possibility to extend Ci properties to CRi properties in institutions as illustrated by the following example.

Corollary 9.25. *$\mathcal{FOL}$ has Craig-Robinson $(Sig, (i**))$ and $((i**), Sig)$-interpolation.*

Proof. By Cor. 9.17, 9.22, and 9.11 $\mathcal{FOL}$ has the corresponding Craig interpolation properties, has implications and is compact (cf. Cor. 6.24). □

Although one may get the feeling that CRi embeds a form of implication and therefore it is expected only in institutions having semantic implications, it is not so. Later on (in Sect. 14.3) we will see that institutions without semantic implications such as $\mathcal{EQL}$ and $\mathcal{HCL}$ may enjoy CRi for a wide class of pushout squares of signature morphisms.

Failure of Craig-Robinson interpolation. In the absence of implications, CRi may fail even in intersection-union squares as shown by the following example. In $\mathcal{EQL}$, consider $\Sigma = \Sigma_2$ single-sorted signatures with four constants a, b, c, d, and $\Sigma_1 = \Sigma'$ their extension with an unary operation symbol f. Let $E_1 = \{fa = b,\ fc = d\}$, $E_2 = \{b = d\}$, and $\Gamma_2 = \{a = c\}$. Then $E_1 \cup \Gamma_2 \models E_2$ but there is no Craig-Robinson interpolant E such that $E_1 \models E$ and $E \cup \Gamma_2 \models E_2$.

Extending interpolation

Sometimes interpolation properties can be established in two stages. At the first stage we establish it for a particular class of commuting squares of signature morphisms. At the second stage we extend them to a larger class of squares of signature morphisms by a general method formulated by Thm. 9.28 below. This technique uses Craig-Robinson interpolation and it constitutes our first application of CRi. We need the following concept.

Logical kernels. A signature morphism $\varphi\colon \Sigma \to \Sigma'$ has a *logical kernel*, when there exists a Σ-theory lk_φ such that

any Σ-model M has a φ-expansion if and only if $M \models lk_\varphi$.

Fact 9.26. *Any logical kernel is a tautology in the target signature, i.e.,* $\models_{\Sigma'} \varphi\, lk_\varphi$.

The following is a typical example of a logical kernel. It just shows how one can recover the model expansion property in the case of the signature morphisms that are injective on the sorts but not necessarily injective on the operation or the relation symbols. The idea is simple, we impose syntactically that the Σ-model does not interpret differently those symbols on which the injectivity of φ fails.

Fact 9.27. *Any* $(i{*}{*})$*-morphism of* $\mathcal{FOL}$ *signatures* $\varphi\colon \Sigma \to \Sigma'$ *such that* Σ *has non-empty sorts has the logical kernel*

$$lk_\varphi = \{(\forall X)\pi X \Leftrightarrow \pi' X \mid \varphi^{\mathrm{rl}}\pi = \varphi^{\mathrm{rl}}\pi'\} \cup \{(\forall X)\sigma X = \sigma' X \mid \varphi^{\mathrm{op}}\sigma = \varphi^{\mathrm{op}}\sigma'\}.$$

In the same way we can even treat the non-empty sorts condition. If φ does not add any new operation symbols whose result is a non-empty sort in Σ, the definition of lk_φ in Fact 9.27 still does it, otherwise we can force the semantic non-emptiness by adding to lk_φ a sentence

$$(\exists x : s)\; x = x$$

for each sort s of Σ that has the issue described above.

Theorem 9.28 (Extending interpolation)**.** *In any institution with model amalgamation consider classes of signature morphisms* $\mathcal{L}_0, \mathcal{L}, \mathcal{R}_0, \mathcal{R}, \mathcal{E} \subseteq \mathit{Sig}$ *such that*

1. *each signature morphism* $\phi \in \mathcal{L}$ *(*$\mathcal{R}$*) can be factored as* $\phi = i;\varphi$ *such that* $\varphi \in \mathcal{E}$ *and* $i \in \mathcal{L}_0$ *(*$\mathcal{R}_0$*), and*
2. *each* $\varphi \in \mathcal{E}$ *is a retract that has a logical kernel.*

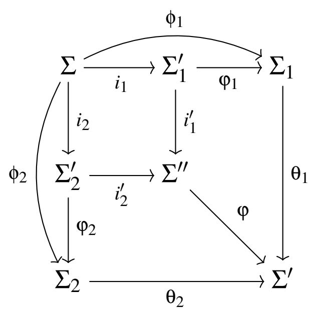
If the institution has the Craig-Robinson $(\mathcal{L}_0, \mathcal{R}_0)$*-interpolation property then it also has the Craig-Robinson* $(\mathcal{L}, \mathcal{R})$*-interpolation property.*

Proof. Consider a pushout θ_1, θ_2 of a span of signature morphisms ϕ_1, ϕ_2 like in diagram (9.11) such that $\phi_1 \in \mathcal{L}$, $\phi_2 \in \mathcal{R}$, and let $E_1 \subseteq \mathit{Sen}\Sigma_1$ and $\Gamma_2, E_2 \subseteq \mathit{Sen}\Sigma_2$ such that

$$\theta_1 E_1 \cup \theta_2 \Gamma_2 \models \theta_2 E_2. \tag{9.10}$$

The main idea of this proof is to derive a CR $(\mathcal{L}_0, \mathcal{R}_0)$-interpolation problem and show that its interpolant is an interpolant for (9.10) too.

- Let $\phi_1 = i_1;\varphi_1$, $\phi_2 = i_2;\varphi_2$ such that $i_1 \in \mathcal{L}_0$, $i_2 \in \mathcal{R}_0$ and $\varphi_1, \varphi_2 \in \mathcal{E}$. Let i'_1, i'_2 be a pushout of i_1, i_2 as in diagram (9.11). By the universal property of pushouts, let φ be the unique signature morphism $\Sigma'' \to \Sigma'$ such that $i'_k;\varphi = \varphi_k;\theta_k$, $k = 1,2$.



(9.11)

- Consider $\overline{\varphi_i}$ a left-inverse to φ_i, $i = 1,2$, and define $E'_1 = \overline{\varphi_1}E_1 \cup lk_{\varphi_1}$, $\Gamma'_2 = \overline{\varphi_2}\Gamma_2 \cup lk_{\varphi_2}$, and $E'_2 = \overline{\varphi_2}E_2$.
- We will show that $i'_1E'_1 \cup i'_2\Gamma'_2 \models i'_2E'_2$. This would represent a CR $(\mathcal{L}_0, \mathcal{R}_0)$-interpolation problem. Consider a Σ''-model M'' such that $M'' \models i'_1E'_1 \cup i'_2\Gamma'_2$. We have to prove that $M'' \models i'_2E'_2$. Define $M'_k = M''\restriction_{i'_k}$, $k = 1,2$. Then

1 $M'_1 \models \overline{\varphi_1}E_1 \cup lk_{\varphi_1}$, $M'_2 \models \overline{\varphi_1}\Gamma_2 \cup lk_{\varphi_2}$ — $M'' \models i'_kE'_k$, $k = 1,2$, Satisfaction Condition.

Because of the logical kernel property, from 1 we obtain a φ_k-expansion M_k of M'_k, $k = 1,2$. Then

2 $M_1 \models \varphi_1(\overline{\varphi_1}E_1) = E_1$ — 1, Satisfaction Condition, $\overline{\varphi_1};\varphi_1 = 1_{\Sigma_1}$

3 $M_2 \models \varphi_2(\overline{\varphi_2}\Gamma_2) = \Gamma_2$ — 1, Satisfaction Condition, $\overline{\varphi_2};\varphi_2 = 1_{\Sigma_2}$

4 $M_k\restriction_{\phi_k} = M_k\restriction_{\varphi_k}\restriction_{i_k} = M'_k\restriction_{i_k} = M''\restriction_{i'_k}\restriction_{i_k}$, $k = 1,2$ — $\phi_k = i_k;\varphi_k$, definitions of M_k, M'_k

5 $M_1\restriction_{\phi_1} = M_2\restriction_{\phi_2}$ — 4, $i_1;i'_1 = i_2;i'_2$

By the model amalgamation property, from 5 we obtain an unique amalgamation $M' \in |Mod\Sigma'|$ of M_1 and M_2. Then

6 $M'\restriction_{\varphi}\restriction_{i'_k} = M'\restriction_{\theta_k}\restriction_{\varphi_k} = M_k\restriction_{\varphi_k} = M'_k = M''\restriction_{i'_k}$ — $i_k;\varphi = \varphi_k;\theta_k$, definitions of M', M_k, M'_k

7 $M'\restriction_{\varphi} = M''$ — 6, uniqueness of model amalgamation

8 $M' \models \varphi(i'_1E'_1 \cup i'_2\Gamma'_2)$ — 7, Satisfaction Condition, $M'' \models i'_1E'_1 \cup i'_2\Gamma'_2$

9 $\varphi(i'_1E'_1) = \theta_1(\varphi_1E'_1) \mathrel{\models\!\!\!|} \theta_1E_1$ — definition of E'_1, $\overline{\varphi_1};\varphi_1 = 1_{\Sigma_1}$, Fact 9.26, 'translation' of $\models$

10 $\varphi(i'_2\Gamma'_2) \mathrel{\models\!\!\!|} \theta_2\Gamma_2$ — similarly to 9

11 $M' \models \theta_2E_2 = \theta_2(\varphi_2E'_2)$ — 8, 9, 10, $\theta_1E_1 \cup \theta_2\Gamma_2 \models \theta_2E_2$, $E_2 = \varphi_2E'_2$ ($\overline{\varphi_2};\varphi_2 = 1_{\Sigma_2}$)

12 $M'' \models i'_2 E'_2$ — 7, 11, $\varphi_2;\theta_2 = i'_2;\varphi$, Satisfaction Condition, $\overline{\varphi_2};\varphi_2 = 1_{\Sigma_2}$.

- We have thus showed that $i'_1 E'_1 \cup i'_2 \Gamma'_2 \models i'_2 E'_2$. Let $E \subseteq Sen\Sigma$ be a CR interpolant such that $E'_1 \models i_1 E$ and $\Gamma'_2 \cup i_2 E \models E'_2$. We show that E is an interpolant for the original consequence $\theta_1 E_1 \cup \theta_1 \Gamma_2 \models \theta_2 E_2$ too. This goes as follows:

  13 $\varphi_1 E'_1 \models \varphi_1(i_1 E)$ — $E'_1 \models i_1 E$, 'translation' of semantic consequence $\models$

  14 $E_1 \cup \varphi_1(lk_{\varphi_1}) \models \phi_1 E$ — 13, $\overline{\varphi_1};\varphi_1 = 1_{\Sigma_1}$, $\phi_1 = i_1;\varphi_2$

  15 $E_1 \models \phi_1 E$ — 14, Fact 9.26

  16 $\varphi_2 \Gamma'_2 \cup \varphi_2(i_2 E) \models \varphi_2 E'_2$ — $\Gamma'_2 \cup i_2 E \models E'_2$, 'translation' of semantic consequence $\models$

  17 $\Gamma_2 \cup \varphi_2(lk_{\varphi_2}) \cup \phi_2 E \models E_2$ — 16, $\overline{\varphi_2};\varphi_2 = 1_{\Sigma_2}$, $\phi_2 = i_2;\varphi_2$

  18 $\Gamma_2 \cup \phi_2 E \models E_2$ — 17, Fact 9.26.

  The relations 15 and 18 show that E is an interpolant for the CR $(\mathcal{L},\mathcal{R})$-interpolation problem (9.10).

□

The result of Thm. 9.28 relies on the CR form of interpolation, it cannot be obtained for just Craig interpolation. The reason for this is the logical kernel lk_{φ_2}, which is necessary to obtain the Σ_2-model M_2. It is an indispensable part of the secondary premise of interpolation, Γ'_2. In an eventual Craig interpolation formulation of the theorem, we can dispense with Γ_2, but not with lk_{φ_2}, which forces us into CR interpolation. However if in some applications we may find CRi being too strong then we can still have a Ci version of Thm. 9.28 by weakening its conclusion. We have to eliminate lk_{φ_2}, which can be achieved by trivialising φ_2, which means $\mathcal{R} = \mathcal{R}_0$. So we can formulate the following Ci version of Thm. 9.28.

Corollary 9.29. *In any institution with model amalgamation consider classes of signature morphisms $\mathcal{L}_0, \mathcal{L}, \mathcal{R}_0, \mathcal{E} \subseteq Sig$ such that*

1. *each signature morphism $\phi \in \mathcal{L}$ can be factored as $\phi = i;\varphi$ such that $\varphi \in \mathcal{E}$ and $i \in \mathcal{L}_0$, and*
2. *each $\varphi \in \mathcal{E}$ is a retract that has a logical kernel.*

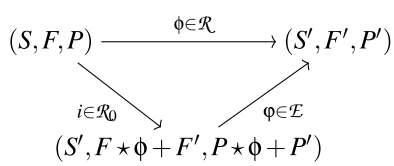
If the institution has the Craig $(\mathcal{L}_0,\mathcal{R}_0)$-interpolation property then it also has the Craig $(\mathcal{L},\mathcal{R}_0)$-interpolation property.

Interpolation in infinitary second order logic. In what follows we illustrate the applicability of the extension Thm. 9.28 by a concrete case. Let $\mathcal{SOL}_{\infty,\omega}$ be the extension of second order logic $\mathcal{SOL}$ which allows infinite conjunctions of sentences.

Proposition 9.30. *$\mathcal{SOL}_{\infty,\omega}$ has Craig $(Sig,\mathcal{R})$ and $(\mathcal{R},Sig)$-interpolation for $\mathcal{R}$ the class of $(i**)$-morphisms of signatures $\varphi\colon \Sigma \to \Sigma'$ for which Σ' is finite and both Σ and Σ' have non-empty sorts.*

Proof. First, let us remember that the category of $\mathcal{SOL}_{\infty,\omega}$ signatures is that of the $\mathcal{FOL}$ signatures, the only difference between the two institutions being at the level of the sentences. Then, $\mathcal{SOL}_{\infty,\omega}$ is quasi-compact because it has infinite conjunctions. This is unlike $\mathcal{SOL}$, and it is important because it allows all the connections between Ci, RC and CRi.

- By the equivalence between Ci and RC provided Theorems 9.15 and 9.16, it is enough to establish Craig $(Sig, \mathcal{R})$-interpolation.
- By Prop. 9.24, in $\mathcal{SOL}_{\infty,\omega}$ Ci is equivalent to CRi.
- Then we can apply the extension Thm. 9.28 where $\mathcal{R}_0$ is the class of the (iii)-morphisms of signatures $\Sigma \to \Sigma'$ such that Σ' is finite and Σ, Σ' have non-empty sorts. For this we need the Craig $(Sig, \mathcal{R}_0)$-interpolation for $\mathcal{SOL}_{\infty,\omega}$. Cor. 9.5 does this for us by arguments similar to those used for obtaining the respective Ci property for $\mathcal{SOL}$ (Cor. 9.6). Though for $\mathcal{SOL}_{\infty,\omega}$ the argument for the existence of Σ_0 finite (see the proof of Cor. 9.6) has to be done on the basis that Σ' is finite rather than that the sentences of the institution are finitary.
- In order to apply Thm. 9.28 mentioned above it still remains to set up the class $\mathcal{E}$ of signature morphisms. This is the class of the (bss)-morphisms. By Fact 9.27 they have logical kernels and, moreover, it is also easy to see that they are also retracts.
- It remains to show that each signature morphism $(\phi\colon\ (S,F,P) \to (S',F',P')) \in \mathcal{R}$ can be factored as $\phi = i;\varphi$ with $i \in \mathcal{R}_0$ and $\varphi \in \mathcal{E}$. This factoring is illustrated by the diagram



  where for each arity w and sort s

  $$(F\star\phi+F')_{\phi w\to\phi s} = F_{w\to s} \uplus F'_{\phi w\to\phi s} \ \text{ and } \ (P\star\phi+P')_{\phi w} = P_w \uplus P'_{\phi w}.$$

  The two formulas above signify a disjoint union between the constituent sets of F and P, with the arities and sorts renamed according to ϕ^{st} one the one hand, and the constituent sets of F' and P', on the other hand. φ is the obvious signature morphism that aggregates ϕ and the identity on (S',F',P').

□

Exercises

9.11. In the counter-example showing how CRi may fail in $\mathcal{EQL}$, we stated that the respective consequence does not admit an interpolant. Prove this. (*Hint:* If $E_1 \models E$ with $E \subseteq Sen^{\mathcal{EQL}}\Sigma$, then each Σ-model satisfies E.)

9.12. Symmetric Birkhoff institutions
A Birkhoff institution $(Sig, Sen, Mod, \models, \mathcal{F}, \mathcal{B})$ is *symmetric* when $\mathcal{B}$ is symmetric. Extend Thm. 9.7 to Craig-Robinson interpolation for symmetric Birkhoff institutions.

9.13. [76] **Lifting interpolation to theories**
For any institution I and a class $\mathcal{S} \subseteq Sig$ of signature morphisms let $\mathcal{S}^{\mathrm{th}}$ be the class of theory morphisms φ such that $\varphi \in \mathcal{S}$ (as a signature morphism). The institution I^{th} of the theories of I has the Craig-Robinson $(\mathcal{L}^{\mathrm{th}}, \mathcal{R}^{\mathrm{th}})$-interpolation if I has the Craig-Robinson $(\mathcal{L}, \mathcal{R})$-interpolation.

9.6 Borrowing interpolation

The borrowing method is the last method to obtain interpolation properties that we discuss in this chapter. In brief, given an institution comorphism $I \to I'$ such that I' has a certain interpolation property, under certain conditions this can be transferred to I along the comorphism. This method is especially useful when I' is a well studied institution in which the interpolation properties are well understood, while our know-how about I may be much weaker. In some cases it would be possible to apply on I our previously developed general interpolation results, but still the borrowing method might require less mathematical effort. The borrowing method requires some special interpolation properties of the involved comorphism.

Interpolation properties of comorphisms. Let $(\Phi, \alpha, \beta) : I \to I'$ be a comorphism of institutions and let $\varphi : \Sigma \to \Omega$ be a signature morphism in I. The naturality of α gives us the following commutative square of signature morphisms:

$$\begin{array}{ccc} Sen\Sigma & \xrightarrow{\alpha_\Sigma} & Sen'(\Phi\Sigma) \\ {\scriptstyle Sen\varphi}\downarrow & & \downarrow{\scriptstyle Sen'(\Phi\varphi)} \\ Sen\Omega & \xrightarrow[\alpha_\Omega]{} & Sen'(\Phi\Omega) \end{array} \tag{9.12}$$

Then the comorphism (Φ, α, β)

- has the *Craig φ-left interpolation property* when for each $E_1 \subseteq Sen\Omega$, $E_2 \subseteq Sen'(\Phi\Sigma)$ such that $\alpha_\Omega E_1 \models' (\Phi\varphi)E_2$ there exists $E \subseteq Sen\Sigma$ such that $E_1 \models \varphi E$ and $\alpha_\Sigma E \models' E_2$, and
- has the *Craig φ-right interpolation property* when for each $E_1 \subseteq Sen'(\Phi\Sigma)$, $E_2 \subseteq Sen\Omega$ such that $(\Phi\varphi)E_1 \models' \alpha_\Omega E_2$ there exists $E \in Sen\Sigma$ such that $E_1 \models \alpha_\Sigma E$ and $\varphi E \models E_2$.

The left and the right interpolation properties are mutually symmetric with respect to the diagonal $Sen\Sigma \to Sen'(\Phi\Omega)$ in the commutative square (9.12). We also extend the definitions of these two properties from single morphisms to classes $\mathcal{S} \subseteq Sig$ by requiring that each morphism from $\mathcal{S}$ has the respective property.

Borrowing interpolation along institution comorphisms

The following is a generic result that can be used for borrowing interpolation properties along institution comorphisms.

Proposition 9.31. *Let* $(\Phi,\alpha,\beta): I \to I'$ *be a conservative institution comorphism such that* Φ *preserves pushouts, and let* $\mathcal{L},\mathcal{R} \subseteq \mathbb{S}ig$ *be classes of signature morphisms such that* I' *has the Craig* $(\Phi\mathcal{L},\Phi\mathcal{R})$*-interpolation. If* (Φ,α,β) *has the Craig* $\mathcal{L}$*-left or* $\mathcal{R}$*-right interpolation, then* I *has Craig* $(\mathcal{L},\mathcal{R})$*-interpolation.*

Proof. Consider a pushout of signature morphisms in I as shown below

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array} \tag{9.13}$$

such that $\varphi_1 \in \mathcal{L}$, $\varphi_2 \in \mathcal{R}$, $E_1 \subseteq Sen\Sigma_1$, $E_2 \subseteq Sen\Sigma_2$ and $\theta_1 E_1 \models \theta_2 E_2$. We have to find an interpolant $E \subseteq Sen\Sigma$ for this $(\mathcal{L},\mathcal{R})$-interpolation problem.

- We have that:

1 $\alpha_{\Sigma'}(\theta_1 E_1) \models' \alpha_{\Sigma'}(\theta_2 E_2)$ — $\theta_1 E_1 \models \theta_2 E_2$, Satisfaction Condition of (Φ,α,β)

2 $(\Phi\theta_1)(\alpha_{\Sigma_1} E_1) \models' (\Phi\theta_2)(\alpha_{\Sigma_2} E_2)$ — 1, naturality of α.

- Since Φ maps the pushout square (9.13) to the following pushout square of signature morphisms in I'

$$\begin{array}{ccc} \Phi\Sigma & \xrightarrow{\Phi\varphi_1} & \Phi\Sigma_1 \\ {\scriptstyle \Phi\varphi_2}\downarrow & & \downarrow{\scriptstyle \Phi\theta_1} \\ \Phi\Sigma_2 & \xrightarrow[\Phi\theta_2]{} & \Phi\Sigma' \end{array} \tag{9.14}$$

  the relation 2 represents a Craig $(\Phi\mathcal{L},\Phi\mathcal{R})$-interpolation problem.

- By the Ci property of I' there exists an interpolant $E_0 \subseteq Sen'(\Phi\Sigma)$ such that

3 $\alpha_{\Sigma_1} E_1 \models' (\Phi\varphi_1)E_0$

4 $(\Phi\varphi_2)E_0 \models' \alpha_{\Sigma_2} E_2$.

  From now we can proceed in two alternative ways depending on which of the left or the right interpolation properties are available for (Φ,α,β).

- In the case of $\mathcal{L}$-left interpolation, from 3 we have that there exists $E \subseteq Sen\Sigma$ such that

5 $E_1 \models \varphi_1 E$

6 $\alpha_\Sigma E \models' E_0$.

We show that E is the desired interpolant for $\theta_1 E_1 \models \theta_2 E_2$. Half of the interpolant property of E is given by 5. We obtain the other half as follows:

7 $(\Phi\varphi_2)(\alpha_\Sigma E) \models' (\Phi\varphi_2)E_0$ — 6, 'translation' of semantic consequence $\models'$

8 $\alpha_{\Sigma_2}(\varphi_2 E) \models' \alpha_{\Sigma_2} E_2$ — 7, 4, naturality of α, 'transitivity' of semantic consequence $\models'$

9 $\varphi_2 E \models E_2$ — 8, (Φ,α,β) conservative.

The relations 5 and 9 show that E is an interpolant for $\theta_1 E_1 \models \theta_2 E_2$.

- In the case of $\mathcal{R}$-right interpolation we can do like we did for $\mathcal{L}$-left interpolation, by using the $\mathcal{R}$-right interpolation property on 4, get E, and then use 5 and the conservativity of the comorphism to show that E is an interpolant for $\theta_1 E_1 \models \theta_2 E_2$.

□

In the applications, the hypothesis on the conservativity of the comorphism can be typically solved as a consequence of the model expansion property of β. The preservation of pushout squares by the signature translation functor Φ is a rather common property in the applications; often Φ is even a left-adjoint. The substantial specific condition of the general borrowing result of Prop. 9.31 is the interpolation property of the comorphism. In what remains of this section we will address this condition. We will also present samples of concrete applications of the borrowing interpolation technique put forward in this section.

Borrowing interpolation between institutions having the same expressive power. Let $(\Phi,\alpha,\beta)\colon\ I \to I'$ be an institution comorphism. We say that it *preserves the expressive power* when for each $\Phi\Sigma$-theory Γ' there exists a Σ-theory Γ such that $\alpha_\Sigma \Gamma \models\!\!\!\dashv \Gamma'$. This property says that any axiomatizable class of models from I' can be axiomatised from I. In this case the interpolation properties for the comorphism can be established rather easily, leading to a rather easy transfer of interpolation from the target institution to the source institution.

Proposition 9.32. *Any institution morphism comorphism* $(\Phi,\alpha,\beta)\colon\ I \to I'$ *which is conservative and preserves the expressive power has both the Craig* Sig^I*-left and right interpolation properties.*

Proof. Under the notations employed when we defined the Craig left and right interpolation we let $E \subseteq Sen\Sigma$ such that

$$\alpha_\Sigma E \models\!\!\!\dashv \begin{cases} E_2, & \text{for the left interpolation} \\ E_1, & \text{for the right interpolation.} \end{cases}$$

Then the Craig Sig^{I}-left and right interpolation properties follow immediately, in each case one of the two properties of the interpolant holds by definition, while in the case of the other one we rely on the conservativity hypothesis. □

A good application of the result of Prop. 9.32 is the derivation of interpolation properties in $\mathcal{PA}$ from those in $\mathcal{FOL}$. This is proposed to the reader in Ex. 9.14.

Interpolation properties for comorphisms through axiomatizabillity

On the one hand, in Sec. 9.3 we developed two interpolation-by-axiomatizability theorems that had a 'left' and 'right' character, respectively. On the other hand, now we have the concepts of 'left' and 'right' interpolation for comorphisms. Can we relate these two situations, and if yes then can we use Theorems 9.7 and 9.12 to obtain some 'left' and 'right' interpolation properties for comorphisms at a general level? In what follows we will answer positively to both questions. This requires an effort which hass the following two main aspects:

- We have to interpret the heterogeneous situation defined by an institution comorphism as a homogeneous situation of a single institution.
- We have to inspect the proofs of the two theorems mentioned above in order to develop a more refined view regarding the role played by each of their conditions.

The former aspect will be understood in its full generality in Chap. 14 only, as it is the simplest non-trivial case of the so-called 'Grothendieck institution' construction. In other words, here we anticipate this important concept by means of a good application.

Flattening comorphisms to institutions. Let us start with a comorphism $(\Phi,\alpha,\beta) : I \to I'$. We can define an institution $I^{\sharp} = (Sig^{\sharp}, Sen^{\sharp}, Mod^{\sharp}, \models^{\sharp})$ as follows.

- $Sig^{\sharp}$ puts the signatures of I and I' on 'one bag', but qualifying them by their origin. So an object of $Sig^{\sharp}$ is either a pair (I,Σ) with $\Sigma \in |Sig|$ or else a pair (I',Σ') with $\Sigma' \in |Sig'|$. The signature morphisms of $Sig^{\sharp}$ are of two kinds:
  - 'internal' ones such as $\varphi : (I,\Sigma) \to (I,\Omega)$ (or $\varphi' : (I',\Sigma') \to (I',\Omega')$), with $\varphi \in Sig(\Sigma,\Omega)$ ($\varphi' \in Sig'(\Sigma',\Omega')$), or
  - 'external' ones, $\varphi : (I,\Sigma) \to (I',\Sigma')$, with $\varphi \in Sig'(\Phi\Sigma,\Sigma')$.

  Note that there are no signature morphisms $(I',\Sigma') \to (I,\Sigma)$. Hence the only possible compositions are between 'internal' morphisms – and this is defined on the basis of the composition of signature morphisms in the respective institution – and between an 'internal' and an 'external' morphism. The latter kind of composition is defined as shown in the diagram below:

$$\begin{array}{ccccc}
(I,\Sigma) & \xrightarrow{\varphi} & (I,\Omega) & & \\
& \searrow^{\Phi\varphi\ ;\ \theta} & \downarrow{\scriptstyle\theta} & \searrow^{\theta;\zeta} & \\
& & (I',\Omega') & \xrightarrow[\zeta]{} & (I',Z)
\end{array}$$

It is easy to check that these definitions yield a category, which is $Sig^\sharp$.

- The sentence functor $Sen^\sharp$ and the model functor $Mod^\sharp$ just extend the sentence and the model functors of I and I'– which are used in the case of the 'internal' signature morphisms – to the 'external' signature morphisms $\varphi : (I,\Sigma) \to (I',\Sigma')$ as follows:
  - $Sen^\sharp\varphi = \alpha_\Sigma \ ; Sen'\varphi$, and
  - $Mod^\sharp\varphi = Mod'\varphi \ ; \beta_\Sigma$.

  We can also easily check that these definitions of $Sen^\sharp$ and of $Mod^\sharp$ are appropriate functors.
- Finally, the satisfaction relation $\models^\sharp$ is defined locally, by

  $$(M \models^\sharp_{(I,\Sigma)} \rho) = (M \models_\Sigma \rho) \ \text{ and } \ (M' \models^\sharp_{(I',\Sigma')} \rho') = (M' \models'_{\Sigma'} \rho').$$

  We can check easily the Satisfaction Condition for the 'internal' morphisms, this being inherited from I and I'. For the 'external' morphisms this involves both the local Satisfaction Condition in I' and the Satisfaction Condition of the comorphism.

The following remark clarifies the formal connection between the 'left' and the 'right' interpolation concepts of Sec. 9.3 and those for comorphisms.

Fact 9.33. *Let $(\Phi,\alpha,\beta) : I \to I'$ be an institution morphism and let $\mathcal{S} \subseteq Sig$ be a class of I-signature morphisms. For any $\varphi \in \mathcal{S}$ let us consider the commutative square of $I^\sharp$-signature morphisms:*

$$\begin{array}{ccc}
(I,\Sigma) & \xrightarrow{1_{\Phi\Sigma}} & (I',\Phi\Sigma) \\
{\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\Phi\varphi} \\
(I,\Omega) & \xrightarrow[1_{\Phi\Omega}]{} & (I',\Phi\Omega)
\end{array} \qquad (9.15)$$

Then (Φ,α,β)

- *has Craig $\mathcal{S}$-right interpolation if and only if for each $\varphi \in \mathcal{S}$, the commutative square (9.15), is a Craig interpolation square in $I^\sharp$, and*
- *has Craig $\mathcal{S}$-left interpolation if and only if for each $\varphi \in \mathcal{S}$ the reflection of the square (9.15) with respect to the diagonal $(I,\Sigma) \to (I',\Phi\Omega)$ is a Craig interpolation square in $I^\sharp$.*

(Φ,β)-amalgamation. In Theorems 9.7 and 9.12 model amalgamation, in its weak form, played a role. This is expected also to happen when establishing interpolation properties for comorphisms using the (technique of those) two theorems. We already have concepts of model amalgamation for comorphisms from Sec. 4.3. We can use them here, but in order to keep the hypotheses of our results as weak / lax as possible, we need model amalgamation at the level of comorphisms only for the classes of signature morphisms that are involved in the interpolation. Therefore we slightly adapt the concepts of model amalgamation for comorphisms from Sec. 4.3 as follows.

For any comorphism $(\Phi,\alpha,\beta)\colon\ I \to I'$ we say that a signature morphism $\varphi\colon\ \Sigma \to \Omega$ in I *has* (Φ,β)*-amalgamation* when for each Ω-model N and each $\Phi\Sigma$-model M' with $N\restriction_{\varphi} = \beta_{\Sigma}M'$ there exists a unique $\Phi\Omega$-model N' such that $\beta_{\Omega}N' = N$ and $N'\restriction_{\Phi\varphi} = M'$. As usual, if we drop the uniqueness requirement on N' we have the weak version of the concept, called *weak* (Φ,β)*-amalgamation*.

Fact 9.34. *Let $(\Phi,\alpha,\beta)\colon\ I \to I'$ be an institution morphism and let $\mathcal{S} \subseteq Sig$ be a class of I-signature morphisms. Then $\mathcal{S}$ has (weak) model (Φ,β)-amalgamation if and only if for each $\varphi \in \mathcal{S}$ the commutative square of $I^{\sharp}$-signature morphisms* (9.15) *has (weak) model amalgamation.*

Left interpolation property for comorphisms. The key to derive a left interpolation property for comorphisms from Thm. 9.12 is to match the following two commutative squares of signature morphisms:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array} \qquad\qquad \begin{array}{ccc} (I,\Sigma) & \xrightarrow{\varphi} & (I,\Omega) \\ {\scriptstyle 1_{\Phi\Sigma}}\downarrow & & \downarrow{\scriptstyle 1_{\Phi\Omega}} \\ (I',\Phi\Sigma) & \xrightarrow[\Phi\varphi]{} & (I',\Phi\Omega) \end{array}$$

The left-hand side one is the interpolation square of Thm. 9.12, while the right-hand side square is the square in $I^{\sharp}$ that through Fact 9.33 bridges ordinary interpolation to comorphism interpolation. Note that the Birkhoff institution structure has been used in the proof of Thm. 9.12 only in connection to φ_1, which means that, in the context of the comorphism interpolation, it is enough to assume it for I only. Let us also consider the conclusion of Fact 9.34 which relates (Φ,β)-amalgamation to ordinary amalgamation in institutions. Thus we can translate Thm. 9.12 to the comorphims environment as follows:

Proposition 9.35. *Consider an institution comorphism $(\Phi,\alpha,\beta)\colon\ I \to I'$ such that $I = (Sig, Sen, Mod, \models, \mathcal{F}, \mathcal{B})$ is a Birkhoff institution. Let $\mathcal{S} \subseteq Sig$ be a class of signature morphisms such that for each $\varphi \in \mathcal{S}$:*

1. *φ has weak (Φ,β)-amalgamation,*
2. *$Mod\varphi$ preserve $\mathcal{F}$-products, and*
3. *φ lifts $\mathcal{B}^{-1}$ and isomorphisms.*

Then the comorphism has the Craig $\mathcal{S}$-left interpolation property.

Right interpolation property for comorphisms. For the right interpolation property we apply the same technique like for the left interpolation, of matching the right-hand side square below to the interpolation square of Thm. 9.7 (the left-hand side square below).

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array} \qquad \begin{array}{ccc} (I,\Sigma) & \xrightarrow{1_{\Phi\Sigma}} & (I',\Phi\Sigma) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \Phi\varphi} \\ (I,\Omega) & \xrightarrow[1_{\Phi\Omega}]{} & (I',\Phi\Omega) \end{array}$$

We also use again the conclusion of Facts 9.33 and 9.34. The proof of Thm. 9.7 uses fully the Birkhoff institution hypothesis only on Σ and Σ_2, which means that it is enough to assume the Birkhoff institution hypothesis only for I. However since the semantic operator $\mathcal{U}$ is defined as $\mathcal{F}(\_)$, we also have to assume $\mathcal{F}$-products for I'. These lead to the following translation of Thm. 9.7 to the comorphism environment:

Proposition 9.36. *Consider an institution comorphism* $(\Phi,\alpha,\beta) : I \to I'$ *such that* $I = (Sig, Sen, Mod, \models, \mathcal{F}, \mathcal{B})$ *is a Birkhoff institution. Let* $\mathcal{S} \subseteq Sig$ *be a class of signature morphisms. We assume that:*

1. φ *has weak* (Φ,β)*-amalgamation for each* $\varphi \in \mathcal{S}$*,*
2. *the categories of models of* I' *have* $\mathcal{F}$*-products,*
3. β_Σ *preserves* $\mathcal{F}$*-products for each* $\Sigma \in |Sig|$*, and*
4. φ *lifts* $\mathcal{B}$ *for each* $\varphi \in \mathcal{S}$*.*

Then the comorphism has the Craig $\mathcal{S}$*-right interpolation property.*

Deja vu concrete interpolation properties, but this time by borrowing. By using the lifting properties provided by Prop. 9.8 and 9.13, from Propositions 9.35 and 9.36, we obtain the interpolation properties for comorphisms as shown in the following table.

I	I'	$\mathcal{S}$-left	$\mathcal{S}$-right
$\mathcal{UNIV}$, $\forall\vee$	$\mathcal{FOL}$	$ie*$	$ii'i$
$\mathcal{HCL}$	$\mathcal{FOL}$	$ie*$	
$\mathcal{EQL}$	$\mathcal{FOL}$	ie	
universal $\mathcal{FOL}$-atoms	$\mathcal{HCL}$		ii'

Based on the interpolation properties for comorphisms listed in the above table we can obtain the interpolation results of Corollaries 9.9 Cor. 9.14 for sub-institutions of $\mathcal{FOL}$, but this time via the borrowing route given by Prop. 9.31.

Exercises

9.14. [76] **Interpolation in $\mathcal{PA}$ by borrowing**
$\mathcal{PA}$ has Craig-Robinson $(Sig^{\mathcal{PA}}, (i{*}{*}))$ and $((i{*}{*}), Sig^{\mathcal{PA}})$-interpolation borrowed from $\mathcal{FOL}$ along

the comorphism $\mathcal{PA} \to \mathcal{FOL}^{\text{th}}$ encoding partial operations as relations (see Sect. 4.1). (*Hint:* Use Ex. 9.13 and Fact 9.32.) Apply this method for obtaining concrete interpolation results in other institutions such as $\mathcal{POA}$, $\mathcal{MBA}$, etc.

9.15. The institution comorphism $\mathcal{FOL} \to \mathcal{FOEQL}$ encoding relations as operations (see Sect. 3.3) has both Craig $(i**)$-left and right interpolation.

9.16. For each injective function $u\colon\ S \to S'$, the institution comorphism $(\overline{\Phi^u}, \overline{\alpha^u}, \overline{\beta^u})\colon\ \mathcal{FOL}^S \to \mathcal{FOL}^{S'}$ (see Ex. 3.28) has both the Sig^S-left and right interpolation properties.

9.17. [76] **Interpolation in $\mathcal{HNK}$**
In any institution with pushouts of signatures, a commuting square of signatures

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is a *quasi-pushout* when the signature morphism $\psi\colon\ \Sigma'' \to \Sigma'$ from the vertex Σ'' of the pushout of φ_1 and φ_2 to Σ' is conservative.

1. Prove a relaxed variant of Prop. 9.31 which replaces the condition that the signature translation functor Φ preserve pushouts by the slightly more general condition that Φ maps pushouts to quasi-pushouts.
2. Apply this upgraded variant of Prop. 9.31 for 'borrowing' interpolation from $\mathcal{FOL}$ to $\mathcal{HNK}$ through the comorphism $\mathcal{HNK} \to \mathcal{FOEQL}^{\text{th}}$ of Ex. 4.12. (*Hint:* The sentence translations $\alpha_{(S,F)}$ of the comorphism $\mathcal{HNK} \to \mathcal{FOEQL}^{\text{th}}$ are bijective.)

9.18. (from [67], corrected) The embedding comorphism $\mathcal{FOEQL} \to \mathcal{POA}$ has both Craig (ie)-left and right interpolation (*Hint:* Use the encoding comorphism $\mathcal{POA} \to \mathcal{FOL}^{\text{th}}$ for translating the left and the right interpolation problems of the given comorphism to interpolation problems in $\mathcal{FOL}$). Give a counterexample for the (ii)-right interpolation by considering the signatures $\Sigma = (\{s_1, s_2\}, \{a, b\colon\ \to s_1\})$ and Σ_2 which extends Σ with the operation $\sigma\colon\ s_1 \to s_2$. (*Hint:* Consider $E_1 = \{a \leq b, (\forall x, y\colon\ s_2)(x \leq y) \Rightarrow (x = y)\}$ and $E_2 = \{\sigma a = \sigma b\}$.)

Notes. The importance of interpolation in logic and model theory can be seen from [224, 42]. A recent monograph dedicated to interpolation in modal and intuitionistic logics is [110]. Interpolation also has numerous applications in computing science especially in formal specification theory [20, 96, 101, 25, 241, 33, 136, 137] but also in data bases (ontologies) [157], automated reasoning [196, 199], type checking [153], model checking [174], and structured theorem proving [6, 173]. This is only a partial account of this phenomenon and furthermore now and then new applications of interpolation in computing science pop up. A survey, but far from being exhaustive, about its applications to modularization of computing systems is [54].

The first pushout-based institution-independent formulation of Craig Interpolation appears in [226] but uses single sentences. This satisfied the need in formal specification theory to generalize interpolation from the conventional framework based on extensions of signatures to a framework involving arbitrary signature morphisms. The formulation of Ci with *sets* of sentences comes from [96] under the influence of Rodenburg's work on equational interpolation [214]; in particular note that (cf. [214]) equational logic satisfies the formulation of Ci with sets of sentences but not the single sentence version. The weak amalgamation square condition of Thm.s 9.3 and 9.7 is weaker than the corresponding assumptions in the literature that the interpolation square is a pushout [227, 96, 33, 32, 102, 213]. The concept of semantic interpolation and Thm. 9.3 have been introduced, respectively, proved in [205]. The interpolation result for Birkhoff institutions (Thm. 9.7) has been developed in [66]. Its equational instance has been developed in an abstract setting in [213]. This work is also the source for the (counter)example showing that the injectivity condition of φ_2 is necessary. The application of Thm. 9.7 to $\mathcal{FOL}$ interpolation by the Keisler-Shelah property (Cor. 9.11) has been noticed in [205].

That the equivalence between Robinson consistency and Craig interpolation relies upon (quasi-)compactness and the existence of negation and of conjunctions, other details of the actual institution being irrelevant, has been noticed within the framework of the so-called 'model-theoretic logics' by [194, 195]. A variant of Robinson consistency was defined for institutions in [226] following a variant of the corresponding property in $\mathcal{FOL}^1$. Our definition of Robinson consistency comes from [140] which follows another well-known definition. The first institution-independent proof of the equivalence between Robinson consistency and Craig interpolation appears in [226]. Robinson consistency (Thm. 9.20) is due to [140] where it has also been used to derive the $\mathcal{FOL}$ interpolation result of Cor. 9.22. This result, which appears also in [34] extended to the limit the previously known interpolation properties of $\mathcal{FOL}$ which appeared in [32]. The counterexample showing the necessity to have injectivity on sorts at least for one signature morphism comes from [32]. The case of many-sorted interpolation shows that the classification of many-sorted logics as "inessential variations" of one-sorted logic [179] is certainly wrong.

Craig-Robinson interpolation plays an important role in specification language theory, see [20, 96, 102]. The name "Craig-Robinson" interpolation has been used for instances of this property in [224, 241, 102] and "strong Craig interpolation" has been used in [96]. Some of the ideas behind Thm. 9.28 come from [102].

The interpolation property for comorphisms was formulated in [67], and the borrowing method for interpolation was developed in [76].

Chapter 10

Definability

The last core topic from model theory that we discuss in the institution-independent framework is definability theory. Traditionally, definability is considered to have a special relationship to interpolation. Partially, this can be explained by the fact that in $\mathcal{FOL}$ the main result of definability theory can be obtained as a consequence of the interpolation property. In this chapter we will understand also other dimensions of this connection, which are revealed only by the institution-independent approach. For instance, the abstraction process from the ordinary concept of definability to the institution-independent one follows a similar route to how we did for interpolation (Chap. 9). Then, axiomatizability properties play for definability a causal role that is strongly reminiscent of one of the interpolation-by-axiomatizability results of Chap. 9.

In the first section of this rather short chapter we explain definability first from the conventional concrete perspective and then from an institution-independent perspective. Without any doubt, the main concept of definability theory is the so-called 'definability property'. This is an equivalence property. One of the implications is obvious in the classical single-sorted $\mathcal{FOL}$ context, but it is not so much in the institution-independent setup. In the second section we develop some widely applicable conditions for this implication to hold. For the other implication, the more substantial one, we first establish it from Craig-Robinson interpolation. Then we establish it on a different basis, in the context of Birkhoff institutions. Like in the case of interpolation, definability can also be establish by the 'borrowing' technique. A proposed exercise is dedicated to this technique.

10.1 What is definability?

Definability theory provides answers to the question to what extent implicit definitions can be made explicit. In order to make a preliminary sense of this it is helpful to discuss an example. Perhaps the most natural way to define the concept of group involves two stages. At the first stage, we consider the concept of monoid. In the additive notational style, this is a single-sorted algebra with two operations, $\_+\_$, of arity 2, and a constant

0. The theory of monoids consists of the associativity equation for $\_+\_$ together with the two identity equations for 0. All these are very familiar to us. At the second stage the signature of the theory of monoids – let us denote it Σ – gets extended with an unary operation, called the 'inverse operation' and denoted $-$, and the theory with a couple of new axioms:

$$(\forall x)x + -x = 0 \text{ and } (\forall x)(-x) + x = 0.$$

Let (Σ', E') denote this theory of groups. We may note two aspects of this extension from the theory of monoids to the theory of groups.

- Given a monoid M, there is *at most* one possibility to expand it to a group. We cannot have two inverses for an element. This utterly simple fact is one of the first exercises that are often given to students of group theory. In model-theoretic terminology we say that the inverse operation is *implicitly defined* by the theory of groups. This is the 'semantic side' of definability.

- There is also a 'syntactic side' of definability, which is less obvious, and refers to the possibility of systematic elimination of the inverse operation from $\mathcal{FOL}$ sentences in the context of group theory. Otherwise said, if we think in terms of $\mathcal{FOL}^{\text{th}}$, then this means that in the theory of groups (Σ', E'), any (Σ', E')-sentence ρ is equivalent to a (Σ', E')-sentence E_ρ that does not contain any occurence of the inverse operation $-$. In other words, E_ρ is a Σ-sentence. There is a subtle 'smell' of interpolation here, isn't it? Moreover, this applies to any extension of Σ with new symbols, in other words ρ and E_ρ may contain other symbols too, but the main point remains: the inverse operation can be eliminated from E_ρ. This property is called the *explicit definability* of the inverse operation. To understand the extent to which this is non-trivial first try to find an E_ρ when ρ is $(\exists x)(-x = -a + x)$. Then show that it does the job. If you failed at the first step try the following choice for E_ρ:

$$(\forall a_1, x_1)(\exists x)(a_1 + a = 0) \wedge (x_1 + x = 0) \Rightarrow (x_1 = a_1 + x).$$

The main issue of definability theory is to establish the equivalence between implicit and explicit definability; this is called the 'definability property'. In the conventional framework of $\mathcal{FOL}$, that explicit implies implicit definability is almost trivial, hence the 'definability property' usually designates the other implication. The main result of the classical definability theory is that the 'definability property' holds in $\mathcal{FOL}^1$ (the single-sorted version of $\mathcal{FOL}$). However, in a more general context we cannot expect such kind of unconditional result, we rather have to formulate adequate conditions for this to work. Now let us formulate all these concepts in an institution-independent manner.

Institution-theoretic definability. After you already studied the concept of institution-theoretic interpolation, you, the reader, with institution-theoretic definability will probably have a strong deja vu feeling.

- The first step towards defining institution-independent definability is to consider arbitrary signature morphisms rather than just signature extensions with a single operation symbol (like in our example of the inverse operation for groups). This also allows for a full abstract approach in the spirit of category theory which treats elements as arrows. Let $\varphi : \ \Sigma \to \Sigma'$ be a signature morphism and E' be a Σ'-theory. Then φ is *defined implicitly* by E' if the reduct functor

$$Mod(\Sigma', E') \longrightarrow Mod\Sigma' \xrightarrow{Mod\varphi} Mod\Sigma$$

  is injective on objects (models).

- In the case of explicit definability we perform some further generalisations as follows.
  - E_ρ is considered to be a set of sentences rather than a single sentence. The motivation for this is similar to that from interpolation when interpolants are considered sets of sentences rather than single sentences.
  - In the group theory example we have discussed that signature of E_ρ can be an 'extension' of Σ. At the abstract level this means that $E_\rho \subseteq Sen\Sigma_1$ for a signature morphism $\theta : \ \Sigma \to \Sigma_1$. Then what about the signature of ρ? We have two different 'extensions' of Σ, one represented by φ and the other one by θ. The way to put them together is to consider a pushout like below:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ \Sigma_1 & \xrightarrow[\varphi_1]{} & \Sigma'_1 \end{array} \qquad (10.1)$$

    Then $\rho \in Sen\Sigma'_1$. Now we have the context for formulating the explicit definability property at the abstract level. A signature morphism $\varphi : \ \Sigma \to \Sigma'$ is *defined explicitly* by a Σ'-theory E' when for each pushout square like (10.1) and each sentence $\rho \in Sen\Sigma'_1$ there exists $E_\rho \subseteq Sen\Sigma_1$ such that $\rho \models\!\!\!\dashv \varphi_1 E_\rho$ in the context of the theory $(\Sigma'_1, \theta' E')$. By the concept of institution of theories, I^{th}, we can write this as $\rho \models\!\!\!\dashv_{(\Sigma'_1, \theta' E')} \varphi_1 E_\rho$. A less compact way to formulate this is that for each Σ'_1-model M such that $M \models \theta' E'$ we have that $M \models \rho$ if and only if $M \models \varphi_1 E_\rho$.

The definability property. This is the main concept of definability theory. In the institution-independent setup it is like this. We say that a signature morphism φ *has the definability property* when any given theory defines φ explicitly if and only if it defines φ implicitly. This extends also to classes $\mathcal{S}$ of signature morphisms.

Exercises

10.1. [204] **Composability of definability**

- In any institution the classes of signature morphisms which are defined implicitly / explicitly form a category. Moreover, if the institution is semi-exact, these classes of signature morphisms are also stable under pushouts.
- In any semi-exact institution with universal $\mathcal{D}$-quantification for a class $\mathcal{D}$ of signature morphisms that are stable under pushouts, for any pushout square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ \Sigma_1 & \xrightarrow[\varphi_1]{} & \Sigma'_1 \end{array}$$

  such that $\theta \in \mathcal{D}$ and has the model expansion property, φ has the definability property with respect to E' whenever φ_1 has the definability property with respect to $\theta' E'$.

10.2. [204] **Borrowing definability**
Let $(\Phi, \alpha, \beta) : I \to I'$ be an institution comorphism. We say that an I-signature morphism $\varphi : \Sigma_1 \to \Sigma_2$ is (Φ, β)-*precise* whenever the function $|Mod'(\Phi\Sigma_2)| \to |Mod'(\Phi\Sigma_1) \times |Mod\Sigma_2|$ mapping each M'_2 to $\langle M'_2 \restriction_{\Phi\varphi}, \beta_{\Sigma_2} M'_2 \rangle$ is injective. We say that the comorphism (Φ, α, β) is *precise* when each I-signature morphism is (Φ, β)-precise.

- What is the connection between (Φ, β)-precise signature morphisms and the signature morphisms that admit the (Φ, β)-amalgamation?
- Establish which of the comorphisms introduced in Sec. 3.3 and in Sec. 4.1 are precise.
- *Borrowing implicit definability.* For any (Φ, α, β)-precise signature morphism φ and theory E', $\Phi\varphi$ is defined implicitly by $\alpha E'$ if φ is defined implicitly by an E'.
- *Borrowing explicit definability.* If
  1. $(\Phi, \alpha, \beta) : I \to I'$ is conservative,
  2. Φ preserves pushouts, and
  3. α is surjective modulo the semantic equivalence $\models\!\mid$,

  then any I-signature morphism φ is defined (finitely) explicitly by a theory E' if $\Phi\varphi$ is defined (finitely) explicitly by $\alpha E'$.
- Under the assumptions at the previous item, any (Φ, α, β)-precise signature morphism φ has the definability property if $\Phi\varphi$ has the definability property.

10.2 Explicit implies implicit definability

In this section we study the implication of implicit definability from the explicit definability. Before formulating sufficient conditions at the general level it is helpful to see how this works in $\mathcal{FOL}$. Then we will get a better understanding of the conditions for this implication to work.

Proposition 10.1. *In $\mathcal{FOL}$ any $(s**)$-morphism of signatures is defined implicitly whenever it is defined explicitly.*

Proof. Let $\varphi : \Sigma \to \Sigma'$ be any $(s**)$-morphism of signature which is defined explicitly by $E' \in Sen\Sigma'$. Let M', N' be (Σ', E')-models that share the same φ-reduct, denoted A. We have to show that $M' = N'$.

- Let s' be any sort of Σ'. Because φ is surjective on the sorts, there exists a sort s of Σ such that $\varphi s = s'$. Then $M'_{s'} = N'_{s'} = A_s$.
- In the case of the operation and of relation symbols we use the explicit definability hypothesis as follows. In the definition of institution-independent explicit definability we set θ to $\iota_\Sigma A$, the elementary extension of A. We let M'_1, N'_1 to be the amalgamations of M with A_A, respectively of N' with A_A. For any $\rho \in Sen\Sigma'_1$ we have that

1	$M'_1, N'_1 \models \theta' E'$	$M', N' \models E'$, Satisfaction Condition
2	$M'_1, N'_1 \in \rho^*$ if and only if $M'_1, N'_1 \in (\varphi_1 E_\rho)^*$	explicit definability
3	$M'_1, N'_1 \in (\varphi_1 E_\rho)^*$ if and only if $A_A \in E_\rho{}^*$	1, Satisfaction Condition
4	$M'_1, N'_1 \in \rho^*$ if and only if $A_A \in E_\rho{}^*$	2, 3.

Thus $M'_1 \equiv N'_1$. Now, by considering in the role of ρ sentences of the form $(\theta'\sigma)(x_1, \ldots, x_k) = y$ and $(\theta'\pi)(x_1, \ldots, x_k)$ (with $x_1, \ldots, x_k, y$ being elements of A of appropriate sorts), we obtain that the operation symbols $(\theta'\sigma)$ and the relation symbols $(\theta'\pi)$ get interpreted the same by M'_1 and N'_1. Consequently, σ and π get also the same interpretations in M' and in N'.

□

What does this example teach us in terms of being able to formulate abstract applicable conditions for the implication of implicit from explicit definability to hold? How can we replicate this proof at an abstract level?

1. The equality $M' = N'$ is established in the signature Σ'_1 (as $M'_1 = N'_1$) where the elements of M' and N' are available as syntactic entities (constants) by using the elementary extension given by their common reduct. M'_1, N'_1 are just the expansions of M', N', respectively, that interpret their elements (which are all shared) by themselves.
2. The elementary equivalence $M'_1 \equiv N'_1$ was proved from the explicit definability hypothesis, by using that both models share the same φ_1-reduct. This piece of the proof has an obvious institution-independent character.
3. The equality $M'_1 = N'_1$ has been derived from the elementary equivalence $M'_1 \equiv N'_1$ by considering the atomic sentences that define the interpretations of the operation and relation symbols by M' and N'. $M'_1 \equiv N'_1$ implies that each atom in the extended signature is satisfied either by none or by both models, which means that each symbol newly added by φ gets the same interpretation in M' and N'. At the institution-independent level this has to be axiomatised as there we cannot consider such kind of sentences becasue these are institution-dependent.

4. It has been crucial that the elements of the shared reduct A cover all elements of M' and N', which can be guaranteed only by the surjectivity of φ on the sort symbols. At the abstract level this is also covered by the axiom that $M'_1 \equiv N'_1$ implies $M'_1 = N'_1$ because without this surjectivity the axiom would simply not hold in concrete situations such as in $\mathcal{FOL}$.

The proof of Prop. 10.1 tells us something else, that it uses only some very basic properties of $\mathcal{FOL}$, such as elementary diagrams and model amalgamation. In the case of the former it uses only the elementary extension, no need for the sentences of the diagrams. This is the sense of this implication being almost trivial, that its proof does no require any substantial model-theoretic result.

Tight signature morphisms

The concept that we are going to introduce now represents the core part in the axiomatisation of the conditions for the derivation of the implicit from the explicit definability at the institution-independent level. In any institution with model amalgamation and with diagrams ι, a signature morphism $\varphi : \Sigma \to \Sigma'$ is ι-*tight* when for all Σ'-models M' and N' with a common φ-reduct A and for any pushout of signature morphisms as in the diagram,

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle \iota_\Sigma A}\downarrow & & \downarrow{\scriptstyle \theta'} \\ \Sigma_A & \xrightarrow[\varphi_1]{} & \Sigma'_1 \end{array}$$

$M'_1 \equiv N'_1$ implies $M' = N'$, where M'_1, N'_1, respectively, are the unique amalgamations of M', N', respectively, with A_A (the initial model of the diagram of A).

Note that the concept of ι-tight signature morphisms does not involve the sets of sentences E_A that constitute de diagrams, so in principle it can be formulated more generally only in terms of the elementary extensions $\iota_\Sigma A$ and of the models A_A. Otherwise said, we can do it only by requiring that for each model Σ-model A there exists a designated signature morphism $\iota_\Sigma A : \Sigma \to \Sigma$ and a designated $\iota_\Sigma A$-expansion A_A of A.

The following helps to characterize concretely the tight signature morphisms in institutions.

Proposition 10.2. *Let $\varphi : \Sigma \to \Sigma'$ be a ι-tight signature morphism in a semi-exact institution with diagrams ι. Then any two Σ'-models that are isomorphic by a φ-expansion of an identity, are equal.*

Proof. Let $h' : M' \to N'$ be a Σ'-isomorphism such that $h'\restriction_\varphi$ is identity. Let $M'\restriction_\varphi = N'\restriction_\varphi = A$. For the diagram from the definition of tight signature morphisms consider the amalgamation of h' with 1_{A_A}; this is also an isomorphism. Therefore $M'_1 \cong N'_1$, hence $M'_1 \equiv N'_1$. By the definition of φ being tight, we get that $M' = N'$. $\square$

Tight signature morphisms in $\mathcal{FOL}$. By virtue of Prop. **??** we expect that $(s**)$-morphisms in $\mathcal{FOL}$ aretight. Cor. 10.3 below says even more, that in $\mathcal{FOL}$ 'tight' is the sharp abstract formulation of $(s**)$-morphisms, which enjoy the implication of implicit from explicit definability. Of course, similar situations, based on similar arguments, are expected in other concrete institutions too.

Corollary 10.3. *A $\mathcal{FOL}$ signature morphism φ is ι-tight (for the standard system of diagrams ι) if and only if φ is an $(s**)$-morphism.*

Proof. Let $\varphi\colon\ \Sigma \to \Sigma'$.

- The surjectivity on the sorts is necessary because otherwise, given a Σ'-model M' we may consider another Σ'-model N' which is like M' but interprets the sorts outside the image of $\varphi\colon\ \Sigma \to \Sigma'$ differently but isomorphically to M'. This gives a non-identity Σ'-isomorphism between different Σ'-models, that expands a Σ-identity, thus contradicting Prop. 10.2.
- The surjectivity on the sorts is also sufficient by an argument that repeats the last paragraph in the proof of Prop. 10.1.

□

Explicit implies implicit definability

Proposition 10.4. *In any institution having model amalgamation and diagrams ι, each ι-tight signature morphism is defined implicitly whenever it is defined explicitly.*

Proof. We employ the familiar notations of this section. The proof aggregates two arguments as follows:

- For $M'_1 \equiv N'_1$ we can copy-paste the corresponding argument from the proof of Prop. 10.1 as that piece of proof did not involve any of the concrete specificities of $\mathcal{FOL}$, being thus institution-independent.
- Then we can use the ι-tight hypothesis.

□

We can see that the $\mathcal{FOL}$ result of Prop. 10.1 follows from the general result of Prop. 10.4 plus the characterisation of the tight $\mathcal{FOL}$ morphisms of signatures given by Cor. 10.3.

Exercises

10.3. A $\mathcal{PA}$ signature morphism is tight if and only if it is surjective on sorts.

10.3 Definability by interpolation

The rest of this chapter is dedicated to the hard implication of the relationship between implicit and explicit definability, i.e., implicit implies explicit definability. The following result obtains this as an application of interpolation. We do it with Craig-Robinson interpolation. This is different from the well-known proof of the corresponding result in $\mathcal{FOL}$ where apparently the weaker Craig form of interpolation in used. But this may be misleading because in the $\mathcal{FOL}$ proof the fact that $\mathcal{FOL}$ has implications is used also. Our reliance on Craig-Robinson interpolation allows for a wider range of applications as Craig-Robinson interpolation is weaker than Craig interpolation plus implications, and there are indeed interesting situations that fall within the former case but outside the latter.

Theorem 10.5. *In any institution with model amalgamation and having Craig-Robinson $(\mathcal{L},\mathcal{R})$-interpolation for classes $\mathcal{L}$ and $\mathcal{R}$ of signature morphisms which are stable under pushouts, any signature morphism in $\mathcal{L}\cap\mathcal{R}$ is defined explicitly when is defined implicitly.*

Proof. Let $(\varphi\colon\ \Sigma \to \Sigma') \in \mathcal{L}\cap\mathcal{R}$ be defined implicitly by $E' \subseteq Sen\Sigma'$. We consider a pushout of φ with an arbitrary signature morphism $\theta\colon\ \Sigma \to \Sigma_1$ and let ρ be any Σ_1'-sentence.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ \Sigma_1 & \xrightarrow[\varphi_1]{} & \Sigma_1' \end{array} \tag{10.2}$$

Now we consider a pushout of φ_1 with itself:

$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{\varphi_1} & \Sigma_1' \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ \Sigma_1' & \xrightarrow[\theta_2]{} & \Sigma'' \end{array}$$

- First, we show that

1 $\theta_1(\theta' E') \ \cup\ \theta_1\rho \ \cup\ \theta_2(\theta' E') \ \models_{\Sigma''}\ \theta_2\rho.$

Consider a Σ''-model M'' such that $M'' \models \theta_1(\theta' E') \cup \theta_1\rho \cup \theta_2(\theta' E')$. We have that

2 $M''\restriction_{\theta_1}\restriction_{\theta'}\restriction_{\varphi} = M''\restriction_{\theta_1}\restriction_{\varphi_1}\restriction_{\theta} = M''\restriction_{\theta_2}\restriction_{\varphi_1}\restriction_{\theta} = M''\restriction_{\theta_2}\restriction_{\theta'}\restriction_{\varphi}$ $\quad\varphi;\theta' = \theta;\varphi_1$, $\varphi_1;\theta_1 = \varphi_1;\theta_2$

3 $M''\restriction_{\theta_k}\restriction_{\theta'} \models E'$, $k = 1,2$ $\quad M'' \models \theta_k(\theta' E')$, Satisfaction Condition

4 $M''\restriction_{\theta_1}\restriction_{\theta'} = M''\restriction_{\theta_2}\restriction_{\theta'}$ $\quad$ 2, 3, φ implicitly defined by E'

5 $M''\restriction_{\theta_1}\restriction_{\varphi_1} = M''\restriction_{\theta_2}\restriction_{\varphi_1}$ $\quad\varphi_1;\theta_1 = \varphi_1;\theta_2$

6 $M''\restriction_{\theta_1} = M''\restriction_{\theta_2}$ $\quad$ 4, 5, unique model amalgamation ((10.2) pushout, model amalgamation square)

7	$M''\upharpoonright_{\theta_1} \models \rho$	$M'' \models \theta_1\rho$, Satisfaction Condition
8	$M'' \models \theta_2\rho$	6, 7, Satisfaction Condition.

- Now we apply Craig-Robinson interpolation property to relation 1:

9	$\varphi_1 \in \mathcal{L} \cap \mathcal{R}$	$\varphi \in \mathcal{L} \cap \mathcal{R}$, $\mathcal{L}$, $\mathcal{R}$ stable under pushouts
10	there exists $E_\rho \subseteq Sen\Sigma_1$ s.th. $\theta' E' \cup \{\rho\} \models \varphi_1 E_\rho$, $\theta' E' \cup \varphi_1 E_\rho \models \rho$	1, 9, CRi.

The conclusion 10 shows that φ is explicitly defined by E'. □

Definability in $\mathcal{FOL}$. By applying the result of Thm. 10.5 to $\mathcal{FOL}$ we obtain a definability property that is more general than what is known from the classical theory of definability.

Corollary 10.6. *In $\mathcal{FOL}$, any $(i**)$-morphism of signatures is explicitly defined when it is implicitly defined. Consequently, in $\mathcal{FOL}$, the definability property holds for the $(b**)$-morphisms of signatures.*

Proof. For the first part of the conclusion we use the $\mathcal{FOL}$ instance of Thm. 10.5 in conjunction with the Craig-Robinson interpolation property of $\mathcal{FOL}$ as given Cor. 9.25. Then by Cor. 10.1 we extend this to the second conclusion. □

Some interesting instances of Thm. 10.5 in sub-institutions of $\mathcal{FOL}$ emerge from a Craig-Robinson interpolation result in Sec. 14.3. Those applications will illustrate well the power of reliance on Craig-Robinson interpolation rather than on Craig interpolation because they refer to institutions that do not have implications.

Exercises

10.4. [204] **Definability by interpolation in $\mathcal{PA}$**
In $\mathcal{PA}$, any $(i**)$-morphism of signatures is defined explicitly if it is defined implicitly. Prove this in two different ways:

1. directly, by Thm. 10.5, and
2. by borrowing along the relational encoding comorphism $\mathcal{PA} \to \mathcal{FOL}^{\mathrm{th}}$ by using the result of Ex. 10.2 (*Hint:* A theory morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is defined implicitly, respectively explicitly, by E'', in the institution of theories I^{th} if and only if $\varphi : \Sigma \to \Sigma'$ is defined implicitly, respectively explicitly, by $E' \cup E''$, in the base institution I.)

10.4 Definability by axiomatizability

In this section we develop another method to obtain definability properties which relies upon the axiomatizability properties of the institution.

Weakly lifting relations. In Sec. 9.3 we have introduced a concept of lifting relations which have been used to derive concrete instances of the general interpolation-by-axiomatizability Thm. 9.7. For the purpose of this section we introduce a weaker variant of this concept. Let $\varphi\colon \Sigma \to \Sigma'$ be a signature morphism and $\mathcal{R} = \langle \mathcal{R}_\Sigma, \mathcal{R}_{\Sigma'} \rangle$ with $\mathcal{R}_\Sigma \subseteq |Mod\Sigma| \times |Mod\Sigma|$ and $\mathcal{R}_{\Sigma'} \subseteq |Mod\Sigma'| \times |Mod\Sigma'|$ be a pair of binary relations. We say that φ *lifts weakly* $\mathcal{R}$ if and only if for each $M', N'' \in |Mod\Sigma'|$ and $N \in |Mod\Sigma|$ if $\langle M'\restriction_\varphi, N''\restriction_\varphi \rangle \in \mathcal{R}_\Sigma$, then there exists $N' \in |Mod\Sigma'|$ such that $N'\restriction_\varphi = N''\restriction_\varphi$ and $\langle M', N' \rangle \in \mathcal{R}_{\Sigma'}$.

$$
\begin{array}{ccccc}
\Sigma & \quad & Mod\Sigma & \quad & M'\restriction_\varphi \xrightarrow{\;\mathcal{R}_\Sigma\;} N''\restriction_\varphi = N'\restriction_\varphi \\
\varphi \big\downarrow & & \big\uparrow Mod\varphi & & \\
\Sigma' & & Mod\Sigma' & & M' \xrightarrow[\mathcal{R}_{\Sigma'}]{} (\exists) N'
\end{array}
$$

Fact 10.7. *A signature morphism lifts weakly (a pair of relations) $\mathcal{R}$ whenever it lifts $\mathcal{R}$.*

The reason for Fact 10.7 is evident: in the weak variant of lifting we restrict the models N to those that already admit a φ-expansion.

Definability in Birkhoff institutions

Theorem 10.8. *Consider a Birkhoff institution $(Sig, Sen, Mod, \models, \mathcal{F}, \mathcal{B})$ with model amalgamation and a class $\mathcal{S} \subseteq Sig$ of signature morphisms which is stable under pushouts and such that for each $\varphi \in \mathcal{S}$*

1. *$Mod\varphi$ preserves $\mathcal{F}$-products, and*
2. *φ lifts weakly $\mathcal{B}^{-1}$.*

Then any signature morphism in $\mathcal{S}$ is defined explicitly if it is defined implicitly.

Proof. Let $\varphi \in \mathcal{S}$. If $\varphi\colon \Sigma \to \Sigma'$ is defined implicitly by E', then we show it is defined explicitly by E' too. Therefore consider any pushout square of signature morphisms for the span $\Sigma_1 \xleftarrow{\theta} \Sigma \xrightarrow{\varphi} \Sigma'$

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\;\varphi\;} & \Sigma' \\
\theta \big\downarrow & & \big\downarrow \theta' \\
\Sigma_1 & \xrightarrow[\varphi_1]{} & \Sigma'_1
\end{array}
$$

and any Σ'_1-sentence ρ. Let $\mathbb{M}'_1 = (\theta' E' \cup \{\rho\})^*$ and $E_\rho = (\mathbb{M}'_1\restriction_{\varphi_1})^*$.

- We first show $\theta' E' \cup \{\rho\} \models \varphi_1 E_\rho$. Consider M'_1 a model of $\theta' E' \cup \{\rho\}$. We have that:

1	$M'_1\restriction_{\varphi_1} \in \mathbb{M}'_1\restriction_{\varphi_1}$	definition of $\mathbb{M}'_1$
2	$M'_1\restriction_{\varphi_1} \models E_\rho$	1, $E_\rho = (\mathbb{M}'_1\restriction_{\varphi_1})^*$

3	$M'_1 \models \varphi_1 E_\rho$	2, Satisfaction Condition.

That was the easy part. Now follows the more difficult part.

- Now we show that $\theta' E' \cup \varphi_1 E_\rho \models \rho$. Consider M'_1 a Σ'_1-model satisfying $\theta' E' \cup \varphi_1 E_\rho$. We prove that $M'_1 \models \rho$. We have that:

4	$M'_1\restriction_{\varphi_1} \models E_\rho$	$M'_1 \models \varphi_1 E_\rho$, Satisfaction Condition
5	$M'_1\restriction_{\varphi_1} \in (\mathbb{M}'_1\restriction_{\varphi_1})^{**}$	4, $E_\rho = (\mathbb{M}'_1\restriction_{\varphi_1})^*$
6	$(\mathbb{M}'_1\restriction_{\varphi_1})^{**} = \mathcal{B}^{-1}_{\Sigma_1}(\mathcal{F}\ \mathbb{M}'_1\restriction_{\varphi_1})$	5, Birkhoff institution
7	$\mathcal{F}\ \mathbb{M}'_1\restriction_{\varphi_1} = \mathrm{Iso}((\mathcal{F}\mathbb{M}'_1)\restriction_{\varphi_1})$	$Mod\varphi_1$ preserves $\mathcal{F}$-products ($\varphi_1 \in \mathcal{S}$, $\mathcal{S}$ stable under pushouts)
8	$\mathcal{F}\mathbb{M}'_1 \subseteq \mathcal{B}^{-1}_{\Sigma'_1}(\mathcal{F}\mathbb{M}'_1) = \mathbb{M}'^{**}_1 = \mathbb{M}'_1$	$\mathcal{B}_{\Sigma'_1}$ reflexive, Birkhoff institution, $\mathbb{M}'_1$ elementary
9	$M'_1\restriction_{\varphi_1} \in \mathcal{B}^{-1}_{\Sigma_1}(\mathbb{M}'_1\restriction_{\varphi_1})$	5, 6, 7, $\mathcal{B}_\Sigma$ closed under isomorphisms, 8

From 9 we obtain the existence of a Σ'_1-model $N'_1 \in \mathbb{M}'_1$ such that $\langle M'_1\restriction_{\varphi_1}, N'_1\restriction_{\varphi_1}\rangle \in \mathcal{B}_{\Sigma_1}$. Since $\mathcal{S}$ is stable under pushouts and $\varphi \in \mathcal{S}$ we have that $\varphi_1 \in \mathcal{S}$ too, hence φ_1 lifts weakly $\mathcal{B}^{-1}$. Thus there exists a Σ'_1-model P'_1 such that $P'_1\restriction_{\varphi_1} = M'_1\restriction_{\varphi_1}$ and $\langle P'_1, N'_1\rangle \in \mathcal{B}_{\Sigma'_1}$. Then:

10	$\mathcal{B}^{-1}_{\Sigma'_1}\mathbb{M}'_1 \ \subseteq\ \mathcal{B}_{\Sigma'_1}(\mathcal{F}\mathbb{M}'_1) \ =\ \mathbb{M}'_1$	$\{\{*\}\} \in \mathcal{F}$, 8
11	$P'_1 \in \mathcal{B}^{-1}_{\Sigma'_1} N'_1 \ \subseteq\ \mathcal{B}^{-1}_{\Sigma'_1}\mathbb{M}'_1$	$\langle P'_1, N'_1\rangle \in \mathcal{B}_{\Sigma'_1}$, $N'_1 \in \mathbb{M}'_1$
12	$P'_1 \in \mathbb{M}'_1$	11, 10
13	$M'_1, P'_1 \models \theta' E'$	12, $\mathbb{M}'_1 = (\theta' E' \cup \{\rho\})^*$, $M'_1 \models \theta' E'$
14	$M'_1\restriction_{\theta'}, P'_1\restriction_{\theta'} \models E'$	13, Satisfaction Condition
15	$M'_1\restriction_{\theta'}\restriction_{\varphi} = M'_1\restriction_{\varphi_1}\restriction_{\theta} = P'_1\restriction_{\varphi_1}\restriction_{\theta} = P'_1\restriction_{\theta'}\restriction_{\varphi}$	$\varphi;\theta' = \theta;\varphi_1$, $M'_1\restriction_{\varphi_1} = P'_1\restriction_{\varphi_1}$
16	$M'_1\restriction_{\theta'} = P'_1\restriction_{\theta'}$	14, 15, φ implicitly defined by E'.

By the uniqueness aspect of the model amalgamation property, from $M'_1\restriction_{\varphi_1} = P'_1\restriction_{\varphi_1}$ and $M'_1\restriction_{\theta'} = P'_1\restriction_{\theta'}$ (16), we get that $M'_1 = P'_1$. Thus $M'_1 \models \rho$ because $P'_1 \in \mathbb{M}'_1$ (12) and $\mathbb{M}'_1 = (\theta' E' \cup \{\rho\})^*$.

We have therefore shown that $\theta' E' \cup \{\rho\} \models \varphi_1 E_\rho$ and $\theta' E' \cup \varphi_1 E_\rho \models \rho$. □

We may note a great similarity between the conditions of the definability Thm. 10.8 and the Craig interpolation Thm. 9.12. The differences are as follows:

- Thm. 10.8 requires a stronger form of model amalgamation than Thm. 9.12.
- Thm. 9.12 requires a stronger form of lifting than Thm. 10.8.
- Thm. 9.12 has a condition on lifting isomorphisms.

How do these translate in the applications? Not much actually, they are merely technical differences. It is rare to find institutions that have weak model amalgamation but not standard model amalgamation. The lifting of isomorphisms generally amounts to the mapping on sorts being injective, which is a condition that pops up when lifting $\mathcal{B}^{-1}$. On the other hand, the weakening of the lifting of $\mathcal{B}^{-1}$ may have more meaning in the applications. For instance if we strengthen the concept of sub-model ($\mathcal{S}_w$ and $\mathcal{S}_c$) to inclusions rather than injections we can get that in $\mathcal{FOL}$, the $(*e*)$-morphisms lift those precisely because an expansion of the model N is already available. This class of signature morphisms is wider than the class of the $(ie*)$-morphisms from the applications of the interpolation Thm. 9.12. But here it is crucial that we consider inclusions rather than injections in order to be able to lift the sub-model homomorphisms also. However, if did that then we could not use the sub-models relations directly as the Birkhoff relation $\mathcal{B}$ because this has to be closed under isomorphisms. But, as it is often the case, technical differences have the potential to translate into differences at the level of the applications, which means that it is still an open issue to find meaningful applications that exploit the weakening of the lifting condition.

In concrete terms, what do we really get from Thm. 10.8? We can use the lifting properties of Prop. 9.13 and get from Thm. 10.8 some concrete definability results as follows.

Corollary 10.9. *In the following institutions, any signature morphism in $\mathcal{S}$ is defined explicitly when it is defined implicitly.*

institution	$\mathcal{S}$
$\mathcal{UNIV}$ universal $\mathcal{FOL}_{\infty,\omega}$-sentences $\mathcal{HCL}$, $\mathcal{HCL}_{\infty,\omega}$, $\forall\vee$, and $\forall\vee_\infty$	$ie*$
universal $\mathcal{FOL}$-atoms	iei
$\mathcal{EQL}$	ie

The results of Cor. 10.9 allow us to have a more clear picture of the impact of the general definability-by-axiomatizability result of Thm. 10.8.

- In the case of the institutions of Cor. 10.9, to great extent, the two sides of the definability property are mutually exclusive. On the one hand, we need the surjectivity of the mapping on the sorts (see Prop. 10.1). On the other hand, we need the injectivity of the same mapping (cf. Cor. 10.9). This means bijectivity. Furthermore, in this context, the encapsulation condition leads also to the bijectivity of the mappings on the operation symbols. This means that the scope of the definability property gets restricted quite severely, only to the relation symbols.
- This situation tells us also that from the general perspective that goes beyond first-order logic, that the traditional view that one of the sides of the definability property is trivial, and somehow taken for granted, namely the implication of the implicit from the explicit definability, is a misconception.

- The strong similarity between the conditions of Thm. 10.8 leads to the following question: would it be possible to obtain meaningful definability properties for institutions such as those from Cor. 10.9 by joining together the results of Theorems 9.12 and 10.5? The immediate answer is no, because because Thm. 9.12 is a Craig interpolation result while Thm. 10.5 requires a Craig-Robinson interpolation property which in the case of the institutions of Cor. 10.9 is not immediately available through something like Prop. 9.24 because these lack implications.
- However, in Sec. 14.3 we will see how we can have a transition from Ci to CRi in the absence of implications, based on the sophisticated technique of the 'Grothendieck institutions'. This means that we will be able to reverse the negative answer at the previous item to a positive one. But in concrete situations, by this alternative route we will not achieve more than Cor. 10.9 because the classes $\mathcal{S}$ of the signature morphisms are identical.

Exercises

10.5. [204] **Definability in $\mathcal{PA}$ by axiomatizability**
By using the results of Ex. 8.13 and 8.9 develop definability properties in $QE_1(\mathcal{PA})$ and $QE_2(\mathcal{PA})$. In the case of $QE_1(\mathcal{PA})$, do this in two alternative ways:

1. Directly by Thm. 10.8, and
2. By borrowing it from $\mathcal{HCL}_{\infty,\omega}$ along the relational encoding comorphism $\mathcal{PA} \to \mathcal{FOL}^{\text{th}}$ by using the result of Ex. 10.2 in the style of Ex. 10.4.

Notes. The material of this chapter is based on [204]. This includes the concept of definability for signature morphisms as well as the definability Theorems 10.5 and 10.8.

The definability by interpolation Thm. 10.5 is a generalization of the conventional concrete Beth definability theorem in $\mathcal{FOL}^1$ of [21]. While traditional proofs of Beth's theorem use Craig interpolation and implication, the proof given in [204] uses only Craig-Robinson interpolation, being thus applicable to institutions without semantic implication.

The reformulation of definability for sets of sentences rather than single sentences of [204] settled the definability concept in a proper form applicable to institutions without conjunctions. The general definability by axiomatizability Thm. 10.8 and its instances in logics such as $\mathcal{HCL}$ owe much to this reformulation of definability.

Part III

Extensions

Chapter 11

Institutions with Proofs

Logic is both the science and the art of reasoning. Not only of reasoning, but rather of *correct* reasoning. These concepts are far from being straightforward. One of the greatest achievements of mathematical logic is that it developed clear approaches to reasoning concepts in form of mathematical objects. The central reasoning concept is that of (logical) *consequence*, which tells us when a certain sentence ρ can be 'deduced' from some set of sentences E. From the point of view of model theory this means $E \models \rho$, i.e. that ρ is a semantic consequence of E. This concept of logical consequence is prevalent in mathematical reasoning because mathematics is semantic. On the other hand, there is also the idea of semantic-free reasoning – called *natural deduction* in some logic literature – that is prevalent especially in the area of formal reasoning and the applications of that, especially in connection to various computing science paradigms. Ideally, this means zero reliance on any aspect that is connected to models and the satisfaction relation. However, this is an utopia, while it may appear like this in the foreground, in the background we need the meta-level of models and satisfaction at least in order to support the concept of *correctness of reasoning*. In principle, formal reasoning does not consider that, but then one cannot ignore the issue of correctness, without which the very concept of reasoning loses its meaning. We insist on the word 'formal' because in common practice, including mathematical practice, reasonings / proofs have a rather strong informal character. Leaving aside the semantic arguments, they still have the aspect of a social event in which the audience just has to be convinced of the validity of the argument, many reasoning steps that otherwise appear in formal arguments being skipped. Without such skips the arguments tend to become monstrously complex and tedious, thus severely hindering understanding. Nevertheless, this informality does not necessarily impede the rigour of the proofs as rigour is not equal to formality (although formalists may disagree with this).

Our aim in this chapter is to develop the mathematical logic theory of formal reasoning from the perspective of institution theory. We will do this as follows:

1. First we introduce the concept of (formal) *proof* in an axiomatic manner as a semantic-free concept. Proofs can be regarded as a refined form of the consequence relations.

Having a proof of ρ from E is more than saying that ρ can be deduced from E because we can have more than one 'proof' of ρ from E. We will also see that in order to construct semantic-free proofs we can aggregate iteratively atomic / primitive proofs in a systematic way. The atomic / primitive proofs, often called *proof rules*, are institution-dependent, but the way of aggregation is institution-independent and can be presented as a free construction (adjunction). This gives a clear separation between the concrete and the general in which the concept of proof is concerned.

2. While the general context of proofs consists of a sentence functor only, the concept of correct proofs is relative to a semantic consequence relation, which requires a semantic level. This is provided as an abstract institution. The correctness of proofs is called *soundness* and is a must-to-have property. Thus, a proof of ρ from E is sound when $E \models \rho$. Its opposite property is called *completeness*, which means that each semantic consequence admits a proof. However, while there is a 'logic life' without completeness, the soundness is absolutely mandatory. Such 'institutions with proofs' can function as a meta-theory for logical systems, that capture both the model and the proof-theoretic sides of logics.
3. One aspect that supports a calculus of proofs, eventually leading to 'mechanisation', is the possibility of finiteness of proofs. This is the proof-theoretic side of our old friend, the compactness. We show how this compactness propagates through the general aggregation of proofs.
4. The semantic logical connectives, such as the propositional ones and the quantifiers, determine properties of the semantic consequence relations. We have met with them in Chap. 5. At the more abstract level of proof systems, these properties can be formulated as category-theoretic axioms, which define the proof-theoretic concepts of logical connectives. In this context we show how proof systems can be enhanced freely with connectives, and moreover, how this enhancement preserve crucial properties such as soundness and compactness.
5. Mathematically, there is a sense in which proof systems are more general than institutions because the semantic consequence in any institution gives rise to a collapsed proof system such that for any E and ρ there exists *at most* one proof of ρ from E. Such proof systems are called 'entailment systems'. Conversely, we show that each entailment system determines also an institution by a generic canonical construction of a model functor. This construction relies on a proof-theoretic concept of theory morphism. Moreover, the institutions thus constructed from entailment systems enjoy soundness and completeness.
6. The final parts of the chapter are devoted to a 'layered' methodology for building sound and complete proof systems for institutions. The main idea of this approach is to start with such a proof system from the simplest, most primitive, sentences of the institution, and then gradually develop it for more complex sentences while preserving the soundness and the completeness. This process may involve several steps. For instance, we may start with a proof system for the atomic sentences, then extend it for quantifier-free Horn sentences, and finally extend it to the quantified Horn sentences.

We develop this example at a general institution-independent level, but many other completeness results can be approached in the same general way. The main message that this methodology sends us is that the sound and complete proof systems have an inherent layered structure that corresponds intimately to the layered structure of the sentences (and implicitly of their satisfaction by the models).

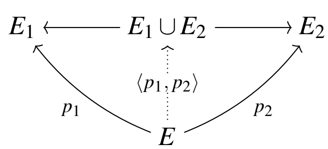
The first five sections of this chapter have almost no model-theoretic content, so they can be studied quite independently of the material from previous chapters. On the hand, the sections 11.6 and 11.7 do have model-theoretic content and require familiarity with some material only from the first of the four parts of the book (until Chap. 5 included).

11.1 What is a proof?

The concept of proof has many facets. Here we develop a concept of proof as a mathematical object that suits institution theory.

Proof systems. So what is a proof? It is a one-way move from a set E to a set E' of sentences, called E *proves* E', and meaning that E' is established 'true' on the basis of E being established 'true'. And there can be several different ways to prove E' from E. Therefore proofs can be conveniently represented as labeled arrows $E \xrightarrow{p} E'$. Proofs between sets of sentences have two natural compositionality properties:

- an associative *horizontal* one, meaning that proofs $E \xrightarrow{p} E'$ and $E' \xrightarrow{p'} E''$ determine a proof $E \xrightarrow{p;p'} E''$, and
- a *vertical* one, meaning that for any E_1, E_2 such that $E_1 \cap E_2 = \emptyset$, any proofs $E \xrightarrow{p_1} E_1$ and $E \xrightarrow{p_2} E_2$ determine uniquely a proof $E \xrightarrow{\langle p_1, p_2 \rangle} E_1 \cup E_2$



such that each p_i can be 'extracted' from $\langle p_1, p_2 \rangle$ by horizontal composition with a canonical *monotonicity* proof $E_1 \cup E_2 \longrightarrow E_i$. Moreover, the monotonicity proofs are unique, i.e. if $E \subseteq \Gamma$ then there exists exactly one monotonicity proof $\Gamma \longrightarrow E$.

Thus horizontal composition gives proofs the structure of a category, whose objects are the sets of sentences of a fixed signature Σ. Let us denote this category by $Pf\Sigma$ and name it the *category of the* Σ*-proofs*. The vertical composition, which we may also call 'union', just says that $Pf\Sigma$ has finite products of disjoint sets of sentences. This disjointness condition enables us to distinguish conveniently between different proofs by designation. Suppose that there existed $e \in E_1 \cap E_2$. Then the proofs p_1 and p_2 determine two different proofs q_1, q_2 of $E_1 \cup E_2$ because e can be proved either through p_1 or p_2. The problem is that in such situations there is no convenient method to trace different proofs.

- It is also natural to assume that any signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ gives a translation from $Pf\Sigma$ to $Pf\Sigma'$ which extends the translation of the sentences to proofs in such a way that both the horizontal and the vertical composition are preserved. The latter property means we have a functor $Pf\varphi\colon\ Pf\Sigma \to Pf\Sigma'$. Moreover, for any signature morphisms $\varphi\colon\ \Sigma \to \Sigma'$ and $\varphi'\colon\ \Sigma' \to \Sigma''$, Pf should preserve their composition.

All these are collected by the following proof-theoretic counterpart for the concept of institution. A *proof system* (Sig, Sen, Pf) consists of

- a category of 'signatures' Sig,
- a 'sentence functor' $Sen\colon\ Sig \to \mathbb{S}et$, and
- a 'proof functor' $Pf\colon\ Sig \to \mathbb{C}at$ (giving for each signature Σ the category of the Σ-proofs)

such that

1. $Sen;\mathcal{P};(-)^{\mathrm{op}}$ is a sub-functor of Pf,

$$\begin{array}{ccc} Pf\Sigma & \xrightarrow{\ Pf\varphi\ } & Pf\Sigma' \\ \big\uparrow & & \big\uparrow \\ (\mathcal{P}(Sen\Sigma),\subseteq)^{\mathrm{op}} & \xrightarrow[\mathcal{P}(Sen\varphi)^{\mathrm{op}}]{} & (\mathcal{P}(Sen\Sigma'),\subseteq)^{\mathrm{op}} \end{array} \tag{11.1}$$

2. the inclusion $(\mathcal{P}(Sen\Sigma),\subseteq)^{\mathrm{op}} \hookrightarrow Pf\Sigma$ is broad, preserves finite direct products and epics (epimorphisms) for each signature Σ, where $\mathcal{P}\colon\ \mathbb{S}et \to \mathbb{C}at$ is the ($\mathbb{C}at$-valued) power-set functor.

Note that:

- The inclusion $(\mathcal{P}(Sen\Sigma),\subseteq)^{\mathrm{op}} \hookrightarrow Pf\Sigma$ is broad means that $Pf\Sigma$ has all subsets of $Sen\Sigma$ as objects.
- For any $E \subseteq \Gamma \subseteq Sen\Sigma$, by $\supseteq_{\Gamma,E}$ let us denote the image of $E \subseteq \Gamma$ in $Pf\Sigma$. This is the *monotonicity proof* corresponding to $E \subseteq \Gamma$.
- Since any arrow in $(\mathcal{P}(Sen\Sigma),\subseteq)$ is monic, the axiom on the preservation of epics means that each monotonicity proof is epic.
- The direct products in $(\mathcal{P}(Sen\Sigma),\subseteq)^{\mathrm{op}}$ are the direct sums of $(\mathcal{P}(Sen\Sigma),\subseteq)$ which are the disjoint unions. Their preservation gives exactly the vertical composition of proofs.

- The commutativity of diagram 11.1, that expresses the sub-functor axiom, gives the preservation of the monotonicity proofs by the sentence translations, i.e. $\varphi \supseteq_{\Gamma,E} = \supseteq_{\varphi\Gamma,\varphi E}$.

Within the context of proof systems, in order to simplify notation, singleton sets $\{\rho\}$ may be sometimes denoted just by their element ρ.

Infinitary proof systems. We can define an *infinitary* variant of the concept of proof system by allowing infinitary vertical compositions of proofs. Although interesting in itself, this extension has limited applicability because finiteness is a crucial aspect of the concept of proof.

Concerning issues of finiteness, in many logic texts, entailments $E \vdash \Gamma$ are considered only for Γ a single sentence (which is equivalent to Γ being finite). Our definition of proof systems represents a middle approach, it allows infinite Γ but only finite vertical compositions (unions). By letting both the premises and the conclusions of proof to be potentially infinite, we achieve a conceptual uniformity which enables the proofs-as-arrows category-theoretic approach.

Entailment systems. Thin proof systems, i.e., such that $Pf\Sigma$ are preorders, are called *entailment systems*. The preorder $Pf\Sigma$ is then called an *entailment relation* while its proofs are called *entailments*. Thus entailment systems can only tell that a certain set of sentences E is *provable* from another set of sentences Γ, without the possibility to distinguish between different proofs. Each proof system can be 'flattened' canonically to an entailment system given by the preorder $\Gamma \vdash E$ on the sets of sentences defined by "there exists at least one proof" from Γ to E, which can also be read as 'Γ entails E'.

An important technical simplification which arises as a consequence of the fact that there exists at most one entailment between any sets of sentences is the fact that the vertical composition of entailments becomes total rather than partial. This is because with entailment systems, the possibility of $q_1 \neq q_2$, that was involved when we explained why the disjointness of E_1 and E_2 is necessary in the general case, is not on the cards anymore.

Fact 11.1. *In any entailment system, for any signature Σ and sets E, E_1, E_2 of Σ-sentences,*

$$E \vdash E_1 \text{ and } E \vdash E_2 \text{ implies } E \vdash E_1 \cup E_2.$$

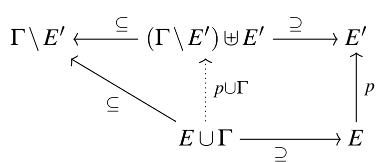
The semantic entailment system. In the light of the properties stated in Proposition 3.7, we can see that in any institution, the semantic consequence relation between sets of sentences gives an example of an infinitary entailment system, which is called the *semantic proof system* or the *semantic entailment system* of the institution. This is the sense in which we can say that proof systems are more abstract than institutions.

Proof-theoretic equivalence. We are already familiar with the concept of semantic equivalence, i.e. $E \models\!\!\mid E'$ meaning $E \models E'$ and $E' \models E$. We can extend this to proof systems.

A weaker version of this extension is when we consider only the provability; given an entailment system, we say that E and E' are *entailment-theoretic equivalent* when $E \vdash E'$ and $E' \vdash E$. This is denoted $E \dashv\vdash E'$. Note that this can be formulated in an equivalent form as $\Gamma \vdash E$ if and only if $\Gamma \vdash E'$, for each set of Σ-sentences Γ. The full concept of proof-theoretic equivalence is as follows. Given a proof system, we say that two sets E, E' of Σ-sentences are *proof-theoretic equivalent* if and only if for each $\Gamma \subseteq Sen\Sigma$, $Pf\Sigma\,(\Gamma, E)$ and $Pf\Sigma\,(\Gamma, E')$ are naturally isomorphic.

Exercises

11.1. In any proof system (Sig, Sen, Pf), for any Σ-proof $E \xrightarrow{p} E'$ and any $\Gamma \subseteq Sen\Sigma$, by $p \cup \Gamma$ we denote the proof $E \cup \Gamma \longrightarrow E' \cup \Gamma$ defined by the following commutative diagram in $Pf\Sigma$:



Show that for any proofs $E \xrightarrow{p} E'$ and $E' \xrightarrow{p'} E''$, for each set of sentences Γ, if $(\Gamma \setminus E'') \subseteq (\Gamma \setminus E')$ then $(p \cup \Gamma); (p' \cup \Gamma) = (p; p') \cup \Gamma$.

11.2 Free proof systems

In general, logic systems have infinite sets of sentences even for single signatures and also infinite sets of proofs. To put proof systems at work it is important to be able to provide finite specifications of such systems. This is actually a common practice in logic, which goes like this: one writes down a finite set of ‘primitive’ proofs and then aggregates them into more complex proofs by using the general properties of proof systems. By such aggregations one obtains the whole proof system. In this section we

- formalise the primitive proofs by the concept of ‘system of (proof) rules’;
- show how such systems of proof rules generate freely proof systems;
- show that the free construction of the proof systems preserve the soundness property. This represents the general method to establish the soundness of concrete institutions with proof systems.

Systems of proof rules

Before discussing the general concept it is helpful to look at a classical example. Readers familiar with conventional logic may recognize the following set of proof rules as defining the proof system of propositional logic $\mathcal{PL}$. The symbols p, q, r below represent $\mathcal{PL}$ sentences of a fixed arbitrary $\mathcal{PL}$ signature.

$(P1)$ $\quad \emptyset \vdash p \Rightarrow (q \Rightarrow p)$
$(P2)$ $\quad \emptyset \vdash (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$
$(P3)$ $\quad \emptyset \vdash (\neg p \Rightarrow \neg q) \Rightarrow (q \Rightarrow p)$
(MP) $\quad \{p, p \Rightarrow q\} \vdash q$

This is a rather typical case of presentations of proof systems as a set of rules written in the form $E \vdash E'$. These rules are the primitive proofs from which the collection of all proofs are generated by closure under the horizontal and vertical compositions of proofs plus the monotonicity proofs. In general, this process can be explained as an adjunction between proof systems and 'systems of rules'. In this example, a proof $\emptyset \longrightarrow (A \Rightarrow A)$ can be obtained as follows:

1 $\quad \emptyset \xrightarrow{p_1} A \Rightarrow ((B \Rightarrow A) \Rightarrow A)$ — instance of $(P1)$
2 $\quad \emptyset \xrightarrow{p_2} (A \Rightarrow ((B \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow A))$ — instance of $(P2)$

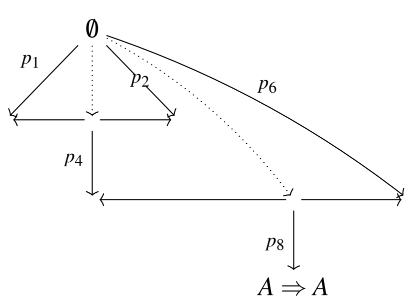
Now let us abbreviate the conclusion of p_1 by c_1 and the conclusion of p_2 by c_2. Then

3 $\quad \emptyset \xrightarrow{p_3} \{c_1, c_2\}$ — vertical composition of p_1 and p_2
4 $\quad \{c_1, c_2\} \xrightarrow{p_4} (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow A)$ — instance of (MP)
5 $\quad \emptyset \xrightarrow{p_5} (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow A)$ — horizontal composition of p_3 and p_4
6 $\quad \emptyset \xrightarrow{p_6} A \Rightarrow (B \Rightarrow A)$ — instance of $(P1)$
7 $\quad \emptyset \xrightarrow{p_7} \{A \Rightarrow (B \Rightarrow A), (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow A)\}$ — vertical composition of p_5 and p_6
8 $\quad \{A \Rightarrow (B \Rightarrow A), (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow A)\} \xrightarrow{p_8} A \Rightarrow A$ — instance of (MP)
9 $\quad \emptyset \xrightarrow{p_9} A \Rightarrow A$ — horizontal composition of p_7 and p_8.

In a compact way we can write the proof p_9 in terms of primitive proofs and aggregations:

$$p_9 = \langle \langle p_1, p_2 \rangle ; p_4, p_6 \rangle ; p_8$$

which can also be represented by the following commutative diagram in which the aggregations can be seen in a categorical form:



where the horizontal arrows represent monotonicity proofs and the dotted arrows represent respective vertical compositions.

Note that the set of rules presented above is in fact infinite since each of the $(P1-3)$ and (MP) are parameterised by the ρ's, hence by giving values to the ρ's as $\mathcal{PL}$ sentences, each of them specifies an infinite set of rules. In developing the proof of $\emptyset \longrightarrow A \Rightarrow A$ we have used this; for instance when considering the primitive proof p_1 we have given A and $B \Rightarrow A$ as values for p and q, respectively. Some other literature may use slightly different terminology, such as calling the generic schemes $(P1-3)$ and (MP) 'rules'. Under that terminology our 'rules' would be called 'rule instances', whilst under our terminology $(P1-3)$ and (MP) may be referred to as 'rule schemes'. The specification of the $\mathcal{PL}$ proof rules above defines an infinite set also for another reason, namely the infiniteness of the class of the signatures in $\mathcal{PL}$. But a crucial aspect of this specification is that in itself it is finite, we can really write it down.

Systems of rules. The following defines the general concept of system of rules. A *system of (proof) rules* (Sig, Sen, Rl, h, c) consists of

- a category of 'signatures' Sig,
- a 'sentence functor' $Sen : Sig \to \mathbb{S}et$,
- a '(proof) rule functor' $Rl : Sig \to \mathbb{S}et$, and
- two natural transformations $h, c : Rl \Rightarrow Sen; \mathcal{P}$, where $\mathcal{P} : \mathbb{S}et \to \mathbb{S}et$ is the $\mathbb{S}et$-valued power-set functor.

Therefore, for each signature Σ, $Rl\Sigma$ gives the set of the Σ-proof rules, $h_\Sigma : Rl\Sigma \to \mathcal{P}(Sen\Sigma)$ gives the hypotheses of the rules, and $c_\Sigma : Rl\Sigma \to \mathcal{P}(Sen\Sigma)$ gives the conclusions. A Σ-rule r can be therefore written as $h_\Sigma r \xrightarrow{r} c_\Sigma r$. The functoriality of Rl and the naturality of the hypotheses h and of the conclusions c, say that the translation of rules along signature morphisms is coherent with the translation of the sentences.

It is also possible to define systems of rules as signature indexed families $(rl\Sigma)_{\Sigma \in |Sig|}$ with $rl\Sigma \subseteq \mathcal{P}(Sen\Sigma) \times \mathcal{P}(Sen\Sigma)$. Note that this can be extended canonically to a system of rules in the previous acceptation by adding freely the translations of the rules by the signature morphisms. However, sometimes this may be redundant, like in the case of the above $\mathcal{PL}$ example when we already have $\varphi(rl\Sigma) \subseteq rl\Sigma'$

Proof-theoretic morphisms and comorphisms

To formulate the freeness of proof systems as a categorical universal propert we have to organise proof systems and systems of rules as categories and define functors between these. For instance, we can easily notice that each proof system can be seen as a system of rules by regarding each proof as a rule (the hypotheses being given by the domain of the proof, and the conclusions by the codomain). This can be understood as a forgetful functor from the category of proof systems to the category of rule systems provided we organize proof systems and systems of rules as categories. Defining mappings between proof systems, and systems of rules, respectively can be done in a straightforward manner just by mimicking the institution-theoretic morphisms and comorphisms and by relying on common sense.

Morphisms and comorphisms of proof systems. Let us consider first the case of comorphisms. A *proof system comorphism* between proof systems (Sig, Sen, Pf) and (Sig', Sen', Pf') consists of

- a 'signature' functor $\Phi\colon\ Sig \to Sig'$,
- a 'sentence translation' natural transformation $\alpha\colon\ Sen \Rightarrow \Phi; Sen'$, and
- a 'proof translation' natural transformation $\gamma\colon\ Pf \Rightarrow \Phi; Pf'$ such that translation of sets of sentences is compatible with translation of single sentences:

$$\begin{array}{ccc} Pf\Sigma & \xrightarrow{\gamma_\Sigma} & Pf'(\Phi\Sigma) \\ \uparrow & & \uparrow \\ Sen\Sigma & \xrightarrow[\alpha_\Sigma]{} & Sen'(\Phi\Sigma) \end{array}$$

Proof systems morphisms are defined by analogy with institution morphisms by reversing the direction of the signature mapping (in the definition of the proof system comorphisms). Let $\mathbb{P}f\mathbb{S}ys$ denote the category of proof system morphisms, and $co\mathbb{P}f\mathbb{S}ys$ denote the category of proof system comorphisms.

Morphisms and comorphisms of systems of rules. A *comorphism of systems of (proof) rules* between systems of rules (Sig, Sen, Rl, h, c) and $(Sig', Sen', Rl', h', c')$ consists of

- a 'signature' functor $\Phi\colon\ Sig \to Sig'$,
- a 'sentence translation' natural transformation $\alpha\colon\ Sen \Rightarrow \Phi; Sen'$,
- a 'rule translation' natural transformation $\gamma\colon\ Rl \Rightarrow \Phi; Rl'$ which is compatible with the hypotheses and the conclusions, i.e., the diagram below commutes:

$$\begin{array}{ccc} Rl & \stackrel{\gamma}{\Longrightarrow} & \Phi; Rl' \\ h \Downarrow\Downarrow c & & h' \Downarrow\Downarrow c' \\ Sen; \mathcal{P} & \underset{\alpha}{\Longrightarrow} & \Phi; Sen'; \mathcal{P} \end{array}$$

Morphisms of systems of rules are defined similarly by reversing the direction of the signature mapping. Let $\mathbb{R}l\mathbb{S}ys$ denote the category of proof rule system morphisms, and $co\mathbb{R}l\mathbb{S}ys$ denote the category of proof rule system comorphisms.

Fact 11.2. *There exist forgetful functors* $\mathbb{P}f\mathbb{S}ys \to \mathbb{R}l\mathbb{S}ys$ *and* $co\mathbb{P}f\mathbb{S}ys \to co\mathbb{R}l\mathbb{S}ys$ *mapping each proof system* (Sig, Sen, Pf) *to the system of rules* (Sig, Sen, Pf, dom, cod) *(i.e., the hypothesis of a* Σ*-proof is its domain and the conclusion is its codomain).*

Free proof systems

The free proof system construction is a left adjoint to the forgetful functor $co\mathbb{P}f\mathbb{S}ys \to co\mathbb{R}l\mathbb{S}ys$ when working with comorphisms, and a right adjoint to $\mathbb{P}f\mathbb{S}ys \to \mathbb{R}l\mathbb{S}ys$ when working with morphisms.

Theorem 11.3. *Each system of proof rules such that its sentence translations are injective generates freely a proof system.*

Proof. The idea of this proof is to encode systems of proof rules as classes of $\mathcal{PA}$ (partial algebra) theories formed by universally quantified quasi-existence equations and then use the initial semantics of those (based on results from Sec. 4.6). The algebras of these theories give proof systems where the proofs are elements in the algebras.

- Let (Sig, Sen, Rl, h, c) be a system of proof rules such that $Sen\varphi$ is injective for each signature morphism $\varphi \in Sig$. For any signature $\Sigma \in |Sig|$ we define the single-sorted $\mathcal{PA}$ signature consisting of the following:
  - total constants: all sets of sentences $E \subseteq Sen\Sigma$, all sets of sentences inclusions $E \supseteq E'$, and all elements of $Rl\Sigma$,
  - unary total operation symbols: h_Σ and c_Σ, and
  - binary partial operation symbols: $\_;\_$ and $\langle \_, \_ \rangle$.
- Then the set of the axioms of any proof system generated by $(Sen\Sigma, Rl\Sigma, h_\Sigma, c_\Sigma)$ can be specified as a quasi-existence equational theory in $\mathcal{PA}$ in the same style of the well known encoding of categories as partial algebras. We will not provide here the full details of this specification; we will rather present only a sample, namely how the vertical composition of proofs is encoded as a set of quasi-existence equations. All other axioms of the proof systems generated by $(Sen\Sigma, Rl\Sigma, h_\Sigma, c_\Sigma)$ can be dealt with in a similar manner.

  Let $E \xrightarrow{p} \Gamma$ abbreviate $(h_\Sigma p \stackrel{e}{=} E) \wedge (c_\Sigma p \stackrel{e}{=} \Gamma)$. The equations $(P1^\Sigma)$ below encode the existence side of the vertical composition, while $(P2^\Sigma)$ encode the uniqueness side.

$$(P1^\Sigma) \quad (\forall p, p')(E \xrightarrow{p} \Gamma) \wedge (E \xrightarrow{p'} \Gamma') \Rightarrow$$
$$\Rightarrow (E \xrightarrow{\langle p, p' \rangle} \Gamma \cup \Gamma') \wedge (\langle p, p' \rangle; (\Gamma \cup \Gamma' \supseteq \Gamma) \stackrel{e}{=} p) \wedge (\langle p, p' \rangle; (\Gamma \cup \Gamma' \supseteq \Gamma') \stackrel{e}{=} p')$$
$$\text{for all } E, \Gamma, \Gamma' \subseteq Sen(\Sigma) \text{ with } \Gamma \cap \Gamma' = \emptyset.$$

$$(P2^\Sigma) \quad (\forall p, p')(E \xrightarrow{p} \Gamma \cup \Gamma') \wedge (E \xrightarrow{p'} \Gamma \cup \Gamma') \wedge$$
$$\wedge (p; (\Gamma \cup \Gamma' \supseteq \Gamma) \stackrel{e}{=} p'; (\Gamma \cup \Gamma' \supseteq \Gamma)) \wedge (p; (\Gamma \cup \Gamma' \supseteq \Gamma') \stackrel{e}{=} p'; (\Gamma \cup \Gamma' \supseteq \Gamma')) \Rightarrow$$
$$p \stackrel{e}{=} p' \text{for all } E, \Gamma, \Gamma' \subseteq Sen(\Sigma) \text{ with } \Gamma \cap \Gamma' = \emptyset.$$

- Let PT^Σ be an initial partial algebra for the resulting quasi-existence theory. The category $Pf\Sigma$ of the Σ-proofs is defined by $|Pf\Sigma| = \mathcal{P}(Sen\Sigma)$ and $(Pf\Sigma)(\Gamma, E) = \{p \in PT^\Sigma \mid PT^\Sigma_h p = \Gamma, PT^\Sigma_c p = E\}$. The composition of proofs is given by $p;p' = p(PT^\Sigma_{\_;\_})p'$ and the monotonicity proofs $\supseteq_{\Gamma,E}: \ \Gamma \to E$ are defined as $PT^\Sigma_{\Gamma \supseteq E}$. Notice also that $(PT^\Sigma)_E = E$.
- Any signature morphism $\varphi: \ \Sigma \to \Sigma'$ induces a morphism $\overline{\varphi}$ between the theories corresponding to Σ and Σ'. The role of the injectivity of $Sen\varphi$ is that $\overline{\varphi}$ maps $P1^\Sigma$ to $P1^{\Sigma'}$ and $P2^\Sigma$ to $P2^{\Sigma'}$. In the absence of this injectivity these mappings are not guaranteed as $\Gamma \cap E = \emptyset$ would not necessarily imply that $\varphi\Gamma \cap \varphi E = \emptyset$. Then we define the functor $Pf\varphi$ as the unique partial algebra homomorphism $PT^\Sigma \to PT^{\Sigma'}\restriction_{\overline{\varphi}}$. With this we have defined a proof system (Sig, Sen, Pf).
- It is a straightforward exercise to show that this is the free proof system over (Sig, Sen, Rl, h, c) as each comorphism $(\Phi, \alpha, \gamma): \ (Sig, Sen, Rl, h, c) \to (Sig', Sen', Pf', dom, cod)$ and each signature $\Sigma \in |Sig|$ determine a partial algebra A of the theory of quasi-existence equations.

The diagram below represents the universal property of the free proof system.

$$\begin{array}{ccc} (Sig, Sen, Rl, h, c) & \xrightarrow{(1_{Sig}, 1_{Sen}, \eta)} & (Sig, Sen, Pf, dom, cod) \\ & {\scriptstyle (\Phi,\alpha,\gamma)} \searrow \quad \swarrow {\scriptstyle (\Phi,\alpha,\gamma')} & \\ & (Sig', Sen', Pf', dom, cod) & \end{array}$$

$\square$

For the actual systems of rules, the injectivity of the sentence translations comes as a consequence of the injectivity of the signature morphisms. For example this can be noticed easily in the case of $\mathcal{PL}$, $\mathcal{FOL}$, etc. This means that we cannot have a proof system for such institutions that is freely generated from rules unless we restrict to the sub-institutions determined by the injective signature morphisms.

Free infinitary proof systems can be obtained by an infinitary version of Thm. 11.3. This requires an extension of partial algebras with infinitary operations for dealing with the infinitary vertical compositions of proofs. This is a straightforward step.

Free entailment systems. Thm. 11.3 and its proof can be downgraded to a theorem on existence of free entailment systems. Of course, this requires downgrading also the system of proof rules to a concept of *system of entailment rules* consisting for each signature Σ of a binary relation $\vdash_\Sigma \subseteq \mathcal{P}(Sen\Sigma) \times \mathcal{P}(Sen\Sigma)$ between sets of Σ-sentences such that for any signature morphism $\varphi: \ \Sigma \to \Sigma'$, if $E \vdash_\Sigma E'$ then $\varphi E \vdash_{\Sigma'} \varphi E'$. In that situation the condition of the injectivity of the sentence translation is not needed because this is used only for translations of families of equations $(P1^\Sigma)$ and $(P2^\Sigma)$. In this highly simplified setting defined by the equations $(\forall p, p')(E \xrightarrow{p} \Gamma \stackrel{e}{=} E \xrightarrow{p'} \Gamma$, $(P2^\Sigma)$ holds trivially without the need of $\Gamma \cap \Gamma' = \emptyset$, while $(P1^{\Sigma})$ can be replaced by $(E \vdash \Gamma) \wedge (E \vdash \Gamma') \Rightarrow (E \vdash \Gamma \cup \Gamma')$, again without the condition $\Gamma \cap \Gamma' = \emptyset$. Hence we can formulate the following:

Corollary 11.4. *Each system of entailment rules generates freely an entailment system.*

Soundness of institutions with free proof systems

We can enhance institutions with proof systems by aggregating the two structures if they agree on the sentence functor. In this way we get together both sides of the general concept of logical system. An *institution with proofs* is a tuple $(Sig, Sen, Mod, \models, Pf)$ such that

- $(Sig, Sen, Mod, \models)$ is an institution, and
- (Sig, Sen, Pf) is a proof system.

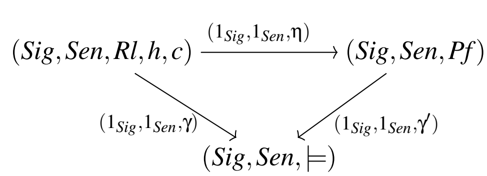
The fundamental coherence relationship between the model theory and the proof theory of any institution with proofs is that of *soundness*: an institution with proofs is *sound* when for each proof $E \longrightarrow E'$ we have that $E \models E'$.

This property is difficult to establish if we think of checking all proofs because in general this is an infinite process. In practice we rely on finite specifications of proof systems by systems of rules. We first check the soundness of each rule (or rules scheme) and then lift this to the free proof system. This lifting is supported by the general result of Prop. 11.5.

Like in the case of institutions with proof systems, an *institution with proof rules* $(Sig, Sen, Mod, \models, Rl, h, c)$ combines an institution $(Sig, Sen, Mod, \models)$ with a system of rules (Sig, Sen, Rl, h, c). An institution with rules is *sound* when for each rule $r \in Rl\Sigma$, $h_{\Sigma} r \models c_{\Sigma} r$.

Proposition 11.5. *The institution with proofs $(Sig, Sen, Mod, \models, Pf)$ such that the proof system (Sig, Sen, Pf) is freely generated by a system of rules (Sig, Sen, Rl, h, c) is sound whenever the institution with rules $(Sig, Sen, Mod, \models, Rl, h, c)$ is sound.*

Proof. Because $(Sig, Sen, Mod, Rl, h, c)$ is sound we consider the canonical comorphism of systems of proof rules $(1_{Sig}, 1_{Sen}, \gamma) : (Sig, Sen, Rl, h, c) \to (Sig, Sen, \models, dom, cod)$ to the institution with semantic proofs that maps any rule $E \xrightarrow{r} E'$ to the semantic proof $E \models E'$.



By the universal property of the free proof system (Sig, Sen, Pf) (Thm. 11.3), $(1_{Sig}, 1_{Sen}, \gamma)$ can be extended to a comorphism of proof systems

$$(1_{Sig}, 1_{Sen}, \gamma') : \ (Sig, Sen, Pf) \to (Sig, Sen, \models).$$

But the existence of such a comorphism represents the soundness of $(Sig, Sen, Mod, \models, Pf)$. $\square$

Completeness. This is the opposite property to soundness. Informally, it says that for each semantic deduction / consequence there exists at least one proof. Usually, it is much more difficult to establish completeness properties than soundness properties. We will understand this in the last parts of this chapter. An institution with proofs $(Sig, Sen, Mod, \models, Pf)$ is *complete* when

$$E \models_\Sigma \Gamma \text{ implies } E \vdash_\Sigma \Gamma$$

for all sets $E, \Gamma \subseteq Sen\Sigma$ with Γ finite. An institution with proof rules is complete if and only if the corresponding institution with proofs freely generated by the system of proof rules is complete.

There is a subtle aspect of completeness given by 'Γ finite'. Soundness does not require this condition, so, strictly technically, completeness is less than the opposite of soundness. Since the common concrete proof systems are generated by systems of finitary rules and also because the vertical composition of proofs is finite, it does make sense to consider a concept of completeness extended to potentially infinite Γs. For instance, in $\mathcal{EQL}$ let us consider a signature with two constants a and b, and a unary operation σ. Then $a = b \models \{\sigma^k a = \sigma^k b \mid k \in \omega\}$. Although we will discuss the Birkhoff proof system for $\mathcal{EQL}$ later on in this chapter, some degree of familiarity with algebraic deduction tells us that under this proof system we cannot have $a = b \vdash \{\sigma^k a = \sigma^k b \mid k \in \omega\}$ in spite of having $a = b \vdash \sigma^k a = \sigma^k b$ for each $k \in \omega$.

Exercises

11.2. From $(P1-3)$ and (MP), prove by natural deduction the *Reductio ad Absurdum* principle by constructing a proof $\emptyset \longrightarrow (\neg\neg A \Rightarrow A)$.

11.3. Develop all details of the proof of Thm. 11.3.

11.4. The rules $(P1-3)$ and (MP) which generate the proof system of $\mathcal{PL}$ are sound in any institution with semantic implications and negations.

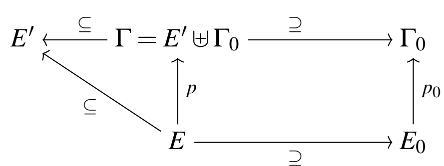
11.3 Compactness

So far we have discussed two semantic notions of compactness: model compactness (m-compactness) and semantic consequence-theoretic compactness. The former one is outside the scope of proof theory, but the latter one can be in fact formulated for any entailment system. Thus, an entailment system $(Sig, Sen, \vdash)$ is compact whenever for each $E \vdash \rho$, where $E \subseteq Sen\Sigma$, $\rho \in Sen\Sigma$, there exists a finite subset $E_0 \subseteq E$ such that $E_0 \vdash \rho$.

In this section we refine this concept of compactness to proof systems in two ways. On the one hand, compactness for proof systems means being able to extract a proof with finite premises from *any* proof with a finite set of conclusions. This is stronger than having that if a sentence is provable then it is also provable from a finite subset of premises.

On the other hand, we extend the concept of compactness to proof with a potentially infinite set of conclusions. What we get is a stronger concept than compactness. This characterises proof systems that are obtained from finitary rules, while the ordinary concept of compactness applies well to semantic entailments. The main result of this section is that the free proof systems generated by finitary rules are compact in the stronger sense announced above. This is crucial especially in the context of the mechanisation of reasoning as it guarantees that any proof obtained from finitary rules has itself a finitary nature. Remember that computing-based reasoning is strictly confined to finite entities.

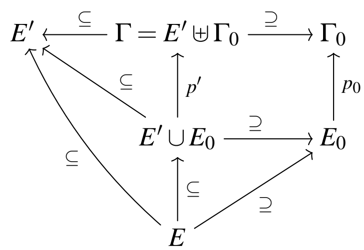
Finitary and quasi-finitary proofs. A proof $E \xrightarrow{p} E'$ is *finitary* when both E and E' are finite. Similarly, a (proof) rule r is finitary when both the hypothesis $h_\Sigma r$ and the conclusion $c_\Sigma r$ are finite for each signature Σ. The main idea of a quasi-finitary proof is that its non-trivial part is finitary. A proof $E \xrightarrow{p} \Gamma$ is *quasi-finitary* when there exists a finitary proof $E_0 \xrightarrow{p_0} \Gamma_0$ such that $E_0 \subseteq E$, $\Gamma_0 \subseteq \Gamma$, $E' = \Gamma \setminus \Gamma_0 \subseteq E$ and $p = \langle \supseteq_{E,E'}, \supseteq_{E,E_0} ; p_0 \rangle$.



This should be understood as follows. E' represents a part of the conclusion that gets a trivial proof thus being also a part of the hypothesis E. By removing E' we are left with a finite set of conclusions Γ_0. Then there exists a proof p_0 of Γ_0 from a finite set of hypotheses (E_0). Moreover p is the aggregation of p_0 with a trivial proof. Note that this does not mean that $E = E' \cup E_0$, nor that E' and E_0 are disjoint. Also, p may have more than one such representations in terms of finitary proofs. For instance, to any p_0 we can aggregate any admissible trivial proof that is finitary, and then a new such representation for p is obtained. The main role of E' is to reduce the conclusion to a finite set, but not necessarily a minimal one. The following result sheds a light on these aspects.

Proposition 11.6. *Any quasi-finitary proof $E \longrightarrow \Gamma$ with Γ finite can be written as a composition between a monotonicity proof and a finitary proof.*

Proof. Consider a quasi-finitary proof $E \xrightarrow{p} \Gamma$ such that Γ is finite. Consider a representation of p like in the definition above of quasi-finitary proofs. Consider the commutative diagram below where $p' = \langle \supseteq_{E' \cup E_0, E'}, \supseteq_{E' \cup E_0, E_0} \,;\, p_0 \rangle$.



- By the uniqueness property of the direct products we get that $p \;\; = \supseteq ; p'$.
- Since Γ is finite it follows that E' is finite too, hence $E' \cup E_0$ is finite. This means p' is finitary.

□

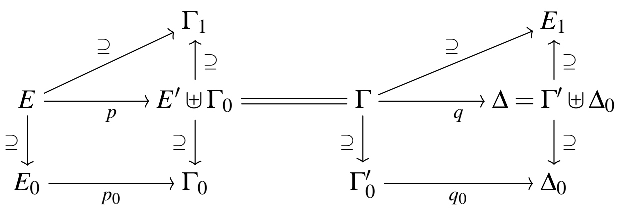
The sub-system of quasi-finitary proofs. A proof system is *quasi-finitary* when each of its proofs is quasi-finitary. It is *compact* when each proof $E \longrightarrow \Gamma$ with Γ finite, is quasi-finitary. So, quasi-finitary proof systems are compact. Viceversa is not necessarily true. For instance, the semantic entailment system of $\mathcal{FOL}$ is compact (Cor. 6.24) but it is not quasi-finitary. To see this, it is enough to consider a signature with two constants a and b, and a unary operation σ. Then $a = b \models \{\sigma^k a = \sigma^k b \mid k \in \omega\}$, which is not quasi-finitary.

The following result is a powerful technical tool that we will use for establishing the quasi-finitary property for some important general proof systems.

Proposition 11.7. *For any proof system (Sig, Sen, Pf), the collection of its quasi-finitary proofs form a proof system, denoted by $(Sig, Sen, C(Pf))$.*

Proof. Note that all monotonicity proofs are trivially quasi-finitary. We therefore have to show that quasi-finitary proofs form a sub-category of all proofs, that this sub-category creates (binary) products of disjoint sets of sentences, and that translations along signature morphisms preserve quasi-finitary proofs. In order to simplify a bit the notations, we will systematically omit the subscripts in the notation of the monotonicity proofs, but only when thy can be understood easily from the context.

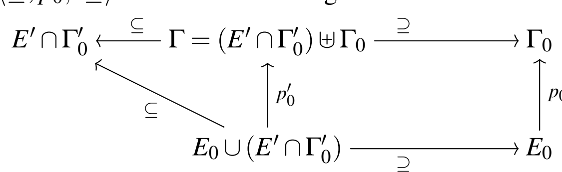
1. *Subcategory (horizontal compositions).* Note that each identity proof is trivially quasi-finitary. Consider two quasi-finitary proofs $E \xrightarrow{p} \Gamma$ and $\Gamma \xrightarrow{q} \Delta$. We represent them in terms of finitary proofs p_0 and q_0, respectively, as shown in the diagram below:



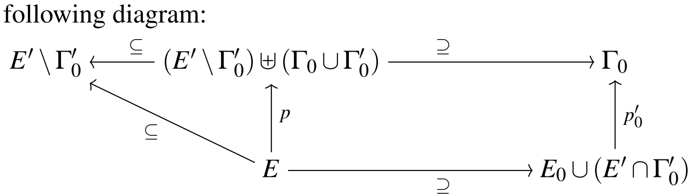
(11.2)

The main idea of our argument is that, based on these representations of p and q we construct new representations for them such that their finitary components compose. Moreover, this composition should provide a representation for $p;q$ in terms of a finitary proof.

- We start with p. We split Γ differently, as $\Gamma = (E \setminus \Gamma') \uplus (\Gamma_0 \cup \Gamma'_0)$ and consider $p'_0 = \langle \supseteq; p_0, \ \supseteq \rangle$ as shown in the diagram below:



All four set-inclusion claims in this diagram are trivial. That Γ_0 and $E' \cap \Gamma'_0$ are disjoint follows from the disjointness of Γ_0 and E'. Now we claim the commutativity of the following diagram:



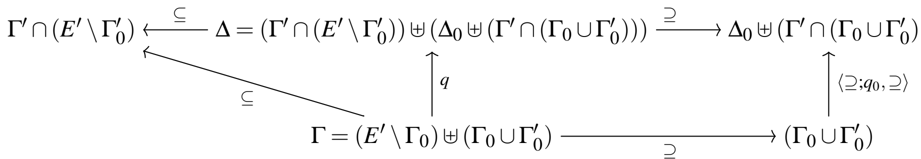
(11.3)

The correctness of this diagram requires the following couple of equalities.

1	$\Gamma = (E' \setminus \Gamma_0) \uplus (\Gamma_0 \cup \Gamma'_0)$	$\Gamma = E' \uplus \Gamma_0, \Gamma'_0 \subseteq \Gamma$
2	$\Gamma_0 \cup \Gamma'_0 = \Gamma_0 \uplus (E' \cap \Gamma'_0)$	$E' \cap \Gamma_0 = \emptyset, \Gamma'_0 \setminus \Gamma_0 \subseteq E' \cap \Gamma'_0$.

While the commutativity of the left-hand side part of (11.3) (the triangle) just follows immediately from the representation of p in terms of p_0 (diagram (11.2)), to establish the commutativity of the right-hand side part of (11.3) (the square) requires a bit of work. We use 2 and compose both side of the square with the monotonicity proofs $\Gamma_0 \uplus (E' \cup \Gamma'_0) \longrightarrow \Gamma_0$ and $\Gamma_0 \uplus (E' \cup \Gamma'_0) \longrightarrow E' \cup \Gamma'_0$. In both cases we get equalities (we skip these straightforward calculations), and thus the uniqueness aspect of the direct products solves the problem.
Now, because Γ'_0, Γ_0, E_0 are finite we have that p'_0 is finitary, hence (11.2) yields a representation of p in terms of the finitary proof p'_0.

- Now, we deal with q. We claim that q can be represented as in the diagram below in terms of the finitary proof $\langle \supseteq; q_0,\ \supseteq \rangle$.



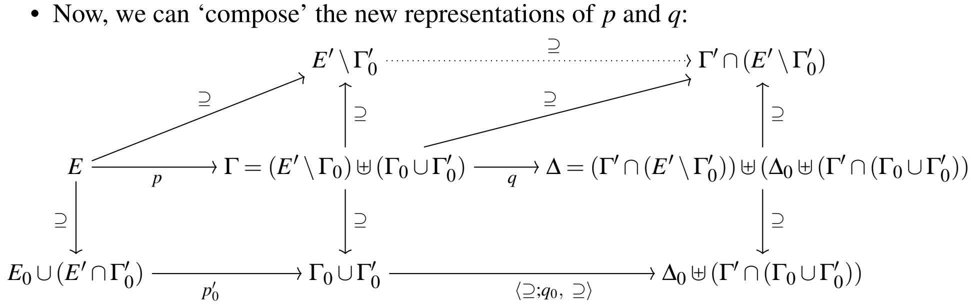
(11.4)

First we have to do the following straightforward checks that involve a bit of set-theoretic calculations (which we skip):

3	$\Delta_0 \cap (\Gamma' \cap (\Gamma_0 \cup \Gamma_0')) = \emptyset$	$\Delta_0 \cap \Gamma' = \emptyset$
4	$(\Gamma' \cap (E' \setminus \Gamma_0')) \cap (\Delta_0 \uplus (\Gamma' \cap (\Gamma_0 \cup \Gamma_0'))) = \emptyset$	$\Delta_0 \cap \Gamma' = \emptyset, E' \cap \Gamma_0 = \emptyset$
5	$\Gamma' = (\Gamma' \cap (E' \setminus \Gamma_0')) \cup (\Gamma' \cap (\Gamma_0 \cup \Gamma_0'))$	$(E' \setminus \Gamma_0') \cup (\Gamma_0 \cup \Gamma_0') = \Gamma.$

The commutativity of the left-hand side of (11.4) (triangle) follows immediately from the representation of q in terms of q_0 (diagram (11.2)). For establishing the commutativity of the right-hand side of (11.4) (square) we compose both sides of the square with the monotonicity proofs $\Delta_0 \uplus (\Gamma' \cap (\Gamma_0 \cup \Gamma_0')) \longrightarrow \Delta_0$ and $\Delta_0 \uplus (\Gamma' \cap (\Gamma_0 \cup \Gamma_0')) \longrightarrow \Gamma' \cap (\Gamma_0 \cup \Gamma_0')$. In both cases we get equalities (again, we skip the straightforward calculations). Then we apply again the uniqueness property of directed products.
Since $\Delta_0, \Gamma_0, \Gamma_0'$ are finite, $\langle \supseteq; q_0,\ \supseteq \rangle$ is finitary.

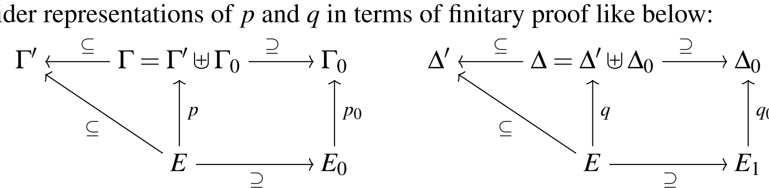
- Now, we can 'compose' the new representations of p and q:



and obtain a representation of $p;q$ in terms of the finitary proof $p_0'; \langle \supseteq; q_0,\ \supseteq \rangle$.

2. *Direct products (vertical compositions).* Let us consider quasi-finitary proofs $E \xrightarrow{p} \Gamma$ and $E \xrightarrow{q} \Delta$ such that $\Gamma \cap \Delta = \emptyset$. We prove that $E \xrightarrow{\langle p,q \rangle} \Gamma \uplus \Delta$ is quasi-finitary.

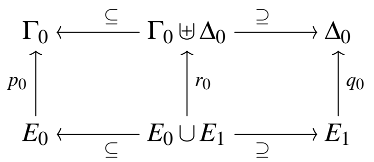
- Consider representations of p and q in terms of finitary proof like below:



(11.5)

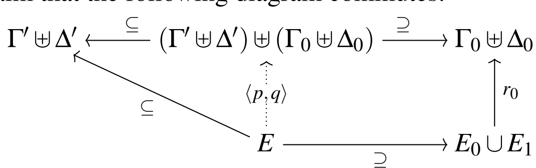
Since $\Gamma \cap \Delta = \emptyset$ it follows $\Gamma_0 \cap \Delta_0 = \emptyset$.

- Consider the proof $r_0 = \langle \supseteq; p_0, \supseteq; q_0 \rangle$ defined in the commutative diagram below:



  Since $\Gamma_0, \Delta_0, E_0, E_1$ are finite, r_0 is finitary.

- We claim that the following diagram commutes:

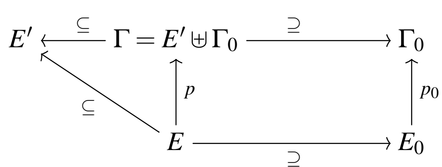


(11.6)

  The left-hand side of (11.6) is obtained by applying the monotonicity proofs $\Gamma' \uplus \Delta' \longrightarrow \Gamma'$ and $\Gamma' \uplus \Delta' \longrightarrow \Delta'$ to the two sides of the triangle and by using the commutativity of the left-hand sides of the diagram (11.5). Through easy diagram chasing we obtain equalities, which by the uniqueness of the mediating arrows to direct products yield the desired commutativity.
  The commutativity of the right-hand side of (11.6) can be obtained similarly, by applying the two monotonicity proofs to both sides of the square.

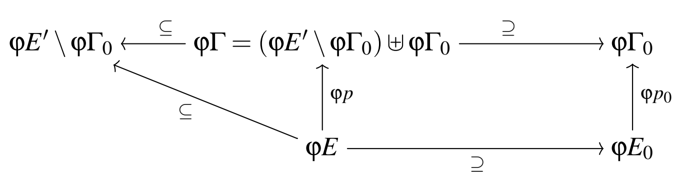
Then (11.6) gives a representation of $\langle p, q \rangle$ in terms of the finitary proof r_0.

3. *Translations of proofs along signature morphisms.* We have to prove that for any signature morphism $\varphi : \Sigma \to \Sigma'$ we have that $\varphi(C(Pf\Sigma)) \subseteq C(Pf\Sigma')$. Let $E \xrightarrow{p} \Gamma$ be a quasi-finitary proof represented by a finitary proof $E_0 \xrightarrow{p_0} \Gamma_0$ as shown by the diagram below:



(11.7)

In principle, a representation of φp by a finitary proof should be obtained by applying $Pf\varphi$ to diagram (11.7). There is a little problem though, the possibility that $\varphi E' \cap \varphi \Gamma_0 \neq \emptyset$. The solution to this is to split $\varphi\Gamma$ as $(\varphi E' \setminus \varphi\Gamma_0) \uplus \varphi\Gamma_0$. Then we get the following representation of φp by the finitary proof φp_0:



□

Quasi-finitary free proof systems. Now we are able to prove the main result of this section, that was announced before. Its proof relies heavily on the exquisite power of Prop. 11.7.

Corollary 11.8. *The proof system freely generated by a system of finitary rules is quasi-finitary. Consequently, it is compact too.*

Proof. Consider a proof system (Sig, Sen, Pf) generated freely by a system of finitary proof rules (Sig, Sen, Rl, h, c), with $(1_{Sig}, 1_{Sen}, \eta)$ universal arrow.

- By Prop. 11.7 let $(Sig, Sen, C(Pf))$ be the proof-system of the quasi-finitary proofs in (Sig, Sen, Pf). Because each proof rule of (Sig, Sen, Rl, h, c) is finitary, it means that $\eta_\Sigma(Rl\Sigma) \subseteq C(Pf)\Sigma$ for each signature Σ, hence $(1_{Sig}, 1_{Sen}, \eta)$ is a comorphism of systems of proof rules $(Sig, Sen, Rl, h, c) \to (Sig, Sen, C(Pf), dom, cod)$.
- By the universal property of $(1_{Sig}, 1_{Sen}, \eta)$ there exists a unique comorphism of proof systems $(1_{Sig}, 1_{Sen}, \gamma) : \ (Sig, Sen, Pf) \to (Sig, Sen, C(Pf))$ such that the diagram below commutes:

$$\begin{array}{ccc} (Sig,Sen,Rl,h,c) & \xrightarrow{(1_{Sig},1_{Sen},\eta)} & (Sig,Sen,Pf,dom,cod) \\ & \searrow^{(1_{Sig},1_{Sen},\eta)} & (1_{Sig},1_{Sen},\gamma)\downarrow \ \uparrow(1_{Sig},1_{Sen},\gamma') \\ & & (Sig,Sen,C(Pf),dom,cod) \end{array}$$

- Let $(1_{Sig}, 1_{Sen}, \gamma')$ be the embedding sub-system comorphism $(Sig, Sen, C(Pf)) \to (Sig, Sen, Pf)$. The above diagram commutes. By the uniqueness part of the universal property for the free proof system, we get that $\gamma;\gamma' = 1$, and because γ' are inclusions, we obtain that $C(Pf) = Pf$, which means that each proof of (Sig, Sen, Pf) is quasi-finitary.

□

The result of Cor. 11.8 can be used to obtain the compactness of institutions. So far, we have a route to compactness through model-theoretic compactness (m-compactness) given by Prop. 6.18, but that does not apply to institutions without negation, such as $\mathcal{EQL}$, $\mathcal{HCL}$, $\mathcal{UNIV}$, etc. Ironically, in $\mathcal{HCL}$ and in $\mathcal{EQL}$, m-compactness is trivialised by each theory being consistent (by initial semantics).

Corollary 11.9. *Any complete institution with proofs such that the proof system is freely generated by a system of finitary rules, is compact.*

When obtaining compactness through Cor. 11.9, the hard part is the completeness property. In Sections 11.6 and 11.7 we will obtain proof systems for $\mathcal{UNIV}$, $\mathcal{HCL}$, $\mathcal{EQL}$ that fit the requirements of Cor. 11.9.

11.4 Proof-theoretic internal logic

In Chap. 5 we introduced an institution-independent semantics for propositional connectives and quantifiers. Here we introduce proof-theoretic definitions for these logical connectives. Moreover, the proof-theoretic connectives can be seen as extensions of their semantic counterparts by considering the proof system of the institution defined by the semantic consequence relation. When doing that, the definition of the proof-theoretic logical connectives appear as properties of the semantic consequence relation.

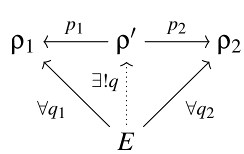
In this section we do the following:

1. We introduce the proof-theoretic interpretation of the propositional and the quantification connectives as universal properties in the categories of proofs.
2. We show how the proof-theoretic interpretations of the logical connectives can be added freely (in the sense of free constructions) to proof systems. This constitutes a an important step of a straightforward method for building sound an complete proof systems for institutions with semantic logical connectives.
3. We show that the free construction of proof system with connectives preserve the crucial properties of soundness and quasi-finitarity of the original proof systems.

Some of the definitions and proofs of results in this section, albeit straightforward, may be technically tedious. Because of this we will afford to skip some details and present only the important ideas. All skipped details can be easily recovered by the interested reader. For this, all that is needed is patience.

Propositional connectives

Conjunctions. A Σ-sentence ρ' is a *proof-theoretic conjunction* of Σ-sentences ρ_1 and ρ_2 when ρ' is the direct product of ρ_1 and ρ_2 in $Pf\Sigma$.



In the language of adjunctions this can be expressed as a natural isomorphism

$$Pf\Sigma\,(E,\rho_1)\times Pf\Sigma\,(E,\rho_2)\;\cong\;Pf\Sigma(E,\rho').$$

Fact 11.10. *In any institution, a semantic conjunction is the same with a proof-theoretic conjunction in the semantic proof system.*

Disjunctions, true, false. These can be defined in the same manner as the conjunctions. As expected, proof-theoretic disjunctions are dual to the conjunctions, disjunctions being co-products in the category of proofs. Proof-theoretic true and false, respectively, are defined as terminal and initial objects, respectively. Identity situations like that described in Fact 11.10 hold also for these three propositional connectives.

Negations. A Σ-sentence ρ' is a *proof-theoretic negation* of a Σ-sentences ρ when for each $\Gamma \subseteq Sen\Sigma$ and for each false sentence false, we have the natural isomorphism:

$$Pf\Sigma\,(\Gamma \cup \{\rho\}, \mathsf{false}) \;\cong\; Pf\Sigma\,(\Gamma, \rho').$$

Proposition 11.11. *If* ρ'' *is a double-negation of a* Σ*-sentence* ρ *then there exists a canonical proof* $\rho \longrightarrow \rho''$.

Proof. Let ρ' be a negation of ρ such that ρ'' is a negation of ρ'. We have the following isomorphisms:

$$Pf\Sigma\,(\rho', \rho') \cong Pf\Sigma\,(\{\rho, \rho'\}, \mathsf{false}) \cong Pf\Sigma\,(\rho, \rho'').$$

Since $Pf\Sigma\,(\rho', \rho') \neq \emptyset$ (it contains the proof $1_{\rho'}$), it follows that $Pf\Sigma\,(\rho, \rho'') \neq \emptyset$. □

When for each sentences ρ and ρ'' such that ρ'' is a double negation of ρ, ρ and ρ'' are proof-theoretic equivalent, then we say that the proof system has $\neg\neg$*-elimination*.

Fact 11.12. *For any institution with semantic negations, its semantic entailment system has* $\neg\neg$*-elimination.*

Implications. A Σ-sentence ρ' is *proof-theoretic implication* of a Σ-sentence ρ from γ when for each $\Gamma \subseteq Sen\Sigma$ there exists a bijective correspondence between $Pf\Sigma\,(\Gamma \cup \{\gamma\}, \rho)$ and $Pf\Sigma\,(\Gamma, \rho')$, which is natural in Γ. The conventional logic version of this property is known under the name of *Deduction Theorem*.

Fact 11.13. *Let* false *be a proof-theoretic false sentence. Then* ρ' *is a negation of* ρ *when it is an implication of* false *from* ρ.

The identity between semantic and proof-theoretic implications holds in the proof-theoretic semantic system of any institution (like in Fact 11.10).

Quantifiers

Let $\chi\colon\ \Sigma \to \Sigma'$ be a signature morphism in a proof system. Then for any Σ-sentence ρ and for any Σ'-sentence ρ', ρ is a *universal* χ*-quantification of* ρ' when for each set of Σ-sentences E we have the natural isomorphism:

$$Pf\Sigma\,(E, \rho) \;\cong\; Pf\Sigma\,(\chi E, \rho').$$

On the other hand, ρ is an *existential* χ*-quantification of* ρ' when:

$$Pf\Sigma\,(\rho, E) \;\cong\; Pf\Sigma'(\rho', \chi E).$$

The isomorphism defining the proof-theoretic universal quantification is known in conventional logic as the ‘Generalization Rule’. Of course, this may appear in other forms, such as an equivalence between entailments, etc. This is rather a property of the proof system than a generating rule, hence calling it ‘meta-rule’ instead of ‘rule’ would be more appropriate.

Finally, the identity between semantic and proof-theoretic quantifications holds in the proof-theoretic semantic system of any institution (like in Fact 11.10).

Designated connectives and adjunctions

Like in the case of the semantic connectives, the proof-theoretic connectives may receive syntactic support at the level of the sentence functor. For instance, in the case of designated conjunctions, this means that for any Σ-sentences ρ_1 and ρ_2, there exists a designated sentence, commonly denoted $\rho_1 \wedge \rho_2$, that is the proof-theoretic conjunction of ρ_1 and ρ_2. Designations of the other connectives are defined similarly. Then the definitions of connectives amount to adjunctions. For instance, the natural isomorphism

$$Pf\Sigma\,(E,\rho_1) \times Pf\Sigma\,(E,\rho_2) \;\cong\; Pf\Sigma\,(E,\rho_1 \wedge \rho_2) \tag{11.8}$$

says that for each pair (ρ_1,ρ_2) there exists an universal arrow from the diagonal functor $\Delta\colon\; Pf\Sigma \to (Pf\Sigma)^2$ to (ρ_1,ρ_2). This consists of the pair of projections proofs

$$\rho_1 \xleftarrow{\;p_1\;} \rho_1 \wedge \rho_2 \xrightarrow{\;p_2\;} \rho_2$$

If we consider proofs only between single sentences then we get the designation of the conjunctions as a right adjoint $\_\wedge\_\colon\; (Pf\Sigma)^2 \to Pf\Sigma$ to the diagonal functor $\Delta\colon\; Pf\Sigma \to (Pf\Sigma)^2$.

Another aspect of these designations is the naturality with respect to the translations of signatures. For instance, the isomorphism (11.8) should be natural also in Σ, not only in E,ρ_1,ρ_2. From the perspective of designation-as-adjunction this means that each signature morphism $\varphi\colon\; \Sigma \to \Sigma'$ gives a morphism between the corresponding adjunctions and, moreover, this mapping is functorial. All these are of course expected coherence properties.

The special case of the quantifications. The adjunction view on proof-theoretic designated quantifications, either universal or existential, comes with a complication at the level of the naturality with respect to signature translations. This is because the context of the quantifications is a signature morphism ($\chi\colon\; \Sigma \to \Sigma'$ in the above definition of the proof-theoretic quantifications) rather than a single signature. Thus, a translation from $\chi\colon\; \Sigma \to \Sigma'$ to $\chi_1\colon\; \Sigma_1 \to \Sigma_1'$ means a pair (θ,θ') of signature morphisms such that the square below commutes:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\;\theta\;} & \Sigma_1 \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi_1} \\ \Sigma' & \xrightarrow[\;\theta'\;]{} & \Sigma_1' \end{array}$$

Moreover, we do not consider all such squares, but only those that are pushout squares. In other words, the signature morphisms that quantify (χ) are not translated to any signature morphism. In this sense $\mathcal{FOL}$ is a good example, the translation squares are always pushout squares and this is intimately related to the model amalgamation property that enables the satisfaction condition for the quantified sentences.

It is also helpful to consider a designated class $\mathcal{D}$ of signature morphisms from which χ and χ_1 are taken. For instance, in $\mathcal{FOL}$ this class $\mathcal{D}$ consists of the signature extensions with finite blocks of variables. In this case, it is technically convenient to axiomatise that $\mathcal{D}$ is stable under pushouts. Note that this is indeed the case in $\mathcal{FOL}$. All these structures and conditions imposed on the quantifications have little direct proof-theoretic relevance, they really come from the model theory side of logics. Since we are interested to aggregate our proof theory to institutions in a meaningful way, these structures and conditions are needed.

Proof systems with connectives everywhere. As a matter of terminology, we say that a proof system has a certain connective when this exists for all sentences. For instance, a proof system has conjunctions if any two Σ-sentences have a conjunction. When we say that it is also designated, then this designation has to obey the coherence conditions just discussed. For instance, it is possible to define $\mathcal{PL}$ only with conjunction and negation as designated connectives, all other $\mathcal{PL}$ connectives being there but not in a designated way.

Free proof systems with connectives

Suppose that we have an institution where the sentences are constructed inductively by using certain connectives, and that the satisfaction relation follows the structure of the sentences. We know that most 'logical' institutions are like that as their satisfaction relation follows the 'Tarskian' definition of semantic truth. Suppose we want to build a proof system for the institution such that it is sound and complete. To achieve completeness, we have to address each connective proof-theoretically. A way to do this is first to build a sound and complete proof system for the atomic part of the institution and then enforce the properties of the respective connectives on the proof system. The resulting proof system should retain the soundness and the completeness. Moreover, this can be further iterated. This process can be regarded as a free construction, as represented by the diagram below:

$$\begin{array}{ccc} (Sig, Sen, Pf) & \xrightarrow{(1_{Sig}, 1_{Sen}, \omega)} & (Sig, Sen, \overline{Pf}) \\ & \searrow^{\forall(\Phi,\alpha,\gamma)} \quad {}^{\exists!(\Phi,\alpha,\overline{\gamma})}\swarrow & \\ & (Sig', Sen', Pf') & \end{array}$$

where $(1_{Sig}, 1_{Sen}, \omega)$ is the universal arrow from the original proof system to the forgetful functor from proof systems with connectives to just proof systems.

The following result is of the same species as Thm. 11.3 (the free construction of proof systems from systems of rules).

Theorem 11.14. *Consider a proof system (Sig, Sen, Pf) such that its sentences have designated syntax for certain connectives (propositional or quantifications). Then there exists a free proof system over (Sig, Sen, Pf) that has the respective connectives in a designated form.*

Proof. The main idea of this proof is shared with the proof of Thm. 11.14, namely to encode the problem as a free construction problem along a morphism of quasi-existence equational theory in $\mathcal{PA}$. This works well because the definitions of all proof-theoretic connectives that we have introduced above give rise to quasi-existence theories. □

The result of Thm. 11.14 applies well to the situations that interest us most, when we have an institution with proofs, that have designated semantic connectives but the existing proof system does not support them. Then we can use this result for enhancing the proof system with new proofs corresponding to the proof-theoretic interpretation of the respective connectives.

The free constructions of Thm. 11.14 preserve the crucial properties of soundness and quasi-finitary as shown by the following couple of corollaries.

Corollary 11.15. *The free proof system with connectives over a quasi-finitary proof system is quasi-finitary too.*

Proof. We replicate the argument of Cor. 11.8

- by taking the quasi-finitary proof (sub-)system $(Sig, Sen, C(\overline{Pf}))$ (which exists cf. Prop. 11.7) of the free proof system with connectives $(Sig, Sen, \overline{Pf})$, and
- by noting that proof system comorphisms preserve quasi-finitary proofs (this fact being similar to the fact that the translations along signature morphisms preserve quasi-finitary proofs; see third part of the proof of Prop. 11.7).

This means that if we assume that (Sig, Sen, Pf) is quasi-finitary, then the universal comorphism $(Sig, Sen, Pf) \to (Sig, Sen, \overline{Pf})$ goes *de facto* to $(Sig, Sen, C(\overline{Pf}))$. □

Proposition 11.16. *Let* $(Sig, Sen, Mod, \models, Pf)$ *be any sound institution with proofs and with certain semantic connectives. Let* $(1_{Sig}, 1_{Sen}, \omega) : (Sig, Sen, Pf) \to (Sig, Sen, Mod, \models, \overline{Pf})$ *be a free proof system with connectives corresponding to the existing semantic connectives. Then* $(Sig, Sen, Mod, \models, \overline{Pf})$ *is a sound institution with proofs.*

Proof. We replicate the proof of Prop. 11.5

- by expressing soundness as comorphisms of proof systems to the semantic proof system $(Sig, Sen, \models)$ (which is possible because for all connectives in the semantic entailment system we have an identity between the respective semantic and the proof-theoretic connective, a fact that has always been mentioned whenever we introduced each proof-theoretic connective), and
- by using the universal property given by Thm. 11.14:

$$\begin{array}{ccc} (Sig, Sen, Pf) & \xrightarrow{(1_{Sig}, 1_{Sen}, \omega)} & (Sig, Sen, \overline{Pf}) \\ & {\scriptstyle (1_{Sig}, 1_{Sen}, \gamma)} \searrow \quad \swarrow {\scriptstyle (1_{Sig}, 1_{Sen}, \overline{\gamma})} & \\ & (Sig, Sen, \models) & \end{array}$$

□

Exercises

11.5. In any entailment system, the entailment-theoretic equivalence $\rho_1 \dashv\vdash \rho_2$ determines for each signature Σ a quotient of the preorder $(Sen\Sigma, \vdash)$ to a partial order $(Sen\Sigma/_{\dashv\vdash}, \leq)$. If the entailment system has conjunctions, disjunctions, true, false, and implications, then $(Sen\Sigma/_{\dashv\vdash}, \leq)$ is a Heyting algebra.

11.6. Any entailment system with conjunctions, negations, and $\neg\neg$-elimination has disjunctions and implications. (*Hint:* Define $\rho_1 \vee \rho_2$ as $\neg(\neg\rho_1 \wedge \neg\rho_2)$) and $\rho_1 \Rightarrow \rho_2$ as $\neg\rho_1 \vee \rho_2$. Use the fact that in any entailment system with negations and $\neg\neg$-elimination we have that $\rho \vdash \rho'$ is equivalent to $\neg\rho' \vdash \neg\rho$.) Does this result generalize to proof systems?

11.7. Collapsing theorem
In any proof system with implication, false, and $\neg\neg$-elimination, there exists at most one proof between any two finite sets of sentences. (*Hint:* By using implications and the initiality of false, we have that for each finite set of sentences E there exists at most one proof $E \cup \{\mathsf{false}\} \longrightarrow E \cup \{\mathsf{false}\}$. Use this for showing that the existence of a proof $E \longrightarrow \mathsf{false}$ implies $E \cong \mathsf{false}$. The conclusion follows by $\neg\neg$-elimination which gives that proofs $E \longrightarrow \rho'$ are in natural bijective corrrespondence to proofs $E \cup \{\neg\rho'\} \longrightarrow \mathsf{false}$.)

11.8. Maximally consistent sets, proof theoretically
Consider an entailment system $(Sig, Sen, \vdash)$ with negations and false. A set of Σ-sentences Γ is *consistent* when $\Gamma \not\vdash \mathsf{false}$. It is maximally consistent when it is consistent and it is maximal with respect to this property, i.e., for any other consistent set Γ' such that $\Gamma \subseteq \Gamma'$ we have that $\Gamma = \Gamma'$. For each signature Σ we let $Mod\Sigma = \{M \subseteq Sen\Sigma \mid M \text{ maximally consistent}\}$ and for each signature morphism $\varphi\colon \Sigma \to \Sigma'$ we let $Mod\varphi\colon Mod\Sigma' \to Mod\Sigma$ be defined by $(Mod\varphi)M' = \varphi^{-1}M'$. We may define a satisfaction relation $\models_\Sigma \subseteq Mod\Sigma \times Sen\Sigma$ by $M \models \rho$ if and only if $\rho \in M$.

1. $(Sig, Sen, Mod, \models, \vdash)$ is an institution with proofs that is sound and has semantic negations.
2. If in addition we assume that $(Sig, Sen, \vdash)$ is compact, then $(Sig, Sen, Mod, \models, \vdash)$ is complete if and only if $(Sig, Sen, \vdash)$ has $\neg\neg$-elimination. (*Hint:* Prove and use the generalization of Lindenbaum's Thm. of Ex. 7.2 to entailment systems.)

11.9. Craig interpolation, proof theoretically
The proof-theoretic concept of interpolation refines the semantic concept of interpolation by considering the interpolant to be a set of sentences together with two corresponding proofs. In any proof system, a square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is a *Craig Interpolation square* if and only if for each set E_1 of Σ_1-sentences and *finite* set E_2 of Σ_2-sentences and each proof $p\colon\ \theta_1 E_1 \to \theta_2 E_2$ there exists a set E of Σ-sentences and proofs $p_1\colon\ E_1 \to \varphi_1 E$ and $p_2\colon\ \varphi_2 E \to E_2$ such that $p = \theta_1 p_1 \,;\, \theta_2 p_2$.

1. Proof-theoretic Craig interpolation squares are closed under both the 'vertical' and the 'horizontal' compositions (in the sense of Ex. 9.2).

2. Formulate a 'single sentence' version for proof-theoretic interpolation and prove that this is a consequence of the 'multiple sentence' version when the proof system is compact and has conjunctions.

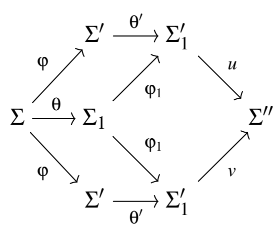
11.10. Craig-Robinson interpolation, proof theoretically
Craig interpolation (abbreviated *Ci*) for proof systems (see Ex. 11.9 above) can be refined to Craig-Robinson interpolation (abbreviated *CRi*) by generalizing the concept of model-theoretic CRi of Sect. 9.5.

1. Extend the result of Prop. 9.24 (which gives sufficient conditions for the equivalence between CRi and Ci) from semantic entailment systems to arbitrary entailment systems.
2. Try to further extend the result at the previous item from entailment systems to proof systems.

11.11. [135] **Proof-theoretic implicit definability**
We say that a signature morphism $\varphi : \Sigma \to \Sigma'$ in a proof system is *defined implicitly* (proof theoretically) by $E' \subseteq Sen\Sigma'$ if for each signature morphism $\theta : \Sigma \to \Sigma_1$ and each Σ'_1-sentence ρ,

$$(\theta';u)E' \cup (\theta';v)E' \cup \{u\rho\} \vdash v\rho \text{ and } (\theta';u)E' \cup (\theta';v)E' \cup \{v\rho\} \vdash u\rho$$

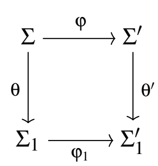
for all pushout squares of the form



In any institution with model amalgamation a signature morphism φ

1. is defined implicitly proof theoretically by E' for the semantic entailment system if it is defined implicitly model theoretically by E' (in the sense of Chap. 10), and
2. is defined implicitly model theoretically by E' if it is defined implicitly proof theoretically by E' when it is ι-tight for a system ι of diagrams of the institutions.

11.12. [135] **Definability by interpolation, proof theoretically**
Given a proof system, we say that a signature morphism φ is *defined explicitly* by $E' \subseteq Sen\Sigma'$ when for each pushout square



and for each $\rho \in Sen\Sigma'_1$, there exists a set of sentences $E_\rho \subseteq Sen\Sigma_1$ such that

$$\theta' E' \cup \{\rho\} \vdash \varphi_1 E_\rho \text{ and } \theta' E' \cup \varphi_1 E_\rho \vdash \rho.$$

The following constitute proof theoretic variants of Prop. 10.4 and Thm. 10.5, respectively.

1. Any signature morphism is defined implicitly by a theory if it is defined explicitly by that theory.

2. If the entailment system has Craig-Robinson $(\mathcal{L},\mathcal{R})$-interpolation (see Ex. 11.10) for classes $\mathcal{L}$ and $\mathcal{R}$ of signature morphisms that are stable under pushouts, any signature morphism in $\mathcal{L}\cap\mathcal{R}$ is defined explicitly if it is defined implicitly.

11.13. Express proof-theoretic designated disjunctions, negations and implications as adjunctions.

11.14. Develop the details of the proof of Thm. 11.14 for each of the propositional and quantification connectives.

11.5 The entailment institution

In any institution the satisfaction relation between models and sentences determine the semantic consequence which has the properties of an entailment system. At this consequence-theoretic level of the semantic entailment system the concepts of models and satisfaction are absent, so this move puts us beyond the model theory realm. In this section we study the possibility to move in the opposite direction, namely to build institutions from entailment systems in a general meaningful way.

Entailment-theoretic theories and their morphisms. Our construction of free institutions over entailment systems requires entailment-theoretic concepts of theory and morphism of theories. These are quite straightforward to come up with if we express the corresponding institution-theoretic concepts in terms of the semantic consequence relation, and then just replace $\models$ by $\vdash$. The entailment-theoretic concept of theory is the same as in institutions, a signature and a set of sentences for that signature. The definition of the proof-theoretic concept of morphism of theories requires more attention. For any $E \subseteq Sen\Sigma$ let $E^\bullet = \{e \mid E \models e\}$. Then a morphism of theories $\varphi : (\Sigma, E) \to (\Sigma', E')$ is defined as a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $\varphi E \subseteq E'^\bullet$. Now, it is important that such morphisms of theories do compose. In general, they do not. The solution to this is to assume compactness.

Proposition 11.17. *In any compact entailment system the composition of theory morphisms, defined by the composition of the underlying signature morphisms, yields a theory morphism.*

Proof. Consider theory morphisms $\varphi : (\Sigma, E) \to (\Sigma', E')$ and $\varphi' : (\Sigma', E') \to (\Sigma'', E'')$. We prove that $\varphi;\varphi' : (\Sigma, E) \to (\Sigma'', E'')$ is a theory morphism, which amounts to proving that for each $e \in E$, $E'' \vdash \varphi'(\varphi e)$. We have that:

1	$E' \vdash \varphi e$	$\varphi : (\Sigma, E) \to (\Sigma', E')$ theory morphism, $e \in E$
2	there exists $E'_0 \subseteq E'$ finite such that $E'_0 \models \varphi e$	1, compactness
3	$\varphi' E'_0 \vdash \varphi'(\varphi e)$	2, translation of entailment
4	$E'' \vdash \varphi' E'_0$	$\varphi' : (\Sigma', E') \to (\Sigma'', E'')$, E'_0 finite, union of entailment
5	$E'' \vdash \varphi'(\varphi e)$	3, 4, transitivity of entailment.

□

An alternative way to define morphisms of theories, that would avoid the compactness condition, would be to ask that $E' \vdash \varphi E$. However, this definition is too strong as there can be theory morphisms in the former acceptation that are not theory morphisms in the latter acceptation. This is so because entailment systems do not always admit infinite unions. In itself this perhaps is not convincing enough, but it becomes a good argument when we lean towards the idea that the provability of everything in φE is what is relevant, and not necessarily the provability of φE as a whole entity.

The canonical institution over an entailment system. We first define the institution, then prove its soundness and completeness properties.

Proposition 11.18 (Entailment institution)**.** *Each compact entailment system $\mathcal{E} = (Sig, Sen, \vdash)$ determines an institution $I(\mathcal{E}) = (Sig, Sen, Mod, \models)$, called the* entailment institution *of $\mathcal{E}$, where for each signature $\Sigma \in |Sig|$,*

- *the Σ-models are pairs (ψ, E'), where $\psi\colon\ \Sigma \to \Sigma'$ is a signature morphism and E' is a Σ'-theory,*
- *a Σ-model homomorphism $\varphi\colon\ (\psi\colon\ \Sigma \to \Sigma',\ E') \to (\psi'\colon\ \Sigma \to \Sigma'',\ E'')$ is a theory morphism $\varphi\colon\ (\Sigma', E') \to (\Sigma'', E'')$ such that $\psi\ ;\ \varphi = \psi'$,*
- *a Σ-model (ψ, E') satisfies a Σ-sentence ρ if and only if $E' \vdash \psi\rho$,*
- *model reducts are obtained just by composition to the left.*

Proof. Apart of the Satisfaction Condition, the other institution axioms are trivial to check on the entailment institution. The compactness hypothesis is necessary for getting the categories of Σ-homomorphisms through the result of Prop. 11.17. For proving the Satisfaction Condition, we consider a signature morphism $\varphi\colon\ \Sigma \to \Sigma'$, any Σ'-model (ψ', E''), and any Σ-sentence ρ. Then the following relations are equivalent:

$(Mod\varphi)(\psi', E'') \models \rho$	
$(\varphi;\psi',\ E'') \models \rho$	definition of reduct
$E'' \vdash \psi'(\varphi\rho)$	definition of satisfaction (Σ)
$(\psi', E'') \models \varphi\rho$	definition of satisfaction (Σ').

□

Soundness and completeness. With the entailment institutions, soundness and completeness come by default.

Proposition 11.19. *Let $\mathcal{E} = (Sig, Sen, \vdash)$ by any compact entailment system and let $I(\mathcal{E}) = (Sig, Sen, Mod, \models)$ be its associated institution (according to Prop. 11.18). Then $(Sig, Sen, Mod, \models, \vdash)$ is sound and complete.*

Proof.

- *Soundness.* Let us assume that $E \vdash \rho$. Consider any Σ-model (ψ, E') such that $(\psi, E') \models E$. We prove that $(\psi, E') \models \rho$ too.

  - $(\psi, E') \models E$ means $\psi : (\Sigma, E) \to (\Sigma', E')$ is a proof-theoretic morphism of theories.
  - $E \vdash \rho$ means $1_\Sigma : (\Sigma, \rho) \to (\Sigma, E)$ is a morphism of theories too.
  - By Prop. 11.17, by using the compactness hypothesis, we obtain that $\psi = 1_\Sigma ; \psi$ is a morphism of theories $(\Sigma, \rho) \to (\Sigma', E')$. This means $E' \vdash \psi\rho$.

- *Completeness.* Let us assume $E \models \rho$. We consider the Σ-model $(1_\Sigma, E)$. Then $(1_\Sigma, E) \models E$, hence $(1_\Sigma, E) \models \rho$. By the definition of $\models$ it follows that $E \vdash \rho$.

□

The following result is an immediate consequence of Prop. 11.19.

Corollary 11.20 (Compactness)**.** *Let $(Sig, Sen, \vdash)$ be a compact entailment system. Then its associated entailment institution is compact too. Moreover, it is trivially model compact.*

Exercises

11.15. Theory morphisms in proof systems
Given a proof system (Sig, Sen, Pf), a theory is a pair (Σ, E) where Σ is a signature and $E \subseteq Sen\Sigma$. This is like in institutions or in entailment systems. A *theory morphism* $(\varphi, p) : (\Sigma, E) \to (\Sigma', E')$ consists of a signature morphism $\varphi : \Sigma \to \Sigma'$ and a family of proofs $p = (p_e)_{e \in E}$ such that $p_e \in Pf\Sigma' \ (E', \varphi e)$.

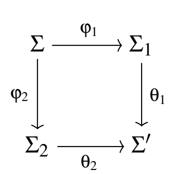
1. Are proof-theoretic morphisms of theories an extension of the concept of entailment-theoretic morphism of theories?
2. Define a composition of proof-theoretic morphisms of theories and then generalise the result of Prop. 11.17 to compact proof systems.

11.16. Special theory morphisms in entailment / proof systems
A theory morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ in a compact entailment system

- is *closed* when $E^\bullet = \varphi^{-1} E'^\bullet$; and
- is *strong* when $(\varphi E)^\bullet = E'^\bullet$.

1. Do the concepts of closed / strong morphisms of theories in a semantic entailment system coincide with the corresponding institution-theoretic concepts (as introduced in Sec. 4.5)?
2. Show that the closed /strong morphisms of theories form sub-categories of the category of theory morphisms.
3. Extend the concepts of closed / strong theory morphisms from entailment systems to proof systems (see Ex. 11.15).

11.17. Modularization squares and interpolation
In any compact entailment system a commutative square of signature morphisms



is said to be a *modularization square* when for any commutative square of theory morphisms like below

$$\begin{array}{ccc} (\Sigma,E) & \xrightarrow{\varphi_1} & (\Sigma_1,E_1) \\ {\scriptstyle \varphi_2}\big\downarrow & & \big\downarrow{\scriptstyle \theta_1} \\ (\Sigma_2,E_2) & \xrightarrow[\theta_2]{} & (\Sigma',\theta_1 E_1 \cup \theta_2 E_2) \end{array}$$

if θ_1 is closed (in the sense introduced in Ex. 11.16) then θ_2 is closed too.
Prove that modularization squares are exactly the Craig-Robinson interpolation squares. Does this equivalence also hold in proof systems?

11.18. Propositional connectives in entailment institutions
Consider an entailment system $\mathcal{E}$ and let $I(\mathcal{E})$ be its entailment institution.

1. If $\mathcal{E}$ has conjunctions, then $I(\mathcal{E})$ has conjunctions too.
2. When $\mathcal{E}$ has disjunctions, implications, negations, or false, $I(\mathcal{E})$ does not necessarily have the corresponding connectives.

11.19. The entailment system of theories
Consider a compact entailment system $\mathcal{E} = (Sig, Sen, \vdash)$. Let *Th* be the category of its theories.

1. Then $\mathcal{E}^{\text{th}} = (Th, Sen^{\text{th}}, \vdash^{\text{th}})$ is the *entailment system of theories* where *Th* is the category of theories, $Sen^{\text{th}}(\Sigma,E) = Sen\Sigma$, and
$$\Gamma \vdash^{\text{th}}_{(\Sigma,E)} \Gamma' \text{ if and only if } \Gamma \cup E \vdash_\Sigma \Gamma'.$$
is an entailment system.
2. $\mathcal{E}^{\text{th}}$ has conjunctions, false, negations, and implications, respectively, when $\mathcal{E}$ has conjunctions, false, negations, and implications, respectively.
3. $\mathcal{E}^{\text{th}}$ has disjunctions if $\mathcal{E}$ has disjunctions and implications.
4. If $\mathcal{E}$ has universal $\mathcal{D}$-quantification, then $\mathcal{E}^{\text{th}}$ has universal $\mathcal{D}^{\text{th}}$-quantification where $\mathcal{D}^{\text{th}} = \{\chi : (\Sigma,E) \to (\Sigma',E') \text{ strong theory morphism } \mid \chi \in \mathcal{D}\}$ (see Ex. 11.16).

11.20. Co-limits of theories in entailment systems
In any compact entailment system the forgetful functor from the category of theory morphisms to the category of signatures lifts limits and co-limits. Do we have a similar property for proof systems?

11.21. Basic model-theoretic properties of $I(\mathcal{E})$
Let $\mathcal{E}$ be a compact entailment system and $I(\mathcal{E})$ be its entailment institution. Then

1. If $\mathcal{E}$ has pushouts for signatures, then $I(\mathcal{E})$ is semi-exact and liberal.
2. In $I(\mathcal{E})$ each sentence is basic.
3. In $I(\mathcal{E})$ each signature morphism is representable.
4. In $I(\mathcal{E})$ the categories of models have the limits / co-limits that the category of the signatures has.
5. $I(\mathcal{E})$ has diagrams such that it is elementary.

11.22. Give an example of an entailment system in which the composition of theory morphisms does not necessarily yield a theory morphism.

11.6 Universal completeness

Until now we have developed a very general categorical theory of proof systems that provides a conceptual environment for developing specific general proof systems for institutions, that enjoy the good properties of soundness and completeness. Now we will do this in relation to 'Birkhoff completness'. In the traditional context, Birkhoff completeness means that the proof system of equational logic ($\mathcal{EQL}$) is complete, its soundness being somehow straightforward. This can be extended to conditional equational logic ($\mathcal{CEQL}$), and then to Horn clause logic ($\mathcal{HCL}$) by letting relations in the picture. These Birkhoff proof systems, which are sound and complete with respect to the respective institutions, are very important for computing as they serve as the basis for the operational semantics of equational and logic programming, two important declarative programming paradigms.

What we will do in what follows is to develop the Birkhoff completeness at a high abstract level, which means first defining a general proof system for a class of abstract Horn sentences, and then prove its soundness and completeness. For this, we will employ a particular method that emphasises a stepwise definition of the proof systems based on the structure of the sentences involved. In the case of Birkhoff completeness this goes as follows:

1. We assume a sound and complete abstract institution with proofs I_1 and extend its proof system to the universal quantifications of the sentences in I_1 such that the resulting institution with proofs (I) is sound and complete.

2. We get a bit more concrete about I_1. In the role of I_1 we take an institution of some quantifier-free Horn clauses that are defined over a sound and complete abstract institution with proofs I_0. The proof system of I_1 is defined as an extension of the proof system of I_0.

3. We set I_0 to some concrete institution with proofs. For instance, if I_0 is the sub-institution of $\mathcal{FOL}$ whose sentences are the atomic equations (i.e. quantifier-free equations), we can think of I_1 as being the quantifier-free sub-institution of $\mathcal{CEQL}$. Then I can be $\mathcal{CEQL}$.

If this is not yet clear, the let us say it explicitly: I, I_1 and I_0 share the same signature category and the same model functor, while $Sen_0 \subseteq Sen_1 \subseteq Sen$ (sub-functor relations) and also $\models_0 \subseteq \models_1 \subseteq \models$.

In this section we develop the first of the two steps above while in the next section we will continue with the second step. An important note is that these two have a life of their own, this being one of the great benefits of this method. For instance I_1 can be any sound and complete institution with proofs that satisfy certain axioms allowing the development of step one. While institutions of quantifier-free Horn sentences fit the role of I_1, there can be also other possibilities for I_1. We will see how some of such choices for I_1 determine interesting sound and complete proof systems for concrete institutions and are by no means Horn institutions. The structure of this section is as follows:

- We define axiomatically the relationship between I_1 and I. This has two components, the institution and the proof system.
- We prove the soundness of I_1, which will be easy.
- We prove the completeness of I_1, which will not be easy and will require some additional specific conditions.
- We develop an analysis of some of the conditions underlying the completeness theorem, in the direction of the applicability to concrete situations.

Universal institutions. Let $I = (Sig, Sen, Mod, \models)$ be an institution and:

- Let Sen_1 be a sub-functor of Sen. This means that for each signature Σ, $Sen_1\Sigma \subseteq Sen\Sigma$ and these inclusions form a natural transformation $Sen_1 \Rightarrow Sen$. The natural transformation property boils down to $\varphi(Sen_1\Sigma) \subseteq Sen_1\Sigma'$ for each signature morphism $\varphi\colon \Sigma \to \Sigma'$.
- Let $\mathcal{D} \subseteq Sig$ be a class of signature morphisms.

We say that I is a *$\mathcal{D}$-universal institution over* $I_1 = (Sig, Sen_1, Mod, \models)$, when the I-sentences are precisely *all* universal $\mathcal{D}$-quantifications of the I_1-sentences.

Let us look at some examples. If $\mathcal{D}$ is the class of all signature extensions with a set of cardinality less than β, $\mathcal{UNIV}_{\alpha,\beta}$ / $\mathcal{HCL}_{\alpha,\beta}$ are $\mathcal{D}$-universal institutions over their quantifier-free sub-institutions.

The generic universal proof system

Now we address the proof system side of the relationship between I_1 and I. In brief, the proof system of I is obtained by inheriting the proof system of I_1 and by enhancing it in two ways.

- By adding a new rule, called 'Substitutivity', which abstracts the common practice to substitute variables by terms. This abstraction relies on the concept of 'representable substitution' of Sec. 5.4. Although 'representable', which is responsible for staying within an essentially first-order context, is not necessary for defining the rule of *Substitutivity*, it is required for the completeness.
- Then we also have to impose the universal quantification property on the proof system, which, as we already discussed in Sec. 11.4, is often traditionally known as the 'meta-rule of Generalization'.

$(\mathcal{D}, Sen_0)$**-substitutivity.** The straightforward abstraction of the common substitutivity can be formulated as

$$(\forall\varphi)\rho \vdash (\forall\chi)\theta\rho, \quad \text{for any } \mathcal{D}\text{-substitution } \theta\colon\ \varphi \to \chi. \tag{11.9}$$

While this works in most cases of interest, there are important situations calling for a more refined approach. For this we need a new parameter, which is a designated sub-functor $Sen_0 \subseteq Sen_1$ satisfying the following axiom:

BA: For each finite $B \subseteq Sen_0\Sigma$ and any $e \in Sen_1\Sigma$, there exists a sentence in $Sen_1\Sigma$ which is semantically equivalent to $B \Rightarrow e$.

A couple of remarks on axiom *BA*:

- The way this axiom is formulated does not require that I_1 must have semantic conjunctions and implications, definitely not in a designated form. What is required is that there exists a sentence which *behaves semantically* like $B \Rightarrow e$, in other words a sentence γ such that $\gamma^* = (Mod\Sigma \setminus B^*) \cup e^*$.
- With the quantifier-free sub-institution of $\mathcal{UNIV}$ in the role of I_1 and Sen_0 the sub-functor of the atomic sentences, we get the axiom *BA* satisfied. This is the case when $B \Rightarrow e$ exists in a designated form as I_1 has all designated propositional connectives.
- The quantifier-free sub-institution of $\mathcal{HCL}$ with the same Sen_0 as above provides another example when the axiom *BA* holds because $B \Rightarrow (H \Rightarrow C)$ is semantically equivalent to $(\wedge B) \wedge H \Rightarrow C$. This is different from the $\mathcal{UNIV}$ case since now I_1 does not enjoy the designated propositional connectives, it does not have designated conjunction nor implications.

Then a $(\mathcal{D}, Sen_0)$-*substitution* is any $\mathcal{D}$-substitution $\theta : (\Sigma \xrightarrow{\varphi} (\Sigma_1, \emptyset)) \to (\Sigma \xrightarrow{\chi} (\Sigma_2, B))$ in I_1^{th} where B is any finite set of Sen_0-sentences.

$$\begin{array}{ccc} (\Sigma_1, \emptyset) & & (\Sigma_2, B) \\ & {}_{\varphi}\nwarrow \quad \nearrow_{\chi} & \\ & \Sigma & \end{array}$$

Given such substitution θ we define the following proof rule:

$$(\forall\varphi)\rho \vdash (\forall\chi)(B \Rightarrow \theta\rho) \qquad (\mathcal{D}, Sen_0)\text{-Substitutivity}$$

Like we often do, this proof rule does not require any designated connectives, such as quantifiers, conjunctions, implications. The rule of $(\mathcal{D}, Sen_0)$-Substitutivity relies on the axiom *BA*. Although (we will see this later on in the section) in many concrete cases of interest the rule of $(\mathcal{D}, Sen_0)$-Substitutivity takes the form (11.9), there are significant situations when its general form is useful because it can cover situations when $\theta\rho$ should be conditioned. A prominent such example comes from $\mathcal{PA}$ (partial algebra). There, the variables are total, and when substituting them by terms we have to make sure that those terms do exist semantically, or in the jargon of $\mathcal{PA}$, that they are defined. In that case B is a set of sentences of the form def t. Ex. 11.27 below builds on this idea.

Universal proof systems. Given a proof system (Sig, Sen_1, Pf_1) for I_1, $\mathcal{D} \subseteq Sig$ sub-category, $Sen_0 \subseteq Sen_1$ sub-functor, the $(\mathcal{D}, Sen_0)$-*universal proof system* for I is the free proof system such that

- it contains (Sig, Sen_1, Pf_1),
- it contains the rules of $(\mathcal{D}, Sen_0)$-substitutivity, and

- has universal $\mathcal{D}$-quantifications.

The existence of this can be established in the style of Theorems 11.3 / 11.14 by encoding to $\mathcal{PA}$ theories and using the $\mathcal{PA}$ initial semantics properties.

Abstract universal soundness

Soundness of I is established from the soundness of I_1 and the soundness of $(\mathcal{D}, Sen_0)$-Substitutivity. While the former is assumed, the latter has to be proved.

Proposition 11.21. *The rule of $(\mathcal{D}, Sen_0)$-Substitutivity is sound.*

Proof. We have to prove that $(\forall\varphi)\rho \models (\forall\chi)B \Rightarrow \theta\rho$. Let M be a model such that $M \models (\forall\varphi)\rho$. Let M_2 be a χ-expansion of M such that $M_2 \models B$. Then

1	$((Mod\theta)M_2)\restriction_\varphi = M_2\restriction_\chi = M$	$\theta: \varphi \to \chi$ substitution, $M_2 \in Mod(\Sigma_2, B)$, $M_2\restriction_\chi = M$
2	$(Mod\theta)M_2 \models \rho$	1, $M \models (\forall\varphi)\rho$
3	$M_2 \models \theta\rho$	2, Satisfaction Condition of θ.

□

Corollary 11.22 (Universal soundness)**.** *If I_1 is sound then I is sound too.*

Proof. By a process similar to that of Prop. 11.5 we lift the soundness from the proof system of I_1 to the proof system that adds freely the rules of $(\mathcal{D}, Sen_0)$-Substitutivity. Then by Prop. 11.16 we lift soundness further to the free proof system with universal $\mathcal{D}$-quantification. □

Note that the soundness of I is obtained for the broadest concept of universal quantifications since we have not imposed any conditions on $\mathcal{D}$. This means the soundness of I holds also in genuine higher-order concrete contexts. But for the completeness we will have to assume conditions on $\mathcal{D}$ that in practice mean a first-order context.

Abstract universal completeness

Completeness of the universal proof systems is significantly more difficult than the soundness property and therefore requires more conceptual infrastructure. With Thm. 11.23 below we distill a set of conditions that enable the completeness of I, the $\mathcal{D}$-universal institution with proofs over I_1. Some of them are straightforward to understand, but other are quite abstract and technical, although these may help us grasp better the fabric of the completeness property. After the proof of the theorem we will have an extensive analysis on how to get them at work in concrete cases, and in the process will appreciate their naturalness.

Theorem 11.23 (Universal completeness)**.** *I is complete if*

1. *I_1 is complete and compact,*
2. *every signature morphism in $\mathcal{D}$ is finitely representable,*

3. *every set of Sen_0-sentences is epic basic,*
4. I_1^{th} *has representable $(\mathcal{D}, Sen_0)$-substitutions, and*
5. *for each $E \subseteq Sen_1\Sigma$, $e \in Sen_1\Sigma$ we have that*

$$E \models e \ \text{ if and only if } \ M_B \models (E \Rightarrow e) \ \text{ for each } \ B \subseteq Sen_0\Sigma$$

(where M_B denotes an initial model of B, thus defining B as basic sets of sentences).

Proof. In this proof by $(\forall\zeta)\gamma'$ we mean any sentence γ which is an universal ζ-quantification of γ'. This is just a convention that eases the writing and the reading, and it does not assume designated quantification. Let us assume that $\Gamma \models_\Sigma (\forall\chi)e'$ for $\Gamma \subseteq Sen\Sigma$ and $e' \in Sen_1'\Sigma'$ where $(\chi : \ \Sigma \to \Sigma') \in \mathcal{D}$. We have to show that $\Gamma \vdash_\Sigma (\forall\chi)e'$. We do this by *Reductio as Absurdum*, by performing the following three steps:

A/ We suppose $\Gamma \not\vdash (\forall\chi)e'$.

B/ We prove that there exists $B \subseteq Sen_0\Sigma'$ such that $M_B\upharpoonright_\chi \not\models (\forall\chi)e'$, where M_B is a basic model for B.

C/ We prove that $M_B\upharpoonright_\chi \models \Gamma$.

The latter two conclusions together contradict $\Gamma \models_\Sigma (\forall\chi)e'$, hence the supposition $\Gamma \not\vdash (\forall\chi)e'$ is false, which proves the theorem.

B/ To prove this we define $\Gamma_1^\chi = \{\rho' \in Sen_1\Sigma' \mid \Gamma \vdash (\forall\chi)\rho'\}$ and prove

1 $\quad \Gamma_1^\chi \not\vdash e'$

By *Reductio ad Absurdum* suppose $\Gamma_1^\chi \vdash e'$. By the compactness of I_1, there exists a finite $\Gamma' \subseteq \Gamma_1^\chi$ such that $\Gamma' \vdash e'$. Then:

2	$\chi\Gamma \vdash \Gamma'$	$\Gamma' \subseteq \Gamma_1^\chi$, union (vertical composition) of $\vdash$
3	$\chi\Gamma \vdash e'$	2, $\Gamma' \vdash e'$, transitivity of $\vdash$
4	$\Gamma \vdash (\forall\chi)e'$	3, the proof system has universal $\mathcal{D}$-quantification

Since 4 contradicts the supposition A/, it follows that the supposition $\Gamma_1^\chi \vdash e'$ is false, hence 1 is proved. Then

5	$\Gamma_1^\chi \not\models e'$	1, completeness of I_1

By the latter hypothesis of the theorem (i.e. (5.)) there exists $B \subseteq Sen_0\Sigma$ such that $M_B \models \Gamma_1^\chi$, $M_B \not\models e'$. Hence $M_B\upharpoonright_\chi \not\models (\forall\chi)e'$.

C/ Let $(\forall\varphi)e_1 \in \Gamma$ with $(\varphi : \ \Sigma \to \Sigma_1) \in \mathcal{D}$. We prove that $M_B\upharpoonright_\chi \models (\forall\varphi)e_1$. Let N be any φ-expansion of $M_B\upharpoonright_\chi$. We show that $N \models e_1$. For this we use the following lemma (which we prove later):

Lemma 11.24. *There exists a finite $B' \subseteq B$ and a homomorphism $h\colon\ M_\varphi \to M_{B'}\restriction_\chi$ such that the diagram below commutes:*

$$\begin{array}{ccc} M_{B'}\restriction_\chi & \xrightarrow{\mu_{B'}\restriction_\chi} & M_B\restriction_\chi = N\restriction_\varphi \\ & {\scriptstyle h}\nwarrow \quad \nearrow{\scriptstyle i_\varphi N} & \\ & M_\varphi & \end{array} \tag{11.10}$$

where $\mu_{B'}$ is the unique homomorphism $M_{B'} \to M_B$ (because B' is epic basic and $B' \subseteq B$).

The representability property of χ in I_1 can be transfered to I_1^{th}, by going down to quasi-representability, and then involving the fact that $Mod(\Sigma', B')$ has $M_{B'}$ as initial model (recall Fact 5.16).

6 $\chi\colon\ \Sigma \to (\Sigma', B')$ representable in I_1^{th} and its representation is $M_{B'}\restriction_\chi$.

We consider the homomorphism h provided by Lemma 11.24 and by the hypothesis that I_1^{th} has representable $(\mathcal{D}, Sen_0)$-substitutions, there exists a substitution $\theta\colon\ (\varphi\colon\ \Sigma \to \Sigma_1) \to (\chi\colon\ \Sigma \to (\Sigma', B'))$ such that the following diagram commutes (as an instance of diagram 5.2 in Prop. 5.19.)

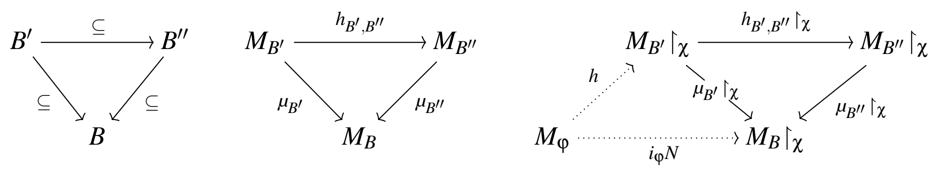
$$\begin{array}{ccc} Mod(\Sigma', B') & \underset{\cong}{\xrightarrow{i_\chi^{B'}}} & M_{B'}\restriction_\chi / Mod\Sigma \\ {\scriptstyle Mod\theta}\big\downarrow & & \big\downarrow{\scriptstyle h/Mod\Sigma} \\ Mod\Sigma_1 & \underset{i_\varphi}{\xrightarrow{\cong}} & M_\varphi / Mod\Sigma \end{array} \tag{11.11}$$

Then:

7	$(Mod\theta)M_B = N$	diagram 11.11, $i_\chi^{B'} M_B = \mu_{B'}\restriction_\chi$, diagram 11.10
8	$(\forall\varphi)e_1 \vdash (\forall\chi)(B' \Rightarrow \theta e_1)$	$(\mathcal{D}, Sen_0)$-substitutivity
9	$\Gamma \vdash (\forall\chi)(B' \Rightarrow \theta e_1)$	$(\forall\varphi)e_1 \in \Gamma$, 8, monotonicity and transitivity of $\vdash$
10	$B' \Rightarrow \theta e_1 \in \Gamma_1^\chi$	9, definition of Γ_1^χ
11	$M_B \models (B' \Rightarrow \theta e_1)$	10, $M_B \models \Gamma_1^\chi$ (definition of M_B)
12	$M_B \models \theta e_1$	11, $M_B \models B'$ ($B' \subseteq B$)
13	$N \models e_1$	7, 12, Satisfaction Condition for θ.

With this, the proof of the theorem is completed modulo the proof of Lemma 11.24, which we do now.

Proof of Lemma 11.24. We write B as the directed co-limit of its finite subsets, as in the left-hand side triangle in the below:



- Then we consider the directed diagram $(h_{B',B''})_{B' \subseteq B'' \subseteq B \text{ finite}}$ determined by the initiality properties of the basic models $M_{B'}$ (as B' are epic basic).
- The co-cone $(\mu_{B'})_{B' \subseteq B \text{ finite}}$ is determined by the same initiality properties, and this is a co-limit of $(h_{B',B''})_{B' \subseteq B'' \subseteq B \text{ finite}}$.
- $(\mu_{B'}\restriction_\chi)_{B' \subseteq B \text{ finite}}$ is a directed co-limit (cf. Prop. 6.9 since χ is representable).
- M_φ is finitely presented since $\varphi \in \mathcal{D}$ is finitely representable (definition).
- Hence there exists a finite $B' \subseteq B$ and a model homomorphism $h\colon\ M_\varphi \to M_{B'}\restriction_\chi$ such that $h\,;\,\mu_{B'}\restriction_\chi = i_\varphi N$.

□

Approaching concrete universal completeness

In what follows we study the applicability of the universal completeness result of Thm. 11.23 by analysing its conditions and by bridging some gaps from abstract to concrete.

1. The completeness of I_1 is the basis for the completeness of I and is a highly expected condition. In practice, the compactness of I_1 is determined by the proof system of I_1 being generated by finitary rules. Moreover, since I adds only the $(\mathcal{D}, Sen_0)$-substitutivity rules, which are also finitary, the compactness of I_1 extends to I, of course by applying also results from Cor. 11.15 about preservation of quasi-finitary under connectives (such as the universal quantifications, but also those eventually involved in the build of the proof system of I_1).

2. Finitary representable signature morphisms represent an abstract way to capture finitary first-order quantifications and from our conventional logic experience we know that in general beyond that the hope for proper completeness is very slim. Hence this condition is a no surprise.

3. In concrete applications Sen_0 consists of atomic sentences. In general, epic basic property covers the atomic sentences.

4. That I_1^{th} has representable substitutions is a mere technical condition and below we provide a general method to establish it easily and that works in most concrete contexts.

5. The latter (fifth) condition of the theorem is technical and, from all conditions, is the one that requires most effort to establish it in concrete situations. We will illustrate how this works in $\mathcal{HCL}$ and $\mathcal{UNIV}$. In this section we will do the $\mathcal{UNIV}$ example and in the next section we will do the $\mathcal{HCL}$ example in an abstract general setup.

6. Furthermore, besides the analysis of the conditions underlying Thm. 11.23 we will also provide general conditions, widely and easily applicable, that allow to replace the $(\mathcal{D}, Sen_0)$-Substitutivity by its simpler form given by (11.9).

Representable $(\mathcal{D}, Sen_0)$-substitutions in I_1^{th}. This rather technical condition can be easily passed to I_1 as follows.

Proposition 11.25. *If in I_1*

1. *$\mathcal{D}$ consists of representable signature morphisms,*
2. *every set of Sen_0-sentences is epic basic, and*
3. *the representation M_φ of any signature morphism $\varphi \in \mathcal{D}$ is projective with respect to $\mathcal{D}$-reducts of model homomorphisms of the form $0_\Sigma \to M_B$ for all sets B of Sen_0-sentences,*

then the institution of theories I_1^{th} has representable $(\mathcal{D}, Sen_0)$-substitutions whenever I_1 has representable $\mathcal{D}$-substitutions.

Proof. We rely on the notations used so far in this section. Let us consider $\varphi\colon\ \Sigma \to \Sigma_1$, $\chi\colon\ \Sigma \to \Sigma_2$ signature morphisms and $B \subseteq Sen_0\Sigma_2$. We have to show that any Σ-homomorphism $h'\colon\ 0_{\Sigma_1}\upharpoonright_\varphi \to M_B\upharpoonright_\chi$ determines a $(\mathcal{D}, Sen_0)$-substitution in I_1^{th}, $\theta\colon\ (\varphi\colon\ \Sigma \to \Sigma_1) \to (\chi\colon\ \Sigma \to (\Sigma_2, B))$, such that the diagram below commutes:

$$\begin{array}{ccc}
Mod(\Sigma_2, B) & \xrightarrow[\cong]{i_\chi^B} & M_B\upharpoonright_\chi \,/\, Mod\Sigma \\
{\scriptstyle Mod\theta}\big\downarrow & & \big\downarrow{\scriptstyle h'/Mod\Sigma} \\
Mod\Sigma_1 & \xrightarrow[i_\varphi]{\cong} & 0_{\Sigma_1}\upharpoonright_\varphi \,/\, Mod\Sigma
\end{array} \tag{11.12}$$

- Because $M_\varphi = 0_{\Sigma_1}\upharpoonright_\varphi$ is projective with respect to $M_\chi = 0_{\Sigma_2}\upharpoonright_\chi \to M_B\upharpoonright_\chi$ there exists a homomorphism h such that the diagram below commutes:

$$\begin{array}{ccc}
M_\chi = 0_{\Sigma_2}\upharpoonright_\chi & \xrightarrow{\;i_\chi M_B\;} & M_B\upharpoonright_\chi \\
{\scriptstyle h}\nwarrow & & \nearrow{\scriptstyle h'} \\
& M_\varphi = 0_{\Sigma_1}\upharpoonright_\varphi &
\end{array} \tag{11.13}$$

- Because I_1 has representable $\mathcal{D}$-substitutions there exists a $\mathcal{D}$-substitution $\theta : \varphi \to \chi$ in I_1 such that the diagram below commutes:

$$\begin{array}{ccc} Mod\Sigma_2 & \xrightarrow[\cong]{i_\chi} & M_\chi/Mod\Sigma \\ {\scriptstyle Mod\theta}\downarrow & & \downarrow{\scriptstyle h/Mod\Sigma} \\ Mod\Sigma_1 & \xrightarrow[i_\varphi]{\cong} & M_\varphi/Mod\Sigma \end{array} \tag{11.14}$$

- Then θ gives the desired $(\mathcal{D}, Sen_0)$-substitution in I_1^{th}. For this we only have to show the commutativity of (11.12). This follows by combining the commutativity of (11.13) and (11.14).

□

The conditions underlying the reduction of the issue of the representability of $(\mathcal{D}, Sen_0)$-substitutions in I_1^{th} to the existence of representable $\mathcal{D}$-substitutions in I_1 are covered by the conditions of the universal completeness result of Thm. 11.23, with the exception of the projectivity condition. In most concrete situations of interest this projectivity is a consequence of the surjectivity of $0_{\Sigma_2} \to M_B$. One notable exception comes from partial algebra with existence equations. In that case, the representable $(\mathcal{D}, Sen_0)$-substitutions in I_1^{th} are obtained by following a route that is different from the one given by Prop. 11.25, a situation that suggests that the conditions of Thm. 11.23 have been formulated at an appropriate level of abstraction.

The fifth condition. This is the only 'interesting' condition of Thm. 11.23, the other ones being expected and easy to 'digest' and to establish in concrete situations. None of these attributes may apply to the fifth condition. As an example case, now we show how this works when I is $\mathcal{UNIV}$ and I_1 is its quantifier-free sub-institution. In the next section we will see how this works for $\mathcal{HCL}$ and similar institutions. We will also understand that in each of these two cases the fifth condition is established in a particular way. Let us recall that $\mathcal{UNIV}$ is the sub-in of $\mathcal{FOL}$ whose sentences are the universal quantifications of the quantifier-free $\mathcal{FOL}$ sentences.

Proposition 11.26. *If Sen_0 is the sub-functor of the $\mathcal{FOL}$ atomic sentences, then the quantifier-free sub-institution of $\mathcal{UNIV}$ satisfies the fifth condition of Thm. 11.23.*

Proof. As the implication from the left to the right is trivial, we do the other implication. We assume that for each $B \subseteq Sen_0\Sigma$, $M_B \models E$ implies $M_B \models e$. By *Reductio ad Absurdum* we suppose $E \not\models e$. Then

- There exists a model M such that $M \models E \cup \{\neg e\}$.
- We factor the unique model homomorphism $0_\Sigma \to M$ through the closed inclusion system of the category of $\mathcal{FOL}$ Σ-models:

$$0_\Sigma \longrightarrow N \longrightarrow M.$$

- Let $B = \{\rho \text{ atom } \mid N \models \rho\}$. Then we can take $M_B = N$ because we can prove that N is an initial model of B.
- Because the $\mathcal{UNIV}$ sentences are preserved by closed sub-models (for instance, cf. Cor. 8.4) it follows that $M_B \models E \cup \{\neg e\}$.
- Hence, $M_B \models E$ and $M_B \not\models e$ which contradicts the right hand side of the fifth condition. Thus our *Reductio ad Absurdum* supposition is false, which means that $E \models e$.

□

Substitutivity revisited. The projectivity condition of Prop. 11.25 allows to replace the $(\mathcal{D}, Sen_0)$-substitutivity rule in the universal completeness Thm. 11.23 by its simpler form (11.9).

Proposition 11.27. *Under the conditions of Thm. 11.23 and Prop. 11.25, the $(\mathcal{D}, Sen_0)$-substitutivity rule is equivalent to its form* (11.9).

Proof. Let us consider a $(\mathcal{D}, Sen_0)$-substitutivity rule

1 $(\forall\varphi)\rho \vdash (\forall\chi)(B \Rightarrow \theta\rho)$

where $B \Rightarrow \theta\rho$ stands for any semantic implication of $\theta\rho$ from B. The I_1^{th} $(\mathcal{D}, Sen_0)$-substitution θ gets extended to a $\mathcal{D}$-substitution, denoted $\overline{\theta}$, by following the proof of Prop. 11.25. First, from θ we get h', then by the projectivity property we get h, and finally by representability we get $\overline{\theta} : \varphi \to \chi$ as a $\mathcal{D}$-substitution in I_1. 'Extension' means just that $Mod\theta$ gets extended from a functor $Mod(\Sigma_2, B) \to Mod\Sigma_1$ to a a functor $Mod\overline{\theta} : Mod\Sigma_2 \to Mod\Sigma_1$. Then:

2	$(\forall\varphi)\rho \vdash (\forall\chi)\overline{\theta}\rho$	assuming (11.9)
3	$\overline{\theta}\rho \cup B \models \overline{\theta}\rho$	monotonicity of $\models$
4	$\overline{\theta}\rho \models B \Rightarrow \overline{\theta}\rho$	3, $B \Rightarrow \overline{\theta}\rho$ semantic implication
5	$\overline{\theta}\rho \vdash B \Rightarrow \overline{\theta}\rho$	4, completeness of I_1
6	$(\forall\chi)\overline{\theta}\rho \vdash (\forall\chi)\overline{\theta}\rho$	monotonicity of $\vdash$
7	$\chi((\forall\chi)\overline{\theta}\rho) \vdash \overline{\theta}\rho$	6, universal $\mathcal{D}$-quantification property of $\vdash$
8	$\chi((\forall\chi)\overline{\theta}\rho) \vdash B \Rightarrow \overline{\theta}\rho$	7, 5, transitivity of $\vdash$
9	$(\forall\chi)\overline{\theta}\rho \vdash (\forall\chi)(B \Rightarrow \overline{\theta}\rho)$	8, universal $\mathcal{D}$-quantification property of $\vdash$.

By taking into account that $Sen\theta = Sen\overline{\theta}$, now 1 is obtained from 2 and 9 by the transitivity of $\vdash$. □

Like with Prop. 11.25, $QE^{\omega}(\mathcal{PA})$ (the institution of quasi-existence equations in $\mathcal{PA}$ with finite premises) is outside the scope of Prop. 11.27. In this case the proof calculus involves $(\mathcal{D}, Sen_0)$-substitutivity in its general form, and this cannot be avoided.

A sound and complete proof system for $\mathcal{UNIV}$. The universal completeness Thm. 11.23 has the potential to generate, with minimal effort, manifold concrete completeness results. In Sec. 11.7 we will use it for obtaining a generic completeness result for Horn clause logics. But now, as an exercise, we sketch the steps for developing a sound and complete proof calculus for $\mathcal{UNIV}$.

1. We set $I = \mathcal{UNIV}$, I_1 to be the quantifier-free sub-institution of $\mathcal{UNIV}$, and Sen_0 the sub-functor of the atomic $\mathcal{FOL}$-sentences.
2. We need a sound and complete proof calculus for I_1. We develop this in two steps as follows:
   - First for $\mathcal{AFOL}$ (the atomic sub-institution of $\mathcal{FOL}$ / $\mathcal{UNIV}$). We will skip this now but will do it after we present the other parts of the $\mathcal{UNIV}$ proof calculus.
   - Then we add to the proof rules of $\mathcal{AFOL}$ any sound and complete system of proof rules for $\mathcal{PL}$. For instance, we may employ the system consisting of the rules $(P1-3)$, MP introduced in Sec. 11.2. This is known to be complete. This trick works because at this layer the $\mathcal{FOL}$-atoms can be assimilated to propositional variables.
3. The other conditions of Thm. 11.23 and of Prop. 11.25 can be checked easily with the exception of the fifth condition of Thm. 11.23, but this one has been already solved in Prop. 11.26.

The end result of this process is a system of proof rules consisting of

1. the rules for $\mathcal{AFOL}$;
2. the rules for $\mathcal{PL}$ (with the propositional variables representing $\mathcal{FOL}$ atoms);
3. $\mathcal{D}$-substitutivity rules where $\mathcal{D}$ is the class of the signature extensions with finite blocks of variables; and
4. which is factored through the meta-rule of universal $\mathcal{D}$-quantification.

A sound and complete $\mathcal{AFOL}$ proof system. The only missing piece in the puzzle of the definition of the $\mathcal{UNIV}$ proof system above is a sound and complete proof calculus for $\mathcal{AFOL}$. We do this now.

Proposition 11.28. *The following system of proof rules for $\mathcal{AFOL}$ is sound and complete:*

(R)	$\emptyset \vdash t = t$	*for each term* t
(S)	$t = t' \vdash t' = t$	*for any terms* t, t'
(T)	$\{t = t', t' = t''\} \vdash t = t''$	*for any terms* t, t', t''
(F)	$\{t_i = t'_i \mid 1 \leq i \leq n\} \vdash \sigma(t_1,\dots,t_n) = \sigma(t'_1,\dots,t'_n)$	*for any operation symbol* σ
(P)	$\{t_i = t'_i \mid 1 \leq i \leq n\} \cup \{\pi(t_1,\dots,t_n)\} \vdash \pi(t'_1,\dots,t'_n)$	*for any relation symbol* π.

Proof. Soundness follows by a simple routine check. Now we prove the completeness.

- for any set E of atoms for a fixed signature we define

$$\equiv_E \; = \; \{(t,t') \mid E \vdash t = t'\}.$$

- By (R), (S), (T) and (F) this is a congruence on the initial (term) algebra.
- Then we define a model M_E as follows:
  - the algebra part of M_E is defined as the quotient of the initial (term) algebra by $\equiv_E$, and
  - for each relation symbol $\pi \in P$, we define $(M_E)_\pi = \{x/_{\equiv_E} \mid E \vdash \pi x\}$.

  The definition of $(M_E)_\pi$ is correct because of the rule (P). Now we note that for each atomic sentence ρ,

$$E \vdash \rho \;\text{ if and only if }\; M_E \models \rho.$$

- If $E \models \rho$ then $M_E \models \rho$ because $M_E \models E$. Hence $E \vdash \rho$. The proof is thus completed.

As a side additional remark, the notation M_E is precisely what it suggests, M_E being indeed a basic (and initial) model of E. □

Exercises

11.23. [135] **Translating $\mathcal{D}$-substitutivity**
Consider an institution I_1 with a sub-category $\mathcal{D}$ of representable signature morphisms such that

1. I has weak model amalgamation,
2. I has representable $\mathcal{D}$-substitutions,
3. $\mathcal{D}$ is stable under pushouts,
4. for each pushout of signature morphisms such that $\chi \in \mathcal{D}$

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ \Sigma_1 & \xrightarrow[\chi_1]{} & \Sigma'_1 \end{array}$$

for any Σ'-sentence ρ' and Σ_1-sentence ρ_1 if $\varphi'\rho' = \chi_1\rho_1$, then there exists a Σ-sentence ρ such that $\rho' = \varphi\rho$ and $\rho_1 = \chi\rho$. (Compare this condition with the co-amalgamation property of Ex. 4.20.)

Then the translation of any $\mathcal{D}$-Substitutivity rule along a signature morphism yields a $\mathcal{D}$-Substitutivity rule.

11.24. Birkhoff calculus for $\mathcal{EQL}$
Derive a sound and complete proof calculus for $\mathcal{EQL}$ from the universal completeness result of Thm. 11.23. (*Hint:* Set I_1 to $\mathcal{AFOL}$ and let Sen_0 to be empty.)

11.7 Birkhoff completeness

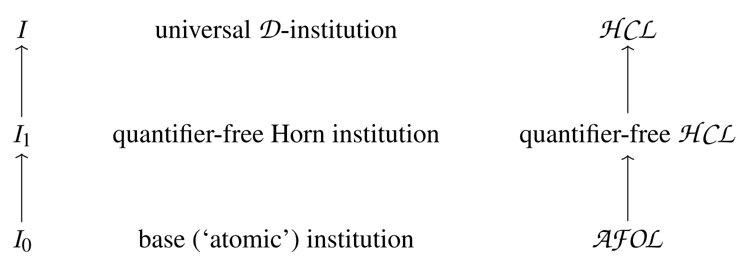
In this section we develop an important application of the universal completeness Thm. 11.23, namely a generic sound and complete system of proof rules for Horn clause institutions. In other words, we treat the proof calculus of $\mathcal{HCL}$ in a general abstract manner by considering an institution-independent abstraction of $\mathcal{HCL}$. This means that the collection of $\mathcal{FOL}$-atoms are abstracted to a sentence functor Sen_0, and the sentences of I_1 are quantifier-free Horn clauses over Sen_0. Then, in the universal institution I we quantify universally over I_1, as we did in the theory of universal completeness. By assuming a sound and complete proof system for Sen_0, we lift it to I_1, and it remains only to make sure that all conditions of Thm. 11.23 are fulfilled. From this succint presentation you may probably guess that Sen_0 here is exactly the Sen_0 that is used in Thm. 11.23.

Horn clause institutions. Previously in this book we introduced an institution-independent abstraction of ordinary Horn clauses (i.e. $\mathcal{HCL}$ sentences) under the name of '$\mathcal{D}$-universal Horn sentences'. This was in Sec. 5.5, for the long purpose of obtaining the axiomatizability by Horn theories of quasi-varieties in abstract institutions (Sec. 8.4). Here we introduce another abstraction of Horn clauses that is motivated by proof-theoretic developments. The difference between these two concepts is slight, it is essentially a matter of different axiomatisations of the same concrete concept. Their strong convergence can be understood even at the abstract level, as, in order to apply Thm. 11.23, Sen_0 is supposed to consist of epic basic sentences. Of course, albeit this strong convergence, in concrete situations the two concepts of Horn sentence / clause may not overlap perfectly.

$I = (Sig, Sen, Mod, \models)$ is a *$\mathcal{D}$-Horn institution* over $I_0 = (Sig, Sen_0, Mod, \models)$ when there exists an institution $I_1 = (Sig, Sen_1, Mod, \models)$ such that

- I is a $\mathcal{D}$-universal institution over I_1, and
- I_0 is the sub-institution of I_1 determined by a sub-functor Sen_0 of Sen_1 such that the I_1-sentences are precisely *all* sentences semantically equivalent to $H \Rightarrow C$ where H is any finite set of Sen_0-sentences and C is any Sen_0-sentence. The sentences of I_1 may be referred to as 'quantifier-free Horn clauses over I_0'.

For example, $\mathcal{HCL}$ is a finitary $\mathcal{D}$-Horn institution over $\mathcal{AFOL}$, where $\mathcal{D}$ is the class of all signature extensions with a finite block of variables. This layered structure can be visualised as follows:



Birkhoff proof systems. Given a $\mathcal{D}$-Horn institution as above, a *Birkhoff proof system* for I is a $(\mathcal{D}, Sen_0)$-universal proof system such that

- the proof system of I_1 is freely generated by a given proof system (Sig, Sen_0, Pf_0) for I_0, such that
- it satisfies the following *Modus Ponens for* Sen_0 property expressed by the following natural isomorphism

  $(MP_0) \quad Pf_1\Sigma\,(\Gamma \cup B, e) \;\cong\; Pf_1\Sigma\,(\Gamma, B \Rightarrow e)$

  where $\Gamma \subseteq Sen_1\Sigma$, B finite $\subseteq Sen_0\Sigma$, $e \in Sen_1\Sigma$.

Note that (MP_0) is a weaker form of proof-theoretic implication since B consists of Sen_0-sentences (rather than Sen_1-sentences). As we often mean, in (MP_0) $B \Rightarrow e$ does not denote necessarily a *designated* semantic implication but rather any sentence which is a semantic implication. What (MP_0) does is that it adds proofs for the quantifier-free Horn clauses $H \Rightarrow C$ on the basis of proof in I_0. Recall also from Thm. 11.3 that to consider free proof systems we should have injective sentence translation functions (which in concrete institutions means injective signature morphisms). But if we work only at the level of entailment systems, then this restriction is not required anymore.

Abstract Birkhoff soundness

Proposition 11.29 (Birkhoff soundness)**.** *If I_0 is sound then I_1 is sound too. Consequently, the $\mathcal{D}$-Horn institution I over I_0 endowed with the Birkhoff proof system, is sound too.*

Proof. The latter conclusion follows from the former by universal soundness (Cor. 11.22). The former soundness is justified like in Prop. 11.16 by regarding (MP_0) as a form of proof-theoretic implication. □

Abstract Birkhoff completeness

To derive the completeness of I through Thm. 11.23 we have to address the conditions of Thm. 11.23 one by one as follows:

1. We establish the completeness and the compactness of I_1 by assuming them for I_0.
2. We assume that every signature morphism in $\mathcal{D}$ is finitely representable. This property does not depend on whether we consider it in I_0, I_1 or in I because it refers to the model functor, which is shared across this hierarchy of institutions.
3. We also have to assume that each set of Sen_0-sentences is epic basic.
4. We will show that if I_0^{th} has representable $(\mathcal{D}, Sen_0)$-substitutions then I_1^{th} has them too. This will be an easy piece.
5. We will prove the fifth condition of Thm. 11.23 under conditions that are already assumed in the context of the completeness of I_1.

Before doing 1., 4., and 5., we develop a couple of technical lemmas that will support our proofs of these conditions.

Two technical lemmas. We assume that any set of Sen_0-sentences is basic. For the purpose of applying Thm. 11.23 they should be epic basic, but in the context of the following two lemmas the epic basic property is unnecessarily strong. For each $\Gamma \subseteq Sen_1\Sigma$ let $\Gamma_0 = \{e \in Sen_0\Sigma \mid \Gamma \vdash e\}$. Since Γ_0 is basic, let M_{Γ_0} denote one of its basic models.

Lemma 11.30. *If I_0 is complete then for each $e \in Sen_0\Sigma$, $M_{\Gamma_0} \models e$ if and only if $\Gamma \vdash e$.*

Proof. The implication from the right to the left holds by the definition of Γ_0. For the reverse implication we consider e such that $M_{\Gamma_0} \models e$. Let us show that

1 $\Gamma_0 \models e$.

Let $M \in |Mod\Sigma|$ such that $M \models \Gamma_0$. Also let M_e denote a basic model for e. Then

2	there exists homomorphism $M_{\Gamma_0} \to M$	$M \models \Gamma_0$, Γ_0 basic
3	there exists homomorphism $M_e \to M_{\Gamma_0}$	$M_{\Gamma_0} \models e$, e basic
4	there exists homomorphism $M_e \to M$	2, 3
5	$M \models e$	4, e basic.

Thus 1 is proved. Then

6	$\Gamma_0 \vdash e$	1, I_0 complete
7	$\Gamma \vdash e$	6, $\Gamma_0 \subseteq \Gamma$, monotonicity and transitivity of $\vdash$.

□

Lemma 11.31. *If I_0 is complete then $M_{\Gamma_0} \models \Gamma$.*

Proof. Let $H \Rightarrow C$ be a quantifier-free Horn clause in Γ. Assume $M_{\Gamma_0} \models H$. Then

1	$\Gamma \vdash H$	$M_{\Gamma_0} \models H$, Lemma 11.30, union (vertical composition) of $\vdash$
2	$\Gamma \vdash H \Rightarrow C$	$H \Rightarrow C \in \Gamma$, monotonicity of $\vdash$
3	$\Gamma \cup H \vdash C$	2, (MP_0)
4	$\Gamma \vdash \Gamma$	monotonicity of $\vdash$
5	$\Gamma \vdash \Gamma \cup H$	1, 4, union (vertical composition) of $\vdash$
6	$\Gamma \vdash C$	3, 5, transitivity of $\vdash$
7	$M_{\Gamma_0} \models C$	6, Lemma 11.30.

Hence $M_{\Gamma_0} \models H \Rightarrow C$. □

Completeness of I_1.

Proposition 11.32. *If I_0 is complete and each set of Sen_0-sentences is basic, then I_1 is complete too.*

Proof. Let $\Gamma \models H \Rightarrow C$ where $\Gamma \subseteq Sen_1\Sigma$, $H \Rightarrow C \in Sen_1\Sigma$. Then

1	$\Gamma \cup H \models C$	$\Gamma \models H \Rightarrow C$, semantic implication
2	$M_{(\Gamma\cup H)_0} \models \Gamma \cup H$	Lemma 11.31
3	$M_{(\Gamma\cup H)_0} \models C$	1, 2
4	$\Gamma \cup H \vdash C$	3, Lemma 11.30
5	$\Gamma \vdash H \Rightarrow C$	4, (MP_0).

□

Compactness of I_1. This is something we have already discussed in Sec. 11.6. Once more, if the proof system of I_0 is quasi-finitary, then (MP_0) preserves this quasi-finitary property (cf. Cor. 11.15).

I_1^{th} **has representable $(\mathcal{D}, Sen_0)$-substitutions.** This property is inherited immediately from I_0^{th}, the argument being almost trivial.

Proposition 11.33. *If I_0^{th} has representable $(\mathcal{D}, Sen_0)$-substitutions then I_1^{th} has them too.*

Proof. Let us consider $\varphi : \Sigma \to \Sigma_1$, $\chi : \Sigma \to \Sigma_2$ signature morphisms and $B \subseteq Sen_0\Sigma_2$. We are under the hypothesis that Sen_0 consists of epic basic sets of sentences, otherwise we cannot talk about representability of $(\mathcal{D}, Sen_0)$-substitutions.

- Any homomorphism $h' : 0_{\Sigma_1}\restriction_\varphi \to M_B\restriction_\chi$ yields an I_0^{th} $(\mathcal{D}, Sen_0)$-substitution; this is what representability means.
- Furthermore, this can be extended canonically to a I_1^{th} $(\mathcal{D}, Sen_0)$-substitution just by extending the sentence translation $Sen_0\theta$ to Sen_1-sentences by defining $(Sen_1\theta)(H \Rightarrow C) = (Sen_0\theta)H \Rightarrow (Sen_0\theta)C$. On the model reducts nothing happens, and the Satisfaction Condition for θ in I_1^{th} follows immediately from I_0^{th}.

□

The fifth condition.

Proposition 11.34. *Under the conditions of Prop. 11.32 and if I_0 is sound, then the fifth condition of Thm. 11.23 does hold.*

Proof. We employ the notations from the statement of Thm. 11.23 and from Lemmas 11.30 and 11.31. Let $e = (H \Rightarrow C)$ with $H \subseteq Sen_0\Sigma$, $C \in Sen_0\Sigma$. We assume $M_B \models (E \Rightarrow e)$ for each $B \subseteq Sen_0\Sigma$. Then

1	$M_{(E\cup H)_0} \models (E \Rightarrow e)$	let $B = (E\cup H)_0$, $M_B \models (E \Rightarrow e)$
2	$M_{(E\cup H)_0} \models E\cup H$	Lemma 11.31
3	$M_{(E\cup H)_0} \models E$	2, monotonicity and transitivity of $\models$
4	$M_{(E\cup H)_0} \models e$	1, 3, semantic implication
5	$M_{(E\cup H)_0} \models H$	2, monotonicity and transitivity of $\models$
6	$M_{(E\cup H)_0} \models C$	4, 5, $e = (H \Rightarrow C)$, semantic implication
7	$E\cup H \vdash C$	6, Lemma 11.30
8	$E\cup H \models C$	7, I_1 sound (cf. Prop. 11.29)
9	$E \models e$	8, $e = (H \Rightarrow C)$, semantic implication.

□

Abstract Birkhoff completeness. Now we can collect all ingredients for a direct formulation of a general Birkhoff completeness result where everything is stated in terms of the base institution I_0.

Corollary 11.35 (Birkhoff completeness)**.** *We consider a $\mathcal{D}$-Horn institution I endowed with a Birkhoff proof system. If*

1. *I_0 is sound and complete and its entailment system is quasi-finitary,*
2. *$\mathcal{D}$ contains only finitely representable signature morphisms,*
3. *every set of Sen_0-sentences is epic basic, and*
4. *I_0^{th} has representable $(\mathcal{D}, Sen_0)$-substitutions,*

then I is sound and complete too.

The sound and complete Birkhoff proof system for $\mathcal{HCL}$. Let us see what we get when applying the general result of Cor. 11.35 to what is perhaps its most emblematic application. The parameters in Cor. 11.35 are set up as expected:

- $I = \mathcal{HCL}$,
- $I_0 = \mathcal{AFOL}$ considered with its proof system as defined by Prop. 11.28, and
- $\mathcal{D}$ is the class of all signature extensions with finite blocks of variables.

Then the Birkhoff proof system for $\mathcal{HCL}$ is obtained as the free proof system generated by

1. the proof rules of $\mathcal{AFOL}$ (as given in Prop. 11.28), and
2. the substitutivity rules $(\forall X)\rho \vdash (\forall Y)\theta\rho$, for any substitution $\theta : X \to T_\Sigma Y$

and which satisfies

3. universal $\mathcal{D}$-quantification, and

4. $Pf\Sigma\ (\Gamma \cup H, C) \ \cong\ Pf\Sigma\ (\Gamma, H \Rightarrow C)$ for any set Γ of quantifier-free Horn clauses, set H of atomic sentences, and atomic sentence C.

Since this Birkhoff proof system for $\mathcal{HCL}$ is not a literal instance of the abstract Birkhoff proof system, some explanations as necessary.

- The $(\mathcal{D}, Sen_0)$-substitutivity rules can be replaced by the simple substitutivity rules above by using the result of Prop. 11.27.
- Although the form of *Modus Ponens* that we are using for the $\mathcal{HCL}$ proof system is apparently more restrictive that (MP_0), in reality they are equivalent as shown by the following natural isomorphisms:

$$Pf\Sigma\ (\Gamma \cup B, H \Rightarrow C) \ \cong\ Pf\Sigma\ (\Gamma \cup B \cup H, C) \ \cong\ Pf\Sigma\ (\Gamma, B \cup H \Rightarrow C) \ \cong\ Pf\Sigma\ (\Gamma, B \Rightarrow (H \Rightarrow C)).$$

Exercises

11.25. Birkhoff calculus for preordered algebras
$\mathcal{HPOA}$, the institution of Horn preordered algebras, gets a sound and complete proof system obtained as the free proof system

- with universal quantification, and
- such that for each quantifier-free Horn sentence $H \Rightarrow C$ and all sets Γ of quantifier-free Horn sentences there exists a natural isomorphism $Pf\Sigma\ (\Gamma \cup H, C) \ \cong\ Pf\Sigma\ (\Gamma, H \Rightarrow C)$

and which is generated by the following system of finitary rules for a signature Σ (where t, t', t'' are any Σ-terms, σ is any operation symbol, ρ is any Σ-sentence, and $\theta : X \to T_\Sigma Y$ is any substitution):

(R)	$\emptyset \vdash t = t$
(RP)	$\emptyset \vdash t \leq t$
(S)	$t = t' \vdash t' = t$
(T)	$\{t = t',\ t' = t''\} \vdash t = t''$
(TP)	$\{t \leq t',\ t' \leq t''\} \vdash t \leq t''$
(F)	$\{t_i = t'_i \mid 1 \leq i \leq n\} \vdash \sigma(t_1, \ldots, t_n) = \sigma(t'_1, \ldots, t'_n)$
(FP)	$\{t_i \leq t'_i \mid 1 \leq i \leq n\} \vdash \sigma(t_1, \ldots, t_n) \leq \sigma(t'_1, \ldots, t'_n)$
$(Comp)$	$\{t'_1 = t_1,\ t_1 \leq t_2,\ t_2 = t'_2\} \vdash t'_1 = t'_2$
$(Subst)$	$(\forall X)\rho \vdash (\forall Y)\theta\rho$.

11.26. [48] Let I_0 be an institution with a sub-category $\mathcal{D}$ of representable signature morphisms such that

1. every set of sentences is epic basic, and
2. for any signature morphism $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$ in $\mathcal{D}$ and any set E_2 of Σ_2-sentences, every homomorphism $h : M_{\chi_1} \to M_{E_2}\restriction_{\chi_2}$ determines an I_0^{th}-substitution $\theta_h : (\chi_1 : \Sigma \to \Sigma_1) \to (\chi_2 : \Sigma \to (\Sigma_2, E_2))$ such that the diagram below commutes:

$$\begin{array}{ccc}
Mod(\Sigma_2, E_2) & \xrightarrow{i_{\chi_2}} & (M_{E_2}\restriction_{\chi_2})/Mod\Sigma \\
{\scriptstyle Mod\theta_h}\downarrow & & \downarrow{\scriptstyle h/Mod\Sigma} \\
Mod\Sigma & \xrightarrow[i_{\chi_1}]{} & Mod\Sigma_1
\end{array}$$

Then the institution of theories I_0^{th} has representable $\mathcal{D}$-substitutions. (Note that by Ex. 5.29 each I_0-theory morphism from $\mathcal{D}$ is representable.)

11.27. [48] **Birkhoff calculus for partial algebras**
$QE^{\omega}(\mathcal{PA})$, the institution of partial algebras with finitary quasi-existence equations as sentences, gets a sound and complete proof system obtained as the free proof system

- with universal $\mathcal{D}$-quantification for $\mathcal{D}$ the class of the injective signature extensions with a finite block of variables, and
- such that for each quantifier-free Horn clause $H \Rightarrow C$ and all sets Γ of quantifier-free Horn clauses there exists a natural isomorphism $Pf\Sigma\,(\Gamma \cup H, C) \;\cong\; Pf\Sigma\,(\Gamma, H \Rightarrow C)$

which is generated by the following system of finitary rules for a signature Σ (where t, t', t'' are any Σ-terms, σ is any operation symbol, τ is any total operation symbol, ρ is any Σ-sentence, and $\theta :\; X \to T_\Sigma Y$ is any substitution):

(S)	$t \stackrel{e}{=} t' \vdash t' \stackrel{e}{=} t$
(T)	$\{t \stackrel{e}{=} t', t' \stackrel{e}{=} t''\} \vdash t \stackrel{e}{=} t''$
(F)	$\{t_i \stackrel{e}{=} t_i' \mid 1 \le i \le n\} \cup \{\mathsf{def}\,\sigma(t_1,\dots,t_n), \mathsf{def}\,\sigma(t_1',\dots,t_n')\} \vdash \sigma(t_1,\dots,t_n) \stackrel{e}{=} \sigma(t_1',\dots,t_n')$
$(\mathit{Totality})$	$\{\mathsf{def}\,t_i \mid 1 \le i \le n\} \vdash \mathsf{def}\,\tau(t_1,\dots,t_n)$
$(\mathit{Subterm})$	$\{\mathsf{def}\,\sigma(t_1,\dots,t_n)\} \vdash \{\mathsf{def}\,t_i \mid 1 \le i \le n\}$
(Subst)	$(\forall X)\rho \vdash (\forall Y)(\bigwedge_{x \in X} \mathsf{def}(\theta x) \Rightarrow \theta\rho)$

(*Hint:* Since the second condition of Prop. 11.25 does not hold, apply the result of Ex. 11.26.)

Notes. Usually, categorical logic works with categories of sentences, where morphisms are (equivalence classes) of proof terms [158]. However, this captures provability between single sentences only, while logic traditionally studies provability from a set of sentences. Proof systems have been defined in [191] that reconcile both approaches by considering categories of sets of sentences. This also avoids one of the big faults of categorical logic, that the definition of implication depends on (the existence of) conjunctions. Systems of proof rules were introduced in [68] which also developed the free proof system Thm. 11.3 and its compactness Cor. 11.8. Note that our concept of proof rules admits multiple conclusions, which constitute a slight generalization of the usual practice in actual logics which use only single conclusion rules. There are several complete systems of rules for $\mathcal{PL}$, by Russel, Frege, Hilbert, Łukasiewicz, Sobociński, etc.; the one presented in Sec. 11.2 is due to Jan Łukasiewicz. Lawvere [160] defined quantification as adjoint to substitutions, while [191] defines quantification as adjoint to sentence translation along signature morphisms. In [68] we developed the free proof system with (universal) quantification and proved its compactness and soundness; that technique works for any logical connective.

Entailment systems were probably defined for the first time under the name of π-institutions in [108], and later modified by [175] in order to formalize the notion of syntactic consequence. [143] gives a similar definition but restricted to finite sets of sentences. Meseguer [175] showed how to construct an institution from an entailment system by producing a model theory directly from a comma category construction on theories, and [96] extends this construction by embedding the category of entailment systems into the category of [ordinary] institutions.

The general soundness results given by Prop. 11.5 and 11.16 are due to [68].

Birkhoff calculus and its completeness result have been developed for a single-sorted conditional version of $\mathcal{EQL}$ in [27]; this has been extended to many sorts in [127], and to arbitrary institutions in [48]. The layered approach to institution-independent completeness was invented by [33] within the framework of specification theory. Later this was extended to Gödel completeness [203] and Birkhoff completeness [48]. The latter work revealed the surprisingly close relationship between the completeness of the institution of universal sentences ($\mathcal{UNIV}$) and the general concept of Birkhoff completeness.

Chapter 12

Models with States

Sometimes in logic and especially in computing we consider models that admit *internal states* that occur as parameters in the satisfaction relation. Recall the modal logic institutions introduced in Sec. 3.2 whose models are Kripke models (W,M); then the possible worlds $|W|$ are the internal states of (W,M) which are structured by the corresponding Kripke frame $W = (|W|, W_\lambda)$. In those institutions there is a *local satisfaction* arising as a ternary relation $(W,M) \models^w \rho$ where $w \in |W|$ was a 'world' of (W,M), in more general terms just an internal state of (W,M). Another class of notorious examples is given by various kinds of automata, when these are viewed with a model theoretic eye. The refined institution theoretic treatment of the phenomenon of models with states is given by the theory of *stratified institutions* which is the very topic of this chapter. But this extension of ordinary institution theory is capable to capture many more situations, than those mentioned above, when models with states are involved. The concept of stratified institution has a higher mathematical / categorical complexity than that of ordinary institution. For instance, a bit of familiarity with 2-categorical thinking is helpful. This higher mathematical complexity reflects an increased difficulty when doing 'non-classical' model theory. Then an important question arises naturally: is it not possible to treat models with states only within the framework of ordinary institution theory? A proper understanding-based answer to this question may emerge while advancing through this chapter. We will see how we can 'reduce' or 'flatten' stratified institutions to ordinary institutions, and how this will provide us with opportunities to import developments from ordinary institution theory to stratified institutions. However, while this reduction is very useful to a large extent, it cannot support the real specificities of models with states. For instance, while the Boolean and the quantifier connectives can be addressed through this reduction, the axiomatic semantics of modalities falls outside its scope. We will also understand other reasons for the theory of stratified institutions. This chapter is structured as follows:

1. We first introduce the concept of stratified institution and related concepts, develop some general basic facts around this, and discuss some concrete examples.

2. We extend the internal logic of Chap. 5 to stratified institutions and we add to that a

semantics for modal logic connectives (such as possibility, necessity, nominals, etc.).

3. In another section we introduce a ‘decomposition technique’ that defines a class of stratified institutions that, on the one hand, cover most concrete examples of interest, and, on the other hand, is technically convenient.

4. Another section is devoted to a general construction that adds explicit Kripke structure to abstract stratified institutions. While the decomposition technique can be regarded as a top-down technique, the general Kripke construction represents its reverse, being a bottom-up technique.

5. The last part of the chapter is concerned with the extension of the method of ultraproducts of Chap. 6 to stratified institutions.

As prerequisites, this chapter relies mainly on matter from Chapters 3, 4, 5 and 6. Matter from Chap. 6 is required only in Sec. 12.5.

12.1 Stratified institutions

In a stratified institution each model M comes equipped with a *set* $[\![M]\!]$. This has to be structurally coherent with respect to the model functor. But more importantly the elements of $[\![M]\!]$ are parameters of the satisfaction relation. In fact this is the main novelty with respect to the ordinary concept of institution. A typical example is given by the Kripke models, where $[\![M]\!]$ is the set of the possible worlds in the Kripke structure M.

A *stratified institution* $\mathcal{S} = (Sig^{\mathcal{S}}, Sen^{\mathcal{S}}, Mod^{\mathcal{S}}, [\![\_]\!]^{\mathcal{S}}, \models^{\mathcal{S}})$ consists of:

– a category $Sig^{\mathcal{S}}$ of signatures,

– a sentence functor $Sen^{\mathcal{S}} : \; Sig^{\mathcal{S}} \to \mathbb{S}et$;

– a model functor $Mod^{\mathcal{S}} : \; (Sig^{\mathcal{S}})^{\mathrm{op}} \to \mathbb{C}at$;

– a “stratification” lax natural transformation $[\![\_]\!]^{\mathcal{S}} : \; Mod^{\mathcal{S}} \Rightarrow SET$, where SET is the constant functor mapping each signature to $\mathbb{S}et$, and

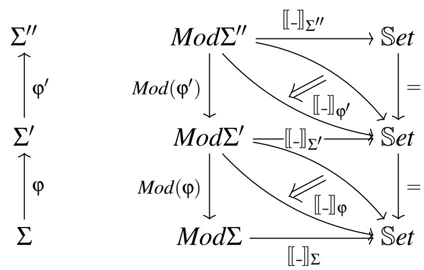
– a satisfaction relation $M \; (\models^{\mathcal{S}})^w_\Sigma \; \rho$ between models and sentences which is parameterized both by signatures and by ‘model states’ $w \in [\![M]\!]^{\mathcal{S}}_\Sigma$ such that

$$(Mod^{\mathcal{S}}\varphi)M \;\; (\models^{\mathcal{S}})_\Sigma^{[\![M]\!]^{\mathcal{S}}_\varphi w} \;\; \rho \;\text{ if and only if }\; M \;\; (\models^{\mathcal{S}})^w_{\Sigma'} \;\; (Sen^{\mathcal{S}}\varphi)\rho \tag{12.1}$$

holds for any signature morphism $\varphi : \; \Sigma \to \Sigma'$, Σ'-model M, $w \in [\![M]\!]^{\mathcal{S}}_{\Sigma'}$, and Σ-sentence ρ.

Like with ordinary institutions, when appropriate we shall also use simplified notations without superscripts or subscripts that are clear from the context.

The lax natural transformation property of $[\![\_]\!]$ is depicted in the diagram below



with the following compositionality property for each Σ''-model M'':

$$[\![M'']\!]_{(\varphi';\varphi)} = [\![M'']\!]_{\varphi'} ; [\![(Mod\varphi')M'']\!]_{\varphi}.$$

Moreover, the natural transformation property of each $[\![\_]\!]_{\varphi}$ is given by the commutativity of the following diagram:

$$\begin{array}{ccccc} M' & \quad & [\![M']\!]_{\Sigma'} & \xrightarrow{[\![M']\!]_{\varphi}} & [\![(Mod\varphi)M']\!]_{\Sigma} \\ {\scriptstyle h'}\downarrow & & {\scriptstyle [\![h']\!]_{\Sigma'}}\downarrow & & \downarrow{\scriptstyle [\![(Mod\varphi)h']\!]_{\Sigma}} \\ N' & & [\![N']\!]_{\Sigma'} & \xrightarrow[{[\![N']\!]_{\varphi}}]{} & [\![(Mod\varphi)N']\!]_{\Sigma} \end{array} \tag{12.2}$$

When the stratification $[\![\_]\!]$ is a strict natural transformation (i.e. $[\![\_]\!]_{\varphi}$ are identities), we say that $\mathcal{S}$ is a *strictly stratified institution*. Note that ordinary institutions are strictly stratified institutions for which $[\![M]\!]_{\Sigma}$ is always a singleton set.

The following notation is useful for what follows. For any Σ-model M and any Σ-sentence ρ we let

$$[\![M,\rho]\!] = \{w \in [\![M]\!]_{\Sigma} \mid M \models^{w} \rho\}.$$

Assumption on closure under isomorphism: As in the case of ordinary institutions, the following very expected property does not follow from the axioms of stratified institutions above, hence we impose it explicitly. In all considered stratified institutions *the satisfaction is preserved by model isomorphisms*, i.e. for each Σ-model isomorphism $h\colon M \to N$, each $w \in [\![M]\!]_{\Sigma}$, and each Σ-sentence ρ,

$$M \models^{w} \rho \quad \text{if and only if} \quad N \models^{[\![h]\!]w} \rho.$$

Semantically equivalent sentences. Two Σ-sentences ρ_1, ρ_2 are called *semantically equivalent* when $[\![M,\rho_1]\!] = [\![M,\rho_2]\!]$ for each Σ-model M.

Examples

We limit the examples presented here only to four. Each of them goes in a particular different direction. Three of these directions admit multiple examples of stratified institutions that share a same central idea.

Modal logics as stratified institutions. The institutions of first order and propositional modal logics discussed in Sec. 3.2 constitute the most emblematic examples of stratified institutions. For instance the stratified institution $\mathcal{MFOL}$ of first-order modal logic is defined by letting $Sig^{\mathcal{MFOL}}$, $Sen^{\mathcal{MFOL}}$, and $Mod^{\mathcal{MFOL}}$ be like in the corresponding example presented in Sec. 3.2. Recall that the $\mathcal{MFOL}$-signatures are structures of the form (S,S_0,F,F_0,P,P_0), $Mod^{\mathcal{MFOL}}$ consists of Kripke frames (W,M) with the corresponding sharing property of the interpretations of the rigid symbols, $Sen^{\mathcal{MFOL}}$ consist of the first order modal sentences with quantification restricted to rigid variables, etc. Then the stratification $[\![\_]\!]$ extracts the possible worlds from the Kripke models, i.e. $[\![(W,M)]\!] = |W|$. The same can be applied to the hybrid modal logics $\mathcal{HPL}$ and $\mathcal{HFOL}$ in order to capture them as stratified institutions.

Note that in these examples $[\![\_]\!]$ is a strict rather than a lax natural transformation since for each signature morphism φ we have that $[\![(W,M)]\!]_\varphi$ are identities. Although this situation is rather common among the concrete examples, there are however meaningful examples when $[\![\_]\!]$ is a proper lax natural transformation, like the following one.

Open first-order logic ($\mathcal{OFOL}$). The main idea of the stratified institution $\mathcal{OFOL}$ is to consider valuations of variables as internal states of $\mathcal{FOL}$-models. Hence an $\mathcal{OFOL}$ signature is a pair (Σ,X) consisting of $\mathcal{FOL}$ signature Σ and a finite block of variables for Σ. An $\mathcal{OFOL}$ signature morphism $\varphi\colon (\Sigma,X) \to (\Sigma',X')$ is just a $\mathcal{FOL}$ signature morphism $\varphi\colon \Sigma \to \Sigma'$ such that $X \subseteq X'$.

We let $Sen^{\mathcal{OFOL}}((S,F,P),X) = Sen^{\mathcal{FOL}}(S,F+X,P)$ and $Mod^{\mathcal{OFOL}}((S,F,P),X) = Mod^{\mathcal{FOL}}(S,F,P)$. For each $((S,F,P),X)$-model M, we let $|M|^X$ denote the set of the S-sorted functions $X \to (M_s)_{s\in S}$. For each $w \in |M|^X$, and each $((S,F,P),X)$-sentence ρ we define

$$(M(\models^{\mathcal{OFOL}}_{(S,F,P),X})^w \rho) \;=\; (M^w \models^{\mathcal{FOL}}_{(S,F+X,P)} \rho)$$

where M^w is the expansion of M to $(S,F+X,P)$ such that $M^w_x = w(x)$ for each $x \in X$. This is a stratified institution with $[\![M]\!]_{\Sigma,X} = |M|^X$ for each (Σ,X)-model M. For any signature morphism $\varphi\colon (\Sigma,X) \to (\Sigma',X')$ and any (Σ',X')-model M', $[\![M']\!]_\varphi\colon |M'|^{X'} \to |M'|^X$ is defined by $[\![M']\!]_\varphi(w') = w'|_X$ (i.e. the restriction of w' to X). Note that $[\![\_]\!]$ is a proper lax natural transformation.

Automata ($\mathcal{SAUT}$). The automata example $\mathcal{AUT}$ of Section 3.2 can be better framed as a stratified institution because of the centrality of the states within the concept of automata. We can re-introduce automata as models in a stratified institution, denoted $\mathcal{SAUT}$. Moreover we take this opportunity to expand the concept of sentence.

- The category of the signatures of $\mathcal{SAUT}$ is $\mathbb{S}et$.
- For each set V (of 'input symbols') a V-sentence is a *regular expression* formed with symbols from V, i.e. defined by the following grammar:

$$S ::= \varepsilon \mid V \mid SS \mid S+S \mid S^*.$$

- The V-models are the V*-automata* $A = ([\![A]\!],\ A_t :\ V \times [\![A]\!] \to [\![A]\!],\ A_F \subseteq [\![A]\!])$.

In automata theory terminology, $[\![A]\!]$ is the set of the 'states' of A (in $\mathcal{AUT}$ denoted A_{state}), A_t is the 'transition function', and A_F is the set of the 'final states'. In this definition note the absence of a designated 'initial state'. For any $\rho \in Sen(V)$ and $w \in [\![A]\!]$ we define

$$A \models^w \rho \ \text{ if and only if } \ A[\rho, w] \subseteq A_F,$$

where $A[\rho, w] \subseteq [\![A]\!]$ is defined by induction on the structure of ρ as follows:

$A[\varepsilon, w] = \{w\}$;

$A[v, w] = \{A_t(v, w)\}$ for each $v \in V$;

$A[\rho_1\rho_2, w] = \bigcup_{w_1 \in A[\rho_1, w]} A[\rho_2, w_1]$;

$A[\rho_1 + \rho_2, w] = A[\rho_1, w] \cup A[\rho_2, w]$;

$A[\rho^*, w] = \bigcup_{n \in \omega} A[\rho^n, w]$ (where $\rho^0 = \varepsilon$ and $\rho^{n+1} = \rho^n\rho$).

Modulo some straightforward missing details this yields a strict stratified institutions.

Other kind of automata or even computational models (such as Turing machines) should be framed as stratified institutions in a similar manner.

Abstract connectives. A *connective signature* C is just a single sorted signature of operation symbols, which are called connectives. Let T_C denote the set of all C-terms. A C*-algebra* A consists of a set $[\![A]\!]$ and a mapping $A :\ T_C \to \mathcal{P}[\![A]\!]$. A C*-homomorphism* $h :\ A \to B$ is a function $h :\ [\![A]\!] \to [\![B]\!]$ such that $2^h \circ A = B$. If $\eta \in [\![A]\!]$ and $\rho \in T_C$ then $A \models^{\eta}_{C} \rho$ holds when $\eta \in A\rho$. All these define the stratified institution of abstract connectives $\mathcal{CON}$ that has the connectives signatures as its signatures, C-algebras as C-models, T_C as the set of C-sentences, the stratification being given by $[\![A]\!]$ and the satisfaction relation defined as above.

$\mathcal{CON}$ is a rather abstract construction, so an example may help clarifying its meaning. Let us consider the stratified institution $\mathcal{MPL}$. Then any $\mathcal{MPL}$ signature P (aka set of propositional variables) determines a connective signature $C(P)$ where $C(P)_0 = P$, $C(P)_1 = \{\neg, \Box\}$, $C(P)_2 = \{\vee, \wedge, \ldots\}$, $C(P)_n = \emptyset$ for $n > 2$. Each $\mathcal{MPL}$ yields a $C(P)$-algebra A defined by $[\![A]\!] = |W|$, $A\rho = [\![(W, M), \rho]\!]$ for each $\rho \in T_{C(P)}$.

Another example can be constructed from $\mathcal{OFOL}$, where each $((S, F, P), X)$-model yields an A such that $[\![A]\!] = |M|^X$.

Morphisms of stratified institutions

They extend the concept of institution morphism from ordinary institutions to stratified institutions. The 2-dimensional aspect of the stratified institutions leads to a higher complexity of the definition of morphisms of stratified institutions. This concept will be instrumental later on in the chapter.

Given two stratified $\mathcal{S}$ and $\mathcal{S}'$, a *stratified institution morphism* $(\Phi,\alpha,\beta)\colon\ \mathcal{S}' \to \mathcal{S}$ consists of

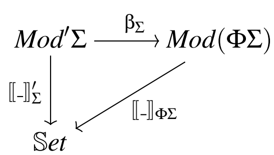
- a functor $\Phi\colon\ Sig' \to Sig$,
- a natural transformation $\alpha\colon\ \Phi;Sen \Rightarrow Sen'$, and
- a lax natural transformation $\beta\colon\ Mod' \Rightarrow \Phi^{\mathrm{op}};Mod$ such that $\beta;\Phi^{\mathrm{op}}[\![\_]\!] = [\![\_]\!]'$,

and such that the following Satisfaction Condition holds for any $\mathcal{S}'$-signature Σ', any Σ'-model M', any $w \in [\![M']\!]_{\Sigma'}$ and any $\Phi\Sigma'$-sentence ρ:

$$M' \models^{w'} \alpha_{\Sigma'}\rho \ \text{ if and only if } \ \beta_{\Sigma'}M' \models^{w'} \rho.$$

When β is strict, (Φ,α,β) is called *strict* too. The condition on β means the following:

- for each $\mathcal{S}'$-signature Σ the following diagram commutes

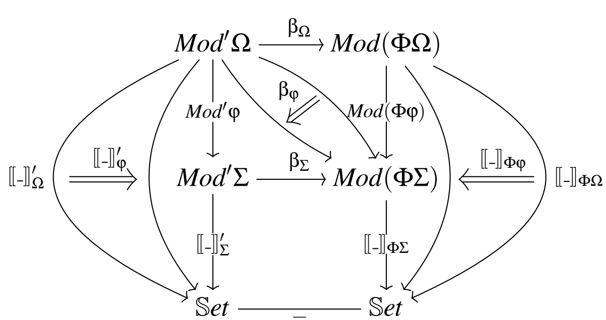


(12.3)

- for each $\mathcal{S}'$-signature morphism $\varphi\colon\ \Sigma \to \Omega$

$$[\![\beta_\Omega\_]\!]_{\Phi\varphi}\ ;\ [\![\beta_\varphi\_]\!]_{\Phi\Sigma} = [\![\_]\!]'_\varphi \tag{12.4}$$

  which can be visualised as the commutativity of the following diagram:



The morphisms of stratified institutions form a category $\mathbb{SI}ns$ under a composition that is defined component-wise like in the case of morphisms of ordinary institutions:

$$(\Phi',\alpha',\beta')\ ;\ (\Phi,\alpha,\beta) = (\Phi;\Phi\ ,\ \alpha\Phi';\alpha'\ ,\ \beta';\beta\Phi^{\mathrm{op}}).$$

Reducing stratified institutions to ordinary institutions

Some situations are greatly helped by the possibility to mathematically transfer concepts and results from ordinary institution theory to stratified institutions. This can be achieved due to a general canonical interpretation of stratified institutions as ordinary institutions, which arises as a left adjoint functor $\mathbb{SI}ns \to \mathbb{I}ns$. Its right adjoint will be instrumental in Sec. 12.3.

The left adjoint $\mathbb{SI}ns \to \mathbb{I}ns$. The idea of this functor is to flatten the satisfaction relation of a stratified institution by bringing the states at the frontend of the concept of model. More precisely, the left adjoint maps each stratified institution $\mathcal{S} = (Sig, Sen, Mod, [\![\_]\!], \models)$ to the ordinary institution $\mathcal{S}^\sharp = (Sig, Sen, Mod^\sharp, \models^\sharp)$ where

- the objects of $Mod^\sharp\Sigma$ are the pairs (M, w), called *pointed models*, such that $M \in |Mod\Sigma|$ and $w \in [\![M]\!]_\Sigma$;
- the Σ-homomorphisms $(M, w) \to (N, v)$ are the pairs (h, w) such that $h\colon M \to N$ and $[\![h]\!]_\Sigma w = v$;
- for any signature morphism $\varphi\colon \Sigma \to \Sigma'$ and any Σ'-model (M', w')
$$(Mod^\sharp\varphi)(M', w') = ((Mod\varphi)M', [\![M']\!]_\varphi w');$$
- for each Σ-model M, each $w \in [\![M]\!]_\Sigma$, and each $\rho \in Sen\Sigma$
$$((M, w) \models^\sharp_\Sigma \rho) \;=\; (M \models^w_\Sigma \rho). \tag{12.5}$$

The right adjoint $\mathbb{I}ns \to \mathbb{SI}ns$. It maps each ordinary institution $\mathcal{B}$ to the stratified institution $\widetilde{\mathcal{B}}$ where

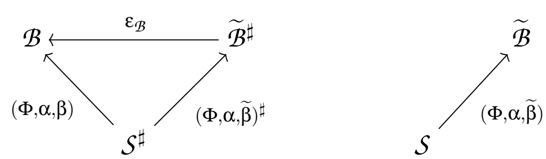
- $Sig^{\widetilde{\mathcal{B}}} = Sig^{\mathcal{B}}$ and $Sen^{\widetilde{\mathcal{B}}} = Sen^{\mathcal{B}}$,
- $|Mod^{\widetilde{\mathcal{B}}}\Sigma| = \{(W, B\colon W \to |Mod^{\mathcal{B}}\Sigma|) \mid W \text{ set}\}$,
- $Mod^{\widetilde{\mathcal{B}}}\Sigma((W,B),(V,N))$ consists of $h = (h_0\colon W \to V, (h^w\colon B^w \to N^{h_0 w})_{w\in W})$,
- for each signature morphism $\varphi\colon \Sigma \to \Sigma'$ and each Σ'-model (W', B'):
$(Mod^{\widetilde{\mathcal{B}}}\varphi)(W', B') = (W',\ B'; (Mod^{\mathcal{B}}\varphi))$,
- $[\![W, B]\!]^{\widetilde{\mathcal{B}}}_\Sigma = W$ and $[\![h]\!]_\Sigma = h_0$,
- $[\![\_]\!]_\varphi$ are identities, and
- $(W, B)\ (\models^{\widetilde{\mathcal{B}}}_\Sigma)^w\ \rho$ if and only if $B^w\ \models^{\mathcal{B}}_\Sigma\ \rho$.

Proposition 12.1. *$(\_)^\sharp$ is a left adjoint to $\widetilde{(\_)}$.*

Proof. The proof that $\widetilde{\mathcal{B}}$ is a stratified institution consists of straightforward verifications. Let us do only the Satisfaction Condition. We consider a signature morphism φ:

$$
\begin{aligned}
(W',B') \models^{w} \varphi\rho &= B'^{w} \models \varphi\rho && \text{definition of satisfaction in } \widetilde{B}\\
&= (Mod^{\mathcal{B}}\varphi)B'^{w} \models \rho && \text{Satisfaction Condition in } \mathcal{B}\\
&= (W',\ B';(Mod\varphi)) \models^{w} \rho && \text{definition of satisfaction in } \widetilde{B}\\
&= (Mod^{\widetilde{\mathcal{B}}}\varphi)(W',B') \models^{w} \rho && \text{definition of reduct.}
\end{aligned}
$$

In order to prove that $\widetilde{(\_)}$ is a right adjoint to $(\_)^{\sharp}$ we first define the co-unit of the adjunction as follows. For each institution $\mathcal{B}$ we let the institution morphism $\varepsilon_{\mathcal{B}} : \ \widetilde{\mathcal{B}}^{\sharp} \to \mathcal{B}$ have identities for the signature and sentence translation functors and maps each $\widetilde{\mathcal{B}}^{\sharp}$ Σ-model $((W,(B^{w})_{w\in W}),w)$ to B^{w}. Then we prove the universal property of $\widetilde{\mathcal{B}}$, namely that for each institution morphism $(\Phi,\alpha,\beta) : \ \mathcal{S}^{\sharp} \to \mathcal{B}$ there exists an unique stratified institution morphism $(\Phi,\alpha,\widetilde{\beta}) : \ \mathcal{S} \to \widetilde{\mathcal{B}}$ such that the following diagram commutes:



(12.6)

Because the signature and the sentences translation functors of $\varepsilon_{\mathcal{B}}$ are identities, there is no other choice for the signature and the sentence translation functors of $(\Phi,\alpha,\widetilde{\beta})$. By (12.3) and by the commutativity (12.6) it follows that for each signature Σ and for each $\mathcal{S}$ Σ-model M, the definition of $\widetilde{\beta}$ is constrained to

$$\widetilde{\beta}_{\Sigma}M = ([\![M]\!]_{\Sigma}, (\beta_{\Sigma}(M,w))_{w\in[\![M]\!]_{\Sigma}}).$$

Similarly, by (12.4) and by (12.6) it follows that for each signature morphism $\varphi : \ \Sigma \to \Sigma'$ and each Σ'-model M'

$$\widetilde{\beta}_{\varphi}M' = ([\![M']\!]_{\varphi}, (1_{\beta_{\Sigma}(\varphi M', [\![M']\!]_{\varphi} w')})_{w'\in[\![M']\!]_{\Sigma'}}).$$

We may skip a few straightforward things related to establishing that $(\Phi,\alpha,\widetilde{\beta})$ is indeed a strict stratified institution morphism and only show its Satisfaction Condition:

$$
\begin{aligned}
\widetilde{\beta}M \models^{w} \rho &= \beta(M,w) \models \rho && \text{definition of } \widetilde{\beta}\text{, of satisfaction in } \widetilde{\mathcal{B}}\\
&= (M,w) \models \alpha\rho && \text{Satisfaction Condition of } (\Phi,\alpha,\beta)\\
&= M \models^{w} \alpha\rho && \text{definition of satisfaction in } \mathcal{S}^{\sharp}.
\end{aligned}
$$

□

Note that the 'local satisfaction' institution of first order modal logic, $\mathcal{MFOL}^{\sharp}$, already introduced in Sec. 3.2 is precisely the result of applying the left adjoint $\mathbb{SI}ns \to \mathbb{I}ns$ to $\mathcal{MFOL}$. $\mathcal{SAUT}^{\sharp}$ adds initial states to automata, so it is $\mathcal{AUT}$ enhanced with more syntax. In general, $\mathcal{S}^{\sharp}$ is called the institution of 'local satisfaction' associated to the stratified institution $\mathcal{S}$.

The institution of 'global satisfaction'. The other institution determined by $\mathcal{MFOL}$, namely the 'global satisfaction' institution $\mathcal{MFOL}^*$, can also be explained as a result of a general construction by letting for any stratfied institution $\mathcal{S}$

$$M \models^* \rho \quad \text{if and only if} \quad [\![M,\rho]\!] = [\![M]\!].$$

Let us say that $[\![\_]\!]$ *is surjective* when for each signature morphism $\varphi\colon \Sigma \to \Sigma'$ and each Σ'-model M', $[\![M']\!]_\varphi\colon [\![M']\!]_{\Sigma'} \to [\![Mod(\varphi)M']\!]_\Sigma$ is surjective.

Fact 12.2. *If the stratification $[\![\_]\!]$ is surjective then $\mathcal{S}^* = (Sig, Sen, Mod, \models^*)$ is an institution, called the* global institution of $\mathcal{S}$.

While the local satisfaction $\models^\sharp$ gives a complete account of the satisfaction relation of the stratified institution, the global one encapsulates information. Therefore it is expected that the local semantic consequence is stronger than the global one:

Fact 12.3. *Let $\mathcal{S}$ be a stratified institution $\mathcal{S}$ with $[\![\_]\!]$ surjective. For each $E \subseteq Sen\Sigma$ and each $\rho \in Sen\Sigma$, we have that*

$$E \models^\sharp \rho \ \ \textit{implies} \ \ E \models^* \rho.$$

Model amalgamation

The concept of model amalgamation in stratified institutions takes two forms. The first one ignores the stratifications and is just the ordinary model amalgamation (as introduced in Sec. 4.3). The second one is more refined as it considers the stratification.

Consider a stratified institution $\mathcal{S}$ and a commutative square of signature morphisms like below:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array} \qquad (12.7)$$

Then this square is a *stratified model amalgamation square* when for each Σ_k-model M_k and each $w_k \in [\![M_k]\!]_{\Sigma_k}$, $k = 1,2$, such that $M_1 \upharpoonright_{\varphi_1} = M_2 \upharpoonright_{\varphi_2}$ and $[\![M_1]\!]_{\varphi_1} w_1 = [\![M_2]\!]_{\varphi_2} w_2$ there exists an unique Σ'-model M' and an unique $w' \in [\![N']\!]_{\Sigma'}$ such that $M' \upharpoonright_{\theta_k} = M_k$ and $[\![M']\!]_{\theta_k} w' = w_k$, $k = 1,2$. The model M' is called the *stratified amalgamation of M_1 and M_2*.

This amalgamation concept can be extended in an obvious manner to other variants of model amalgamation such as (semi-)exactness, weak amalgamation, etc.

Stratified model amalgamation is just ordinary model amalgamation in the institution of local satisfaction:

Fact 12.4. *A commutative square of signature morphisms like* (12.7) *is a stratified model amalgamation square in $\mathcal{S}$ if and only if it is a model amalgamation square in $\mathcal{S}^\sharp$.*

A couple of categorical conditions *almost* characterise stratified model amalgamation. They are sufficient but fall short of being also necessary conditions. In particular this situation shows that plain model amalgamation cannot be derived from the seemingly more refined concept of stratified model amalgamation.

Proposition 12.5. *A commutative square of signature morphisms like* (12.7) *is a stratified model amalgamation square if*

- $$\begin{array}{ccc} Mod\Sigma & \xleftarrow{Mod\varphi_1} & Mod\Sigma_1 \\ {\scriptstyle Mod\varphi_2}\uparrow & & \uparrow{\scriptstyle Mod\theta_1} \\ Mod\Sigma_2 & \xleftarrow[Mod\theta_2]{} & Mod\Sigma' \end{array}$$

  is a pullback in $|\mathbb{C}at|$, *and*
- *for each* Σ'*-model* M'

$$\begin{array}{ccc} [\![M'\restriction_\theta\restriction_\varphi]\!]_\Sigma & \xleftarrow{[\![\theta_1 M']\!]_{\varphi_1}} & [\![M'\restriction_{\theta_1}]\!]_{\Sigma_1} \\ {\scriptstyle [\![\theta_2 M']\!]_{\varphi_2}}\uparrow & & \uparrow{\scriptstyle [\![M']\!]_{\theta_1}} \\ [\![M'\restriction_{\theta_2}]\!]_{\Sigma_2} & \xleftarrow[{[\![M']\!]_{\theta_2}}]{} & [\![M']\!]_{\Sigma'} \end{array}$$

is a pullback in $\mathbb{S}et$.

Proof. Note that the first condition just says that (12.7) is a model amalgamation square. We consider M_1, w_1, M_2 and w_2 like in the definition of stratified model amalgamation. Then we consider M' to be the unique amalgamation of M_1 and M_2 and apply the second condition for w_1 and w_2. □

Note that stratified model amalgamation implies the second condition of Prop. 13.39 (by considering $M_k = M'\restriction_{\theta_k}$) but it does not technically imply the first condition. When the stratification is strict then the concept of stratified model amalgamation collapses to that of ordinary model amalgamation because the latter pullback of Prop. 13.39 is trivial. For instance this is the case in $\mathcal{MPL}$, $\mathcal{MFOL}$, etc. Otherwise, ordinary model amalgamation and stratified model amalgamation are different concepts. Let us look in some detail into the $\mathcal{OFOL}$ case where the stratification is a proper lax natural transformation. Let us consider a pushout square of $\mathcal{FOL}$ signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array} \qquad (12.8)$$

and sets of variables X, X_1, X_2, X' such that $X = X_1 \cap X_2$ and $X' = X_1 \cup X_2$. Then

$$\begin{array}{ccc} (\Sigma, X) & \xrightarrow{\varphi_1} & (\Sigma_1, X_1) \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ (\Sigma_2, X_2) & \xrightarrow[\theta_2]{} & (\Sigma', X') \end{array}$$

is a stratified model amalgamation square in $\mathcal{OFOL}$ because

- it is an ordinary model amalgamation square since (12.8) is a model amalgamation square in $\mathcal{FOL}$ as $\mathcal{FOL}$ is semi-exact, and
- for each (Σ', X')-model M' (aka $\mathcal{FOL}$ Σ'-model) and each $w_k : X_k \to |M_k|$, $k = 1, 2$, such that $w_1 x = w_2 x$ for each $x \in X$, $w' : X' \to |M'|$ defined by $w' x = w_k x$ when $x \in X_k$ is unique such that $[\![M']\!]_{\theta_k} w' = w_k$, $k = 1, 2$. (Note that $|M_1| = |M_2| = |M'|$). Then we apply Prop. 13.39.

Exercises

12.1. In $\mathcal{SAUT}$, for any V-sentences a, b, c prove that $(ab)c \models\!\!\!\models a(bc)$, $(a^*)^* \models\!\!\!\models a^*$, $(a+b)c \models\!\!\!\models ac + bc$.

12.2. The satisfaction relation of a stratified institution can be presented as a natural transformation $\models : Sen \Rightarrow [\![Mod(\_) \to \mathbb{S}et]\!]$ where the functor $[\![Mod(\_) \to \mathbb{S}et]\!] : Sig \to \mathbb{S}et$ is defined by

– for each signature $\Sigma \in |Sig|$, $[\![Mod\Sigma \to \mathbb{S}et]\!]$ denotes the set of all the mappings $f : |Mod\Sigma| \to \mathbb{S}et$ such that $fM \subseteq [\![M]\!]_\Sigma$; and

– for each signature morphism $\varphi : \Sigma \to \Sigma'$, $([\![Mod\varphi \to \mathbb{S}et]\!] f)M' = [\![M']\!]_\varphi^{-1} f((Mod\varphi)M')$.

Then the Satisfaction Condition (12.1) appears exactly as the naturality property of $\models$.

12.3. Define a suitable concept of comorphism for stratified institutions by replicating ideas from the definition of morphisms of institutions.

12.2 The internal logic of stratified institutions

We start by extending the definition of the semantics of Boolean connectives and quantifiers from ordinary institutions of Chap. 5 to stratified institutions. After this, based on the stratified structure of stratified institutions, we define semantics for modalities and for hybrid features (i.e. nominals, @) at the level of abstract stratified institutions. Unlike in the case of the Boolean connectives, in each of the latter cases a minimally sufficient particular condition is imposed on the stratification.

Boolean connectives

Given a signature Σ in a stratified institution, a Σ-sentence ρ' is a *semantic*

- *negation* of ρ when $[\![M,\rho']\!] \;=\; [\![M]\!] \setminus [\![M,\rho]\!]$;
- *conjunction* of ρ_1 and ρ_2 when $[\![M,\rho']\!] \;=\; [\![M,\rho_1]\!] \cap [\![M,\rho_2]\!]$;
- *disjunction* of ρ_1 and ρ_2 when $[\![M,\rho']\!] \;=\; [\![M,\rho_1]\!] \cup [\![M,\rho_2]\!]$;
- *implication* of ρ_1 and ρ_2 when $[\![M,\rho']\!] \;=\; ([\![M]\!] \setminus [\![M,\rho_1]\!]) \;\cup\; [\![M,\rho_2]\!]$;
- etc.,

for each Σ-model M.

A stratified institution *has (semantic) negation* when each sentence of the institution has a negation. It has *(semantic) conjunctions* when each two sentences (of the same signature) have a conjunction. Similar definitions can be formulated for disjunctions, implications, and equivalences. Like in ordinary institution theory, distinguished negations are usually denoted by $\neg\_$, distinguished conjunctions by $\_\wedge\_$, distinguished disjunctions by $\_\vee\_$, distinguished implications by $\_\Rightarrow\_$, distinguished equivalences by $\_\Leftrightarrow\_$, etc.

Note that $\mathcal{MFOL}$, $\mathcal{MPL}$ together with their hybrid extensions $\mathcal{HFOL}$, $\mathcal{HPL}$, as well as $\mathcal{OFOL}$ have all these semantics Boolean connectives. $\mathcal{SAUT}$ has conjunctions $(\rho_1 + \rho_2)$.

Fact 12.6. *When they exist, the negations, conjunctions, disjunctions, implications, negations, coincide in $\mathcal{S}$ and $\mathcal{S}^\sharp$.*

This identity between the Boolean connectives at the level of the stratified institution $\mathcal{S}$ and its associated local institution $\mathcal{S}^\sharp$ does not carry in general to the global institution.

Quantifiers

Given a morphism of signatures $\chi\colon\; \Sigma \to \Sigma'$, a Σ-sentence ρ is a *semantic*

- *universal χ-quantification* of a Σ'-sentence ρ' when

$$[\![M,\rho]\!] = \bigcap_{M'\restriction_\chi = M} \{w \in [\![M]\!]_\Sigma \mid [\![M']\!]_\chi^{-1} w \subseteq [\![M',\rho']\!]\}, \text{ and}$$

- *existential χ-quantification* of a Σ'-sentence ρ' when

$$[\![M,\rho]\!] = \bigcup_{M'\restriction_\chi = M} [\![M']\!]_\chi([\![M',\rho']\!]),$$

for any Σ-model M.

A stratified institution *has (semantic) universal $\mathcal{D}$-quantification* for a class $\mathcal{D}$ of signature morphisms when for each $(\chi\colon\; \Sigma \to \Sigma') \in \mathcal{D}$, each Σ'-sentence has a universal

χ-quantification. A similar definition applies to existential quantification. Distinguished universal / existential quantifications are denoted by $(\forall\chi)\rho'$ / $(\exists\chi)\rho'$.

The following table shows the situation of the semantic quantification in some concrete stratified institutions. In all cases we have both universal and existential quantifications; these can be checked easily by making the corresponding definitions explicit.

	χ-quantification	
$\mathcal{MPL}$	none	
$\mathcal{MFOL}$	$\chi : \Sigma \to \Sigma + X$	X block of rigid first-order variables
$\mathcal{HPL}$	$\chi : (\mathrm{Nom}, \Sigma) \to (\mathrm{Nom} + N, \Sigma)$	N block of nominal variables
$\mathcal{HFOL}$	$\chi : (\mathrm{Nom}, \Sigma) \to (\mathrm{Nom} + N, \Sigma + X)$	X block of rigid first-order variables, N block of nominal variables
$\mathcal{OFOL}$	$\chi : \Sigma \to \Sigma + Y$	Y block of first-order variables

Fact 12.7. *When they exist, the universal and the existential χ-quantifications, respectively, coincide in $\mathcal{S}$ and $\mathcal{S}^\sharp$.*

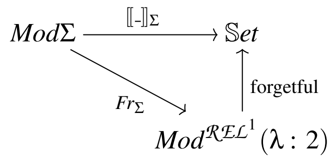
On the one hand, the concepts of semantic Boolean connectives and quantifications in ordinary institutions arise as an instance of those of stratified institutions when the underlying set of each $[\![M]\!]_\Sigma$ is a singleton set. On the other hand, Facts 12.6 and 12.7 shows that the stratified institution concepts of Boolean connectives and quantifications are in substance no more general than their ordinary institution theoretic correspondents. Therefore an alternative equivalent way to introduce the stratified institution semantics of Boolean connectors and quantifications would be to turn Facts 12.6 and 12.7 into definitions and then infer the current definitions as properties.

Modalities

Stratified institutions allow for a very abstract interpretation of modalities as semantic connectives. This is beyond ordinary institution theory, something for which the technique of flattening to $\mathcal{S}^\sharp$ is useless.

Frame extractions (binary). In order to define semantic possibility ($\Diamond$) and necessity ($\Box$) in a stratified institution we have to be able to 'extract' Kripke frames from the stratification. Let $\mathcal{REL}^1$ denote the single sorted version of $\mathcal{REL}$ in which we retain only the signatures without constants.

A *binary frame extraction* assumes that for each signature Σ the stratification $[\![\_]\!]_\Sigma$ is a composition between a functor $Fr_\Sigma : Mod\Sigma \to Mod^{\mathcal{REL}^1}(\lambda : 2)$ and the forgetful functor $Mod^{\mathcal{REL}^1}(\lambda : 2) \to \mathbb{S}et$, where $Mod^{\mathcal{REL}^1}(\lambda : 2)$ is the category of the $\mathcal{FOL}$-models for a single sorted signature with one binary relation symbol λ.



Note that the models of $Mod^{\mathcal{REL}^1}(\lambda:\, 2)$ are exactly the Kripke frames $W = (|W|, W_\lambda)$ of the modal logic examples (introduced in Sec. 3.2). Since $|Fr_\Sigma M| = [\![M]\!]_\Sigma$ we can write $Fr_\Sigma M = ([\![M]\!]_\Sigma, (Fr_\Sigma M)_\lambda)$. The Fr_Σ functors are also required to form a lax natural transformation from *Mod* to the constant functor mapping any signature to the category $Mod^{\mathcal{REL}^1}(\lambda : 2)$.

Concretely, in the stratified institutions $\mathcal{MFOL}$, $\mathcal{MPL}$, $\mathcal{HFOL}$, $\mathcal{HPL}$, the Fr maps the Kripke models (W, M) to their underlying Kripke frames $W = (|W|, W_\lambda)$.

Frame extractions (general). In the most general situation, when we allow *polyadic* modalities, i.e. modalities with more than one argument, first we need a functor $L :\ Sig^{\mathcal{S}} \to Sig^{\mathcal{REL}^1}$ such that $L\Sigma$ represents the relation symbols corresponding to the modalities of Σ (we allow a flexible approach where the modalities may change with the signature). Then we have a more general concept of frame extraction. In the binary case $L\Sigma$ is always $\{\lambda : 2\}$ and hence no reason to have λ as part of the signatures.

A *(general) frame extraction* (L, Fr) is a stratified institution morphism

$$(L, \emptyset, Fr) :\ \mathcal{S} \to \mathcal{REL}^1$$

where $\mathcal{REL}^1$ is considered as a stratified institution with no sentences and for each $\mathcal{REL}^1$-model M, $[\![M]\!]$ is the underlying set of M and the satisfaction is invariant with respect to the states, i.e. $M \models^w \rho$ is $M \models \rho$. Commonly, in concrete examples, it happens that frame extractions are in fact strict institution morphisms.

Semantic modalities. In any stratified institution endowed with a binary frame extraction Fr, a Σ-sentence ρ' is a *semantic*

- *possibility* ($\Diamond$) of ρ when $[\![M, \rho']\!] = (Fr_\Sigma M)_\lambda^{-1} [\![M, \rho]\!]$;
- *necessity* ($\Box$) of ρ when $[\![M, \rho']\!] = \{i \mid (Fr_\Sigma M)_\lambda i \subseteq [\![M, \rho]\!]\}$,

for each Σ-model M.

Obviously, in $\mathcal{MPL}$, $\mathcal{MFOL}$, $\mathcal{HPL}$, $\mathcal{HFOL}$ we have that each $\Diamond\rho$ / $\Box\rho$ is a semantic possibility / necessity of ρ in the sense of our definitions above.

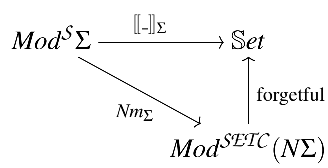
The concept of semantic possibility / necessity admits an obvious extension to polyadic modalities by using general frame extractions.

Nominals

Nominals extraction. In order to define the semantics of hybrid features such as nominals and the satisfaction operator (@) in stratified institutions we need to be able to extract nominals data from the corresponding stratification. Let $\mathcal{SETC}$ be the sub-institution of $\mathcal{REL}$ that restricts the signatures to single-sorted ones and without relation symbols, so only constants being admitted.

A *nominals extraction* assumes two additional data:

- a functor $N : Sig^{\mathcal{S}} \to Sig^{\mathcal{SETC}}$, i.e. each $N\Sigma$ is a single-sorted $\mathcal{FOL}$ signature having only constants; and
- that for each signature Σ the stratification $[\![\_]\!]_\Sigma$ is a composition between a functor $Nm_\Sigma : Mod^{\mathcal{S}}\Sigma \to Mod^{\mathcal{SETC}}(N\Sigma)$ and the forgetful functor $Mod^{\mathcal{SETC}}(N\Sigma) \to \mathbb{S}et$,



such that the Nm_Σ functors are also required to form a lax natural transformation $Mod^{\mathcal{S}} \Rightarrow N^{\mathrm{op}} ; Mod^{\mathcal{SETC}}$.

Hence, a nominals extraction (N, Nm) is a stratified institution morphism

$$(N, \emptyset, Nm) : \mathcal{S} \to \mathcal{SETC}$$

where $\mathcal{SETC}$ is considered as a stratified institution in the same manner we considered $\mathcal{REL}^1$ as stratified institution.

Concretely, in the stratified institutions of the hybrid modal logics $\mathcal{HFOL}$, $\mathcal{HPL}$, we have that N maps each signature (Nom, Σ) to the single-sorted signature of constants Nom, and that $Nm_{(\mathrm{Nom},\Sigma)}$ maps each Kripke model (W, M) to the $Mod^{\mathcal{SETC}}(\mathrm{Nom})$-model $(|W|, (W_i)_{i\in \mathrm{Nom}})$, so from the Kripke models it forgets both the M part as well as the accessibility relation W_λ.

Semantic nominals. In any stratified institution endowed with a nominals extraction N, Nm, for each signature Σ and each $i \in N\Sigma$,

- a Σ-sentence ρ' is an *i-sentence* when $[\![M, \rho']\!] = \{(Nm_\Sigma M)_i\}$;
- a Σ-sentence ρ' is the *satisfaction of* ρ *at* i when

$$[\![M, \rho']\!] = \begin{cases} [\![M]\!], & (Nm_\Sigma M)_i \in [\![M, \rho]\!] \\ \emptyset, & (Nm_\Sigma M)_i \notin [\![M, \rho]\!] \end{cases}$$

for each Σ-model M.

In $\mathcal{HPL}$ and $\mathcal{HFOL}$ we have that each nominal i of the signature is an i-sentence and each sentence $@_i\rho$ is a satisfaction at i in the sense of the above definitions. In general, for the designated i-sentences and satisfaction at i we will use the notations i-sen and $@_i\rho$, respectively. Moreover we say that a stratified institution has *explicit local satisfaction* when there exists a satisfaction at i for each sentence and each appropriate i.

Exercises

12.4. In any stratified institution $\mathcal{S}$ with surjective stratification,

1. any semantic conjunction in $\mathcal{S}$ is a semantic conjunction in $\mathcal{S}^*$ too, and
2. any semantic universal χ-quantifications in $\mathcal{S}$ is a semantic universal χ-quantifications in $\mathcal{S}^*$ too.

12.5. Give an example of a stratified institution $\mathcal{S}$ that has semantic disjunctions but such that $\mathcal{S}^*$ does not have them.

12.6. Extend the definition of $\mathcal{SAUT}$ with all Boolean connectives, quantifications, and nominal related sentences.

12.7. [87] **Local versus global interpolation.**
Let $\mathcal{S}$ be any stratified institution. Formulate a set of general conditions such that Craig interpolation in $\mathcal{S}^\sharp$ implies Craig interpolation in $\mathcal{S}^*$. (*Hint:* In $\mathcal{HPL}$, the stratified institution of hybrid propositional logic, for any signature (Nom,P), if $i \notin \mathrm{Nom}$ then we can add it to Nom and obtain a signature inclusion $\iota : (\mathrm{Nom},P) \to (\mathrm{Nom}\cup\{i\},P)$. Then for each (Nom,P)-model (W,M) and each $w \in [\![(W,M)]\!]$ there exists a ι-expansion (W',M') of (W,M) such that

$$[\![(W',M')]\!]_i(Nm_{(\mathrm{Nom}\cup\{i\},P)}(W',M'))_i = w.$$

Moreover, for each $\theta_1 : (\mathrm{Nom}_1,P_1) \to (\mathrm{Nom}',P')$ and $\iota' : (\mathrm{Nom}',P') \to (\mathrm{Nom}'\cup\{i'\},P')$ there exists $\iota_1 : (\mathrm{Nom}_1,P_1) \to (\mathrm{Nom}_1\cup\{i\},P_1)$, $i \notin \mathrm{Nom}_1$, and $\theta'_1 : (\mathrm{Nom}_1\cup\{i\},P_1) \to (\mathrm{Nom}',\cup\{i'\},P')$ such that $(\iota_1,\theta_1,\theta'_1,\iota')$ is a stratified model amalgamation square. At the general level these properties can be expressed as axioms. Further assume universal quantifications for the signature extensions ι.)

12.8. **Preservation of semantic connectives along signature morphisms.**
In any institution I, if ρ is a semantic conjunction of Σ-sentences ρ_1 and ρ_2 then for any signature morphism $\varphi : \Sigma \to \Sigma'$ we can prove easily that $\varphi\rho$ is a semantic conjunction of $\varphi\rho_1$ and $\varphi\rho_2$. Similar preservation properties hold also for the other Boolean connectives and for quantifications. Replicate these results for stratified institutions. What does it take to establish such preservation properties also for semantic modalities and nominals? (*Hint:* For the Boolean connectives and quantifications we may use the result of Fact 12.7.)

12.3 Decompositions of stratified institutions

An analysis of the structure of conventional Kripke semantics reveals the following situation for an individual Kripke model:

- There is a *family* of models in a "lower" logical system, usually propositional or first order logic. The indexing of the family is what is usually referred to as "worlds".

- Then there is a certain structure imposed upon this family of models. This happens at the level of the "worlds" commonly in the form of relations.

We address this general structure of Kripke semantics from an abstract axiomatic perspective. The result is a general abstract class of stratified institutions that does not necessarily consider explicitly Kripke frames nor modalised sentences, but which retains in an abstract form the essential idea of a stratified institution in which a "header" institution structures a certain multiplication of a "base" institution.

Bases for stratified institutions

In a concrete stratified institution $\mathcal{S}$ with Kripke semantics, if M is a Kripke model and $w \in [\![M]\!]$ then the model (M,w) of $\mathcal{S}^\sharp$ represents a "localisation" in M of the "world" w. This corresponds to a model in "lower" institution. However the construction of $\mathcal{S}^\sharp$, being fully abstract, is independent of the fact that M is really a Kripke model. On the other hand, we should be able to have the (syntax of the) "lower" logic available at the level of $\mathcal{S}$. These ideas are captured by the following definition: a *base for a stratified institution* $\mathcal{S}$ is an institution morphism $(\Phi,\alpha,\beta): \ \mathcal{S}^\sharp \to \mathcal{B}$. Let us look into a couple of examples of this.

A base for $\mathcal{MPL}$. For the stratified institutions that are based on some form of Kripke semantics we may consider $\mathcal{B}$ to be the institution that at the syntactic level removes from $\mathcal{S}$ all syntactic entities that involve modalities, and whose models are the individual "worlds" of the respective Kripke semantics. For instance, in the case of $\mathcal{MPL}$:

- $\mathcal{B} = \mathcal{PL}$ and Φ is the identity functor on $\mathbb{S}et$,
- α_P is the inclusion $Sen^{\mathcal{PL}}P \subseteq Sen^{\mathcal{MPL}}P$,
- $\beta_P(M,w) = M^w$, etc.

A base for $\mathcal{MFOL}$.

- We let $\mathcal{B} = \mathcal{AFOL}$ (the sub-institution of $\mathcal{FOL}$ determined by its atomic sentences) and let Φ forget the rigid symbols, i.e. $\Phi(S,S_0,F,F_0,P,P_0) = (S,F,P)$.
- α consists of the canonical inclusions of sets of sentences.
- $\beta_{(S,S_0,F,F_0,P,P_0)}(M,w) = M^w$ (as (S,F,P)-model).

As a matter of notation, in what follows, for any base $(\Phi,\alpha,\beta): \ \mathcal{S}^\sharp \to \mathcal{B}$ we will denote its correspondent through the natural isomorphism $\mathbb{I}ns(\mathcal{S}^\sharp,\mathcal{B}) \cong \mathbb{SI}ns(\mathcal{S},\widetilde{\mathcal{B}})$ by $(\Phi,\alpha,\widetilde{\beta})$. The idea here is that while β gives the 'local' / 'base' model corresponding to certain point / world from a Kripke model, $\widetilde{\beta}$ gives the whole bunch of 'local' / 'base' models of a Kripke model. Of course, this intuition applies to Kripke semantics, while the natural isomorphism from above is more abstract.

Decompositions of stratified institutions

Model constraints. In many Kripke semantics examples the models are subject to certain constraints. For instance,m in $\mathcal{MFOL}$ the interpretations of the rigid symbols are shared. This means that $\widetilde{\beta}M$ is not any bunch of 'base' models, but a bunch of models that is subject to a certain constraint, in this example the mentioned sharing. At the level of $\widetilde{\mathcal{B}}$ such constraints are treated abstractly by the concept of sub-fucntor. Let $\mathcal{S}$ be a strict stratified institution and $(\Phi,\alpha,\beta): \ \mathcal{S}^\sharp \to \mathcal{B}$ be a base for $\mathcal{S}$. A *constraint model sub-functor* $Mod^C \subseteq Mod^{\widetilde{\mathcal{B}}}$ is a sub-functor such that for each signature Σ,

$$\widetilde{\beta}_\Sigma(Mod^{\mathcal{S}}\Sigma) \subseteq Mod^C(\Phi\Sigma).$$

By $\widetilde{\mathcal{B}}^C$ we denote the stratified sub-institution of $\widetilde{\mathcal{B}}$ induced by Mod^C.

Decompositions. A *decomposition of $\mathcal{S}$* consists of two strict stratified institution morphisms like below

$$\mathcal{S}^0 \xleftarrow{(\Phi^0,\alpha^0,\beta^0)} \mathcal{S} \xrightarrow{(\Phi,\alpha,\widetilde{\beta})} \widetilde{\mathcal{B}}^C$$

such that for each $\mathcal{S}$-signature Σ

$$\begin{array}{ccccc} Mod^0(\Phi^0\Sigma) & \xleftarrow{\beta^0_\Sigma} & Mod^{\mathcal{S}}\Sigma & \xrightarrow{\widetilde{\beta}_\Sigma} & Mod^C(\Phi\Sigma) \\ & {\scriptstyle [\![-]\!]^0_{\Phi^0\Sigma}} \searrow & \downarrow {\scriptstyle [\![-]\!]^{\mathcal{S}}_\Sigma} & \swarrow {\scriptstyle [\![-]\!]^{\widetilde{\mathcal{B}}}_{\Phi\Sigma}} & \\ & & \mathbb{S}et & & \end{array}$$

is a pullback in $\mathbb{C}at$.

Let us note the following aspects emerging from the concept of decomposition.

- The models of $\mathcal{S}$ can be represented as pairs of $\mathcal{S}^0$-models and families of $\mathcal{B}$-models satisfying certain constraints (hence $\widetilde{\mathcal{B}}^C$ models) such that the "worlds" of the corresponding $\widetilde{\mathcal{B}}^C$ model constitutes the stratification of the corresponding $\mathcal{S}^0$-model. This means that at the semantic level $\mathcal{S}$ is completely determined by the two components of the decomposition.
- The situation at the syntactic level is different. The syntax (signatures and sentences) of each of the two components is represented in the syntax of $\mathcal{S}$, but the latter is not completely determined by the former syntaxes. In other words $\mathcal{S}$ may have signatures and sentences that do not originate from either of the two components. This is what the definition gives us. However, while there are hardly any examples / applications where all sentences come from either one of the two components, in many examples the signatures of $\mathcal{S}$ are composed from the signatures of $\mathcal{S}^0$ and those from $\mathcal{B}$.

Now we present some examples of decompositions of some of the most common stratified institutions from modal logic.

A decomposition of $\mathcal{MPL}$. We let

- $\mathcal{S}^0 = \mathcal{REL}^1$ (regarded as a trivially stratified institution like in the definition of frame extractions), $\Phi^0 P = \{\lambda : 2\}$, α^0 is empty, $\beta^0_P(W,M) = W$.
- $\mathcal{B} = \mathcal{PL}$, $\Phi P = P$, α_P is the inclusion $Sen^{\mathcal{PL}} P \subseteq Sen^{\mathcal{MPL}} P$, Mod^C is just $Mod^{\widetilde{\mathcal{B}}}$ (there are no constraints), and $\beta_P(W,M) = (|W|, (M^w)_{w \in |W|})$.

A decomposition of $\mathcal{HPL}$. This is quite similar to the decomposition of $\mathcal{MPL}$. The differences are:

- Now $\mathcal{S}^0 = \mathcal{RELC}^1$, which is the single-sorted sub-institution of $\mathcal{FOL}$ determined by the signatures without operation symbols other than constants. Consequently $\Phi^0(\mathrm{Nom}, P) = (\mathrm{Nom}, \lambda : 2)$.
- α^0 is not empty anymore, it is rather defined by

$$\alpha^0_{(\mathrm{Nom},P)} \lambda(i,j) = @_i \Diamond j (= @_i \neg \Box \neg j)$$

  for the atoms, and then for any sentence by induction on the structure of the respective sentence such that $\alpha^0_{(\mathrm{Nom},P)}$ commutes with the connectives (Boolean and quantifiers).

A decomposition of $\mathcal{MFOL}$. Like with $\mathcal{MPL}$, $\mathcal{S}^0$ is $\mathcal{REL}^1$, and consequently $(\Phi^0, \alpha^0, \beta^0)$ is similar to that in the decomposition of $\mathcal{MPL}$. But on the side of $\widetilde{\mathcal{B}}^C$ the situation is more complicated than in the previous examples. Although we may be tempted to use $\mathcal{AFOL}$ as the base for $\mathcal{MFOL}$ (as we have done above), that would not work because $\mathcal{AFOL}$ does not have enough syntax to specify the sharing of the rigid symbols. So we have to upgrade it essentially by allowing the $\mathcal{MFOL}$ signatures at the level of the base institution too. So, let the institution $\mathcal{AFOLR}$ be defined by

- $Sig^{\mathcal{AFOLR}} = Sig^{\mathcal{MFOL}}$,
- $Sen^{\mathcal{AFOLR}}(S, S_0, F, F_0, P, P_0)$ consists of all (S,F,P)-atoms,
- $Mod^{\mathcal{AFOLR}}(S, S_0, F, F_0, P, P_0) = Mod^{\mathcal{FOL}}(S,F,P)$, and
- the satisfaction relation is inherited from $\mathcal{FOL}$.

Then we let $\mathcal{B} = \mathcal{AFOLR}$ and $Mod^C(S, S_0, F, F_0, P, P_0)$ consist of the $\widetilde{\mathcal{B}}$-models (W,M) such that for each $w, v \in W$ and each x rigid symbol, $M^w_x = M^v_x$.

All these can be extended to a meaningful decomposition of $\mathcal{HFOL}$ by defining $(\Phi^0, \alpha^0, \beta^0)$ like in the example of the decomposition of $\mathcal{HPL}$.

The concept of decomposition has a theoretical potential related to the $\mathcal{S}^0$ component that may generate situations much beyond Kripke semantics in its common acceptations. For instance we may consider $\mathcal{S}^0$ to be an institution of algebras, which will mean algebraic operations on the “worlds” in $\mathcal{S}$ models. To unleash the full potential of this concept in this direction is an interesting topic of further investigation.

Implicit frame and nominal structures via decompositions. In the applications the eventual frame / nominal structures of $\mathcal{S}$ may come from $\mathcal{S}^0$ as follows:

Fact 12.8. *Consider a decomposition of a stratified institution*

$$\mathcal{S}^0 \xleftarrow{(\Phi^0,\alpha^0,\beta^0)} \mathcal{S} \xrightarrow{(\Phi,\alpha,\widetilde{\beta})} \widetilde{\mathcal{B}}^C.$$

Then any frame / nominals extraction of $\mathcal{S}^0$ *induces canonically a frame / nominals extraction of* $\mathcal{S}$ *by composition with* $(\Phi^0,\alpha^0,\beta^0)$.

Note how the decompositions of $\mathcal{MPL}$, $\mathcal{HPL}$, $\mathcal{MFOL}$, $\mathcal{HFOL}$ discussed above fall within the scope of Fact 12.8.

Exercises

12.9. [88] **Preservation of pushouts in a decomposition**
If the decomposition of the stratified institution has the property that

$$Sig^0 \xleftarrow{\Phi^0} Sig^{\mathcal{S}} \xrightarrow{\Phi} Sig^{\mathcal{B}}$$

is a product in $\mathbb{C}at$, then both Φ^0 and Φ preserve pushouts. Moreover any pair of pushout squares of signatures, one from $\mathcal{S}^0$ and the other one from $\mathcal{B}$, determine canonically a pushout square of $\mathcal{S}$ signatures. Is it possible to use this result to explain pushout squares of $\mathcal{MFOL}$ signatures?

12.10. [88] **Model amalgamation via decomposition**
Let $\mathcal{B}$ be any institution. A constraint model sub-functor $Mod^C \subseteq Mod^{\widetilde{\mathcal{B}}}$ *preserves amalgamation* when for any pushout square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

and for any $\widetilde{\mathcal{B}}$ Σ'-model (W,B'), $(W,B')\upharpoonright_{\theta_k} \in |Mod^C \Sigma_k|$, $k = 1,2$, implies $(W,B') \in |Mod^C(\Sigma')|$.

Consider a decomposition of a stratified institution $\mathcal{S}$ such that (1) $\mathcal{S}^0$ is strict, (2) Φ and Φ^0 preserve pushouts, (3) $\mathcal{B}$ and $\mathcal{S}^0$ are semi-exact, and (4) Mod^C preserves amalgamation. Then $\mathcal{S}$ is semi-exact too.

12.4 Modalised institutions

In this section we develop an generic class of examples of stratified institutions by endowing abstract strictly stratified institutions $\mathcal{S}$, referred to as *base stratified institutions*, with Kripke semantics. On the one hand, this means an extension of the syntax of $\mathcal{S}$ with modalities, and on the other hand it means to have Kripke models based on the models of $\mathcal{S}$. This process will produce a new stratified institution that will be denoted $\mathcal{K}(\mathcal{S})$. For example, the stratified institutions $\mathcal{MFOL}$, $\mathcal{MPL}$ may arise as such $\mathcal{K}(\mathcal{S})$ under a suitable choice for the parameters of our construction. The $\mathcal{K}(\mathcal{S})$ construction can be regarded as a bottom-up counterpart of the decompositions of stratified institutions. In $\mathcal{K}(\mathcal{S})$ the stratified institution $\mathcal{S}$ plays a role similar to that played by $\mathcal{B}$ in decompositions.

Let us make the definition of $\mathcal{K}(\mathcal{S})$ explicit by taking three steps: first the syntax, then the semantics, and finally the satisfaction relation.

Modal syntax in stratified institutions

Signatures. The $\mathcal{K}(\mathcal{S})$ signatures are just the $\mathcal{S}$ signatures, i.e. $Sig^{\mathcal{K}(\mathcal{S})} = Sig^{\mathcal{S}}$.

Sentences. For any signature Σ we let $Sen^{\mathcal{K}(\mathcal{S})}\Sigma$ be constructed from $Sen^{\mathcal{S}}\Sigma$ (whose elements are considered atomic sentences at the level of $\mathcal{K}(\mathcal{S})$) by the usual Boolean connectives $\neg$, $\wedge$, etc., by the unary modal connectives $\Diamond$ and $\Box$, and by designated universal and existential quantifications. The latter means that we fix a *quantification system* $\mathcal{D} \subseteq Sig^{\mathcal{S}}$ (recall the concept from Sec. 5.2). The quantification system $\mathcal{D}$ is a parameter of our construction but which, for reasons of simplicity, is omitted from the notation $\mathcal{K}(\mathcal{S})$. Thus, more formally, $Sen^{\mathcal{K}(\mathcal{S})}\Sigma$ is the least set such that

- $Sen^{\mathcal{S}}\Sigma \subseteq Sen^{\mathcal{K}(\mathcal{S})}\Sigma$;
- is closed under the unary connectives $\neg$, $\Diamond$, $\Box$ and under the binary connective $\wedge$; and
- is closed under quantifications $(\forall\chi)\rho'$ and $(\exists\chi)\rho'$ where $\chi : \Sigma \to \Sigma' \in \mathcal{D}$ and $\rho' \in Sen^{\mathcal{K}(\mathcal{S})}\Sigma'$.

Sentence translations. For any signature morphism $\varphi : \Sigma \to \Sigma_1$ the sentence translation $Sen^{\mathcal{K}(\mathcal{S})}\varphi$ extends canonically the sentence translation $Sen^{\mathcal{S}}\varphi$, the only step that requires a bit more explanation being the translation of the quantifiers. For any $\chi : \Sigma \to \Sigma' \in \mathcal{D}$ and any $\rho' \in Sen^{\mathcal{K}(\mathcal{S})}\Sigma'$ we define

$$(Sen^{\mathcal{K}(\mathcal{S})}\varphi)\ (\forall\chi)\rho' \;=\; (\forall\chi(\varphi))(Sen^{\mathcal{K}(\mathcal{S})}\varphi[\chi])\rho'.$$

(Recall the notational convention for the designated pushout squares of the quantification systems

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ \varphi\ } & \Sigma_1 \\ {\scriptstyle \chi}\big\downarrow & & \big\downarrow{\scriptstyle \chi(\varphi)} \\ \Sigma' & \xrightarrow[\ \varphi[\chi]\ ]{} & \Sigma'_1 \end{array}$$

). The functoriality of $Sen^{\mathcal{K}(\mathcal{S})}$ follows now easily from the functoriality of $Sen^{\mathcal{S}}$ and from the axioms of the quantification system $\mathcal{D}$.

$\mathcal{MFOL}$ **syntax as** $\mathcal{K}(\mathcal{S})$ **syntax.** In this example the role of $\mathcal{S}$ is played by $\mathcal{AFOLR}$ considered trivially as a stratified institution, each $[\![M]\!]$ being the same singleton set $\{*\}$. The quantification system extends the quantification system of $\mathcal{FOL}$ presented in Section 5.2 by letting $\mathcal{D}$ to consist of the signature extensions $(S,S_0,F,F_0,P,P_0) \to (S,S_0,F+X,F_0+X,P,P_0)$ with finite blocks X of *rigid* variables. Under this setup we have that $Sen^{\mathcal{MFOL}}$ is just $Sen^{\mathcal{K}(\mathcal{S})}$.

Kripke models in abstract stratified institutions

We define them in two steps: first for the unconstrained case, and then for the constrained case.

Unconstrained models. For any signature Σ a *unconstrainted Kripke model over* $\mathcal{S}$ is a pair (W,M) where

- $W = (|W|, W_\lambda)$ is a Kripke frame like in $\mathcal{MFOL}$ or $\mathcal{MPL}$, i.e. a model in $Mod^{\mathcal{REL}^1}(\lambda : 2)$; and
- M is a function $|W| \to |Mod^{\mathcal{S}}\Sigma|$ such that $[\![M^w]\!]^{\mathcal{S}}_{\Sigma} = [\![M^v]\!]^{\mathcal{S}}_{\Sigma}$ for all $w,v \in |W|$. This condition is essential for having a stratification for (W,M) in a natural and simple way that works.

Note that although we call these Kripke models 'unconstrained' their components still share the same set of 'internal states'. This allows us to abbreviate $[\![M^w]\!]^{\mathcal{S}}_{\Sigma}$ by $[\![M]\!]^{\mathcal{S}}_{\Sigma}$.

Stratification. For the Kripke models of *KMod* we define a stratification by

$$[\![(W,M)]\!]^{\mathcal{K}(\mathcal{S})}_{\Sigma} = |W| \times [\![M]\!]^{\mathcal{S}}_{\Sigma}.$$

Model homomorphisms. A *homomorphism* of unconstrainted Kripke models $h : (W,M) \to (W',M')$ consists of a pair aggregating

- a model homomorphism $h_0 : W \to W'$ in $Mod^{\mathcal{REL}^1}(\lambda : 2)$, i.e. a function $h_0 : |W| \to |W'|$ such that $h_0 W_\lambda \subseteq W'_\lambda$; and
- a $|W|$-indexed family of Σ-model homomorphisms $h_1 = (h_1^w : M^w \to M'^{h_0 w})_{w \in |W|}$ such that $[\![h^w]\!]^{\mathcal{S}}_{\Sigma} = [\![h^v]\!]^{\mathcal{S}}_{\Sigma}$ for all $w,v \in |W|$.

The composition of homomorphisms is defined component-wise by $(h;h')_0 = h_0;h'_0$ and $(h;h')^w_1 = h^w_1;(h'_1)^{h_0 w}$. It is easy to check that unconstrained Kripke models and their homomorphism form a category, denoted $KMod^{\mathcal{S}}\Sigma$.

Model reducts. Given a signature morphism $\varphi\colon\ \Sigma \to \Sigma'$, the corresponding reduct functor $KMod^{\mathcal{S}}\Sigma' \to KMod^{\mathcal{S}}\Sigma$ is defined on the basis of models reducts in $\mathcal{S}$, i.e.

$$(KMod^{\mathcal{S}}\varphi)(W', M') = (W',\ M'; Mod^{\mathcal{S}}\varphi).$$

In order for $(W',\ M'; Mod^{\mathcal{S}}\varphi)$ to qualify as Kripke model we need that $[\![(Mod^{\mathcal{S}}\varphi)M'^{w}]\!]^{\mathcal{S}}_{\Sigma} = [\![(Mod^{\mathcal{S}}\varphi)M'^{v}]\!]^{\mathcal{S}}_{\Sigma}$. This holds by the natural transformation property of $[\![\_]\!]^{\mathcal{S}}$, the strictness assumption, and the respective property for M' as follows:

$$[\![(Mod^{\mathcal{S}}\varphi)M'^{w}]\!]^{\mathcal{S}}_{\Sigma} = [\![M'^{w}]\!]_{\Sigma'} = [\![M'^{v}]\!]_{\Sigma'} = [\![(Mod^{\mathcal{S}}\varphi)M'^{v}]\!]^{\mathcal{S}}_{\Sigma}.$$

Constrained models. When dealing with constrained models, we adopt the same strategy like in the theory of decompositions, namely to stay abstract by employing abstract sub-functors. A *constrained Kripke model functor* is a sub-functor $Mod^{\mathcal{K}(\mathcal{S})} \subseteq KMod$ that satisfies the following condition:

$(CAMG)$ *any designated pushout of the quantification system $\mathcal{D}$ is a model amalgamation square for $Mod^{\mathcal{K}(\mathcal{S})}$.*

Note that due to its abstractness, $Mod^{\mathcal{K}(\mathcal{S})}$ is an implicit parameter of our construction alongside the base stratified institution $\mathcal{S}$, and the quantification system $\mathcal{D}$. The axiom $(CAGM)$ will be necessary for the Satisfaction Condition of $\mathcal{K}(\mathcal{S})$. This kind of reliance on model amalgamation is typical for institutions with quantifications, which we have first met with, in an implicit form, when we proved the Satisfaction Condition of $\mathcal{FOL}$in Chap. 3. But here this has to be treated axiomatically.

Δ-rigid Kripke models. The constraints on the Kripke model can be of various types, this is why it is appropriate to treat them abstractly. For instance, some constraints may come as property of the frames, such as T, $S4$, or $S5$ in $\mathcal{MPL}$ or $\mathcal{MFOL}$, etc. Other constraints can be of a very different nature. As we have already seen, the sharing constraints are very important, they are even crucial in the context of quantifications. These can be defined at the general level by assuming an additional structure, namely a morphism of strict stratified institutions:

$$\Delta = (\Phi^{\Delta}, \emptyset, \beta^{\Delta})\colon\ \mathcal{S} \to \mathcal{S}^{\Delta} = (Sig^{\Delta}, \emptyset, Mod^{\Delta}, [\![\_]\!]^{\Delta}, \emptyset)$$

(the sentences functor of $\mathcal{S}^{\Delta}$ is empty which also implies the emptiness of the satisfaction relation and of the sentence translation). Δ is subject to the following axiom, which causes $(CAGM)$ to hold:

$(\Delta\text{-}AMG)$ Φ^{Δ} *maps any designated pushout square of the quantification system $\mathcal{D}$ to a model amalgamation square.*

We then define $Mod^{\mathcal{K}(\mathcal{S})} \subseteq KMod^{\mathcal{S}}$ as the sub-functor of the Δ-*rigid Kripke models*, which are the Kripke models that satisfies the following sharing conditions:

(SH) *for each Kripke model (W, M) and all $w, w' \in |W|$,* $\beta^{\Delta}_{\Sigma}M^{w} = \beta^{\Delta}_{\Sigma}M^{w'}$; and

(SHH) *for each homomorphism* $h\colon\ (W,M) \to (W',M')$ *of Kripke models and all* $w,w' \in |W|$, $\beta^{\Delta}_{\Sigma} h_1^{w} = \beta^{\Delta}_{\Sigma} h_1^{w'}$.

With these definitions we have that the Δ-rigid models qualify indeed as constrained Kripke models.

Proposition 12.9. *Suppose that any designated pushout of* $\mathcal{D}$ *is a model amalgamation square in* $\mathcal{S}$. *Then the* Δ*-rigid Kripke models functor* $Mod^{\mathcal{K}(\mathcal{S})}$ *satisfies the amalgamation condition* $(CAMG)$.

Proof. Consider a designated pushout square in $\mathcal{D}$ like below:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ \varphi\ } & \Sigma_1 \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi(\varphi)} \\ \Sigma' & \xrightarrow[\ \varphi[\chi]\ ]{} & \Sigma'_1 \end{array}$$

and $(W,M_1) \in |Mod^{\mathcal{K}(\mathcal{S})}\Sigma_1|$, $(W,M') \in |Mod^{\mathcal{K}(\mathcal{S})}\Sigma'|$ such that $(W,M_1)\restriction_{\varphi} = (W,M')\restriction_{\chi}$.

- For each $w \in |W|$ we have $M_1^{w}\restriction_{\varphi} = M'^{w}\restriction_{\chi}$. In $\mathcal{S}$, let M'^{w}_1 be the unique amalgamation of M_1^{w} and M'^{w}.
- First we have to show that $(W,M'_1) \in |Mod^{\mathcal{K}(\mathcal{S})}\Sigma'_1|$, which means that that we have to prove the sharing condition (SH) for (W,M'_1). Let $w,v \in |W|$. Then

1	$(Mod^{\Delta}(\Phi^{\Delta}\chi(\varphi)))(\beta^{\Delta}_{\Sigma'_1} M'^{w}_1) = \beta^{\Delta}_{\Sigma_1}((Mod^{\mathcal{S}}\chi(\varphi))M'^{w}_1)$	naturality of β^{Δ}
2	$(Mod^{\Delta}(\Phi^{\Delta}\chi(\varphi)))(\beta^{\Delta}_{\Sigma'_1} M'^{v}_1) = \beta^{\Delta}_{\Sigma_1}((Mod^{\mathcal{S}}\chi(\varphi))M'^{v}_1)$	naturality of β^{Δ}
3	$(Mod^{\mathcal{S}}\chi(\varphi))M'^{w} = M_1^{w}$	definition of M'^{w}_1
4	$(Mod^{\mathcal{S}}\chi(\varphi))M'^{v} = M_1^{v}$	definition of M'^{v}_1
5	$\beta^{\Delta}_{\Sigma_1} M_1^{w} = \beta^{\Delta}_{\Sigma_1} M_1^{v}$	(W,M_1) satisfies (SH)
6	$(Mod^{\Delta}(\Phi^{\Delta}\chi(\varphi)))(\beta^{\Delta}_{\Sigma'_1} M'^{w}_1) = (Mod^{\Delta}(\Phi^{\Delta}\chi(\varphi)))(\beta^{\Delta}_{\Sigma'_1} M'^{v}_1)$	1, 2, 3, 4, 5
7	$(Mod^{\Delta}(\Phi^{\Delta}\varphi[\chi]))(\beta^{\Delta}_{\Sigma'_1} M'^{w}_1) = (Mod^{\Delta}(\Phi^{\Delta}\varphi[\chi]))(\beta^{\Delta}_{\Sigma'_1} M'^{v}_1)$	like 6
8	$\beta^{\Delta}_{\Sigma'_1} M'^{w}_1 = \beta^{\Delta}_{\Sigma'_1} M'^{v}_1$	6, 7, $(\Delta\text{-}AMG)$.

□

***$\mathcal{MFOL}$* models as Δ-rigid Kripke models.** We continue the presentation of $\mathcal{MFOL}$ as a $\mathcal{K}(\mathcal{S})$. At the syntax part we have already set $\mathcal{S}$ to $\mathcal{AFOLR}$. Now we also set $\mathcal{S}^{\Delta}$ to the reduced variant of $\mathcal{AFOLR}$, without any sentences and regarded trivially as a stratified institution (the stratification consists of a singleton set $\{*\}$). We also let

- Φ^{Δ} maps each $\mathcal{AFOLR}$ signature (S,S_0,F,F_0,P,P_0) to its rigid part (S_0,F_0,P_0);

- each $\beta^{\Delta}_{(S,S_0,F,F_0,P,P_0)}$ is the $\mathcal{FOL}$-model reduct $Mod^{\mathcal{FOL}}(S,F,P) \to Mod^{\mathcal{FOL}}(S_0,F_0,P_0)$; and

Under this setup, clearly the Δ-rigid Kripke models are precisely the $\mathcal{MFOL}$ models. However it remains to establish the model amalgamation condition (Δ-AMG), but this is straightforward by using the semi-exactness of $\mathcal{FOL}$ (see Ex. 12.11).

The $\mathcal{K}(\mathcal{S})$ Satisfaction Relation

Now, we define a stratified satisfaction relation $\models$ in $\mathcal{K}(\mathcal{S})$ between the Kripke models of $Mod^{\mathcal{K}(\mathcal{S})}$ and the $\mathcal{K}(\mathcal{S})$-sentences by induction on the structure of $\mathcal{K}(\mathcal{S})$-sentences as follows. The induction steps of this definition are similar to those from the definition of the satisfaction relation in $\mathcal{MFOL}$, the only difference being that now quantifiers are more abstract that in $\mathcal{MFOL}$. Let Σ be a signature, (W,M) be any Kripke Σ-model, $w^1 \in |W|$, and $w^0 \in [\![M]\!]^{\mathcal{S}}_{\Sigma}$. We define:

- $(W,M) \models^{w^1,w^0} \rho$ iff $M^{w^1} (\models^{\mathcal{S}})^{w^0} \rho$; when $\rho \in Sen^{\mathcal{S}}\Sigma$,
- $(W,M) \models^{w^1,w^0} \rho_1 \wedge \rho_2$ iff $(W,M) \models^{w^1,w^0} \rho_1$ and $(W,M) \models^{w^1,w^0} \rho_2$,
- $(W,M) \models^{w^1,w^0} \neg\rho$ iff $(W,M) \not\models^{w^1,w^0} \rho$,
- $(W,M) \models^{w^1,w^0} \Box\rho$ iff for any $(w^1,v^1) \in W_\lambda$ we have that $(W,M) \models^{v^1,w^0} \rho$,
- $(W,M) \models^{w^1,w^0} \Diamond\rho$ iff there exists $(w^1,v^1) \in W_\lambda$ such that $(W,M) \models^{v^1,w^0} \rho$,
- $(W,M) \models^{w^1,w^0} (\forall\chi)\rho$ iff $(W,M') \models^{w^1,w^0} \rho$ for any (W,M') such that $(Mod^{\mathcal{K}(\mathcal{S})}\chi)(W,M') = (W,M)$,
- $(W,M) \models^{w^1,w^0} (\exists\chi)\rho$ iff $(W,M') \models^{w^1,w^0} \rho$ for some (W,M') such that $(Mod^{\mathcal{K}(\mathcal{S})}\chi)(W,M') = (W,M)$.

Note that with respect to this definition, the eventual constraints on the Kripke models do not really play any role, not even at the level of the quantification because once the constraint model functor $Mod^{\mathcal{K}(\mathcal{S})}$ is established, the semantics of the quantifications is taken care only by the model reducts. However, the sharing constraints have the effect of representing model expansions (W,M') as uniform valuations of the variables. For instance, in $\mathcal{MFOL}$, an expansion (W,M') of (W,M) along a signature extension with a block of rigid variables is the same with a mapping $X \to M^w$, for all w. In the absence of the sharing constraint, a single variable is evaluated differently according to w.

$\mathcal{MFOL}$ satisfaction. Under the setup that explained $\mathcal{MFOL}$ syntax and semantics in terms of $\mathcal{K}(\mathcal{S})$ note that the $\mathcal{MFOL}$ satisfaction relation is that defined for $\mathcal{K}(\mathcal{S})$. Since in this case the stratification of $\mathcal{S}$ consists of the singleton set $\{*\}$, each $[\![(W,M)]\!] = |W| \times [\![M]\!] \cong |W|$, which allows to simplify the notation $\models^{w^1,*}$ to $\models^{w^1}$.

The $\mathcal{K}(\mathcal{S})$ Satisfaction Condition. In order to complete the argument that $\mathcal{K}(\mathcal{S})$ is a stratified institution we still have to establish its Satisfaction Condition.

Theorem 12.10. *For any signature morphism $\varphi\colon\ \Sigma \to \Sigma'$, any $\mathcal{K}(\mathcal{S})$ Kripke Σ'-model (W,M'), any $w^1 \in |W|$, any $w^0 \in [\![M'^{w^1}]\!]^{\mathcal{S}}_{\Sigma'}$, and any $\mathcal{K}(\mathcal{S})$ Σ-sentence ρ,*

$$(Mod^{\mathcal{K}(\mathcal{S})}\varphi)(W,M') \models^{w^1,w^0} \rho \ \textit{ if and only if } \ (W,M') \models^{w^1,w^0} (Sen^{\mathcal{K}(\mathcal{S})}\varphi)\ \rho.$$

Proof. We follow the same routine of induction on the structure of ρ like in most proofs of Satisfaction Conditions, such as that for $\mathcal{FOL}$ in Sec. 3.1. The base step relies on the Satisfaction Condition of the base institution $\mathcal{S}$, while the induction steps corresponding to the Boolean connectives and to the modalities are trivial. Like in the proof of the $\mathcal{FOL}$ Satisfaction Condition, the only interesting step corresponds to the quantifiers. This follows the same general ideas like in the $\mathcal{FOL}$ proof. In particular, the amalgamation axiom (*CAMG*) plays a crucial role, and in fact this is the sole reason of this axiom. □

Corollary 12.11. *$\mathcal{K}(\mathcal{S})$ is a stratified institution that has semantic conjunctions, negations, implications, disjunctions, equivalence, possibility and necessity (under the Kripke frame extraction $Fr(W,M) = W$), and universal and existential $\mathcal{D}$-quantifications.*

Extensions of the modalisation procedure

The modalisation procedure on institutions can extended easily to more sophisticated features from the realm of modal logics. We hint very briefly to how this can be done in a couple of cases, the full details being left to the reader.

Polyadic modalities. This consists of the generalisation that allows modalities to have any arities instead of just being unary connectives like the standard modalities $\Box$ and $\Diamond$. At the level of syntax this means that we fix a family $\Lambda = (\Lambda_n)_{1\leq n}$ of modalities, and for each $\lambda \in \Lambda_n$ we have two $(n-1)$-ary connectives: λ-*possibility* denoted $\langle\lambda\rangle(\rho_1,\dots,\rho_{n-1})$, and λ-*necessity* denoted $[\lambda](\rho_1,\dots,\rho_{n-1})$. At the level of the Kripke models (W,M) we upgrade the Kripke frames W to $\mathcal{FOL}$ models in $Mod^{\mathcal{REL}^1}\Lambda$, meaning that each $\lambda \in \Lambda_n$ gets an interpretation as an n-ary relation $W_\lambda \subseteq |W|^n$. These upgrades reflect in the definition of the $\mathcal{K}(\mathcal{S})$ satisfaction relation as follows:

- $(W,M) \models^{w,v} [\lambda](\rho_1,\dots,\rho_{n-1}) \ = \ \bigwedge_{(w,w_1,\dots,w_{n-1})\in W_\lambda} \bigvee_{1\leq k\leq n-1}((W,M) \models^{w_k,v} \rho_k)$; and
- $(W,M) \models^{w,v} \langle\lambda\rangle(\rho_1,\dots,\rho_{n-1}) \ = \ \bigvee_{(w,w_1,\dots,w_{n-1})\in W_\lambda} \bigwedge_{1\leq k\leq n-1}((W,M) \models^{w_k,v} \rho_k)$.

One may go even further by allowing Λ to vary across signatures. This means that $\mathcal{K}(\mathcal{S})$ signatures now ought to be pairs (Λ,Σ), where Σ is an $\mathcal{S}$ (base institution) signature. There are several technical consequences to this. An important one is that by resorting to the usual institution theoretic approach to quantifiers through signature extensions and model reducts, this generalisation allows for quantifications over modalities. Moreover, this upgrade of $\mathcal{K}(\mathcal{S})$ has general frame extractions and semantic polyadic modalities.

Hybridisations. We may add features of hybrid logics to our modalisation procedure such that $\mathcal{HFOL}$ and $\mathcal{HPL}$ arise as instances of $\mathcal{K}(\mathcal{S})$. At the level of syntax we upgrade signatures to pairs (Nom,Σ) where Nom is a set of nominal constants and Σ is a signature in the base institution $\mathcal{S}$. Then in the building process of the $\mathcal{K}(\mathcal{S})$ sentences we add for each $i\in \mathrm{Nom}$, i itself as an atomic sentence, $@_i$ as an unary connective, and quantifications by nominals $(\exists i)\rho$. The concept of Kripke model (W,M) is upgraded accordingly, by letting the frame W be a $\mathcal{FOL}$ model in $Mod^{\mathcal{RELC}^1}(\mathrm{Nom},\lambda:\ 2)$, meaning that W interprets λ as a binary relation $W_\lambda \subseteq |W|\times|W|$ as in the standard case but also interprets each nominal $i\in\mathrm{Nom}$ as a constant $W_i\in|W|$. In the definition of $\mathcal{K}(\mathcal{S})$ satisfaction we add

- $((W,M)\models^{w,v} i) \;=\; (w=W_i)$ for each $i\in\mathrm{Nom}$; and
- $((W,M)\models^{w,v} @_i\rho) \;=\; ((W,M)\models^{W_i,v}\rho)$; and
- $(W,M)\models^{w,v}_{(\mathrm{Nom},\Sigma)} (\exists i)\rho$ if and only if $(W',M)\models^{w,v}_{(\mathrm{Nom}+i,\Sigma)}\rho$ for some $(\mathrm{Nom}+i)$-expansion W' of W.

Note that any $\mathcal{K}(\mathcal{S})$ that includes the hybrid logic features has semantic i-sentences and satisfaction at i by letting the nominals extraction N and Nm be defined like for $\mathcal{HFOL}$ and $\mathcal{HPL}$. Moreover, by employing the institution theoretic approach to quantifiers through signature extensions and model reducts, the presence of the nominals as entities of signatures allows for quantifications over nominals.

Decompositions of $\mathcal{K}(I)$ (I ordinary institution)

Let I be an ordinary institution, which we consider as a trivially stratified institution by $[\![M]\!]=\{*\}$ for each model M. For any $\mathcal{K}(I)$ we always have a decomposition

$$\mathcal{S}^0 \xleftarrow{(\Phi^0,\alpha^0,\beta^0)} \mathcal{K}(I) \xrightarrow{(\Phi,\alpha,\widetilde{\beta})} \widetilde{I}^C$$

that generalises corresponding concrete decompositions of discussed in Sec. 12.3. For instance if $\mathcal{K}(I)$ is a hybridisation of I then

- $(\Phi^0,\alpha^0,\beta^0)$ is like in the decomposition of $\mathcal{HPL}$,
- $\Phi(\mathrm{Nom},\Sigma)=\Sigma$, $\alpha_{(\mathrm{Nom},\Sigma)}$ is the inclusion $Sen^I\Sigma\subseteq Sen^{\mathcal{K}(I)}\Sigma$, and $\beta_{(\mathrm{Nom},\Sigma)}((W,M),w)=M^w$.

Exercises

12.11. Prove the model amalgamation condition (Δ-AMG) in the context of the presentation of $\mathcal{MFOL}$ models as Δ-rigid models.

12.12. Develop the details of the proof of Thm. 12.10.

12.13. Limits and co-limits of Kripke models
If $Mod^{\mathcal{S}}\Sigma$ has J-(co-)limits and β^Δ_Σ lifts J-(co-)limits then the category of the Δ-rigid Kripke Σ-models has J-(co-)limits.

12.14. Let us consider a modalised institution $\mathcal{K}(\mathcal{S})$ with a Δ-rigid Kripke model functor $Mod^{\mathcal{K}(\mathcal{S})}$. Let $\chi : \Sigma \to \Sigma'$ be a signature morphism such that (1) χ is quasi-representable in $\mathcal{S}$, (2) $Mod^{\Delta}(\Phi^{\Delta}\chi)$ is faithful, and (3) the naturality diagram below is a model amalgamation square

$$\begin{array}{ccc} Mod^{\Delta}(\Phi^{\Delta}\Sigma) & \xleftarrow{\beta^{\Delta}_{\Sigma}} & Mod^{\mathcal{S}}\Sigma \\ {\scriptstyle Mod^{\Delta}(\Phi^{\Delta}\chi)}\uparrow & & \uparrow{\scriptstyle Mod^{\mathcal{S}}(\chi)} \\ Mod^{\Delta}(\Phi^{\Delta}\Sigma') & \xleftarrow[\beta^{\Delta}_{\Sigma'}]{} & Mod^{\mathcal{S}}\Sigma' \end{array}$$

Then χ is quasi-representable in $\mathcal{K}(\mathcal{S})$ too. Consequently, in $\mathcal{MFOL}$ any signature extension with a block of rigid variables is quasi-representable. This gives an important example of quasi-representability that falls beyond representability.

12.15. Consider $\mathcal{MPL}$ in the role of the base institution $\mathcal{S}$. Unfold the details of the modalisation $\mathcal{K}(\mathcal{MPL})$, denoted $\mathcal{M}^2\mathcal{PL}$, where the Kripke models are considered unconstrained.

12.16. [97] Develop a modalisation of the institution $\mathcal{PA}$ of partial algebra that has Δ-rigid Kripke models in which the partial algebras share the interpretations of rigid sorts, rigid total functions, and of the domains of rigid partial functions.

12.17. Extend the definition of semantic possibility / necessity of Sec. 12.2 to the case of polyadic modalities.

12.18. Extend the single sorted variant $\mathcal{OFOL}^1$ of $\mathcal{OFOL}$ with modalities as follows:

- for any relation symbol $\pi \in P_{n+1}$ of arity $n+1$, if $\rho_1, \ldots, \rho_n$ are sentences then $\langle\pi\rangle(\rho_1, \ldots, \rho_n)$ is a sentence too; and
- $M \models^w \langle\pi\rangle(\rho_1, \ldots, \rho_n) \;=\; \bigvee_{(w,w_1,\ldots,w_n)\in(M^X)_\pi} \bigwedge_{1\leq i\leq n}(M \models^{w_i} \rho_i)$.
  (Here M^X denotes the X-power of M in the category of $\mathcal{FOL}$ (F,P)-models.)

Let us denote this extension by $\mathcal{MOFOL}$. Show that $\mathcal{MOFOL}$ can be presented as a modalised institution $\mathcal{K}(\mathcal{S})$ where $Mod^{\mathcal{K}(\mathcal{S})}$ is a constrained Kripke model functor which however cannot be presented as a Δ-rigid one.

12.19. Extend the construction of Ex. 12.15 to hybrid features by letting the base institution $\mathcal{S}$ be $\mathcal{HPL}$ and $\mathcal{K}(\mathcal{HPL})$ be its hybridisation where the Kripke models (W,M) are constrained by the requirement that the M^ws share the same interpretation of the (base) nominals. Let us denote this stratified institution by $\mathcal{H}^2\mathcal{PL}$.

12.5 Ultraproducts in stratified institutions

In Chap. 6 we developed the method of ultraproducts in an institution-independent setting. In this section we extend this to stratified institutions. While some of the developments in this section can be reduced to the ultraproducts in $\mathcal{S}^\sharp$, some of them cannot and have to be done directly in the stratified context.

1. First, we discuss how filtered products of models can be obtained in concrete stratified institutions. There are several ways to do this, depending on the actual context. Moreover, we will see how filtered products in $\mathcal{S}$ give corresponding filtered products in $\mathcal{S}^\sharp$.

This is useful for understanding the method of ultraproducts in stratified institutions in relation to the method of ultraproducts in ordinary institutions.

2. Then we extend the concepts of preservation by filtered products / factors from ordinary to stratified institutions. This allows for a formulation of Łoś ultraproducts theorem in abstract stratified institutions.

3. Concrete versions of Łoś theorem can be obtained by applying iteratively preservation results corresponding to particular connectives, just as we did in Chap. 6 for ordinary institutions. We develop such preservation results for Boolean connectives, quantifications, modalities, and hybrid features, by following their stratified institution semantics of Sec. 12.2.

4. The 'induction' base for obtaining Łoś theorems in concrete stratified institutions cannot be considered in terms of basic sentences like we did for ordinary institution (in Chap. 6). The reason is that the atomic sentences in the common concrete stratified institutions do not enjoy the injectivity property that underlies the concept of basic sentence. Therefore, here we address this problem differently by transferring from bases of stratified institutions.

Filtered products of models in stratified institutions

In principle, filtered products of models in stratified institutions are no different from those of ordinary institutions as they are just categorical filtered products. However, in concrete situations, models of stratified institutions are often significantly more complex structures than those of common ordinary institutions. For example an important class of stratified institution have various kinds of Kripke structures as their models. For such situations it is helpful to have some general support. This may come from two directions:

1. *From the decomposition technique.* We establish filtered products at the level of the components of a decomposition of a stratified institution $\mathcal{S}$ and then aggregate them as filtered products in $\mathcal{S}$. For the aggregation part we can have a general result. Although this method applies well to Kripke semantics-based concrete stratified institutions, its applicability goes beyond that as it does not commit to any specific form of a decomposition.

2. *From the $\mathcal{K}(\mathcal{S})$ construction.* This applies strictly to Kripke semantics based stratified institutions, and it may work in situations when the decomposition technique cannot be applied. The two methods overlap when $\mathcal{K}(\mathcal{S})$ is based on an ordinary institution $\mathcal{S}$ rather than a properly stratified institution, since in the latter case finding a decomposition for $\mathcal{K}(\mathcal{S})$ is problematic. In the $\mathcal{K}(\mathcal{S})$ framework we can have a general but yet quite explicit construction of filtered products in $Mod^{\mathcal{K}(\mathcal{S})}\Sigma$ when we assume sharing constraints through the concept of Δ-rigidity.

Exercises 12.22 and 12.24 are about general results supporting the applicability of these two methods in concrete stratified institutions.

Transfering filtered products from a stratified institution $\mathcal{S}$ to its local institution $\mathcal{S}^\sharp$. In general the existence of filtered products in stratified institutions $\mathcal{S}$ transfer to its local institution $\mathcal{S}^\sharp$, which allows for a reuse of parts of the method of ultraproducts of Chap. 6 to the stratified institution context.

Given a class $\mathcal{F}$ of filters, a stratified institution *has (concrete) $\mathcal{F}$-products* when for each signature Σ, $Mod\Sigma$ has $\mathcal{F}$-products (and $[\![\_]\!]_\Sigma : Mod\Sigma \to \mathbb{S}et$ preserves them). In particular situations it is common for the $\mathcal{F}$-products to be concrete.

The following result gives a representation of filtered products in the local institution $\mathcal{S}^\sharp$ from the filtered products in the stratified institution $\mathcal{S}$.

Proposition 12.12. *Let F be a filter. If a stratified institution $\mathcal{S}$ has concrete F-products, then $\mathcal{S}^\sharp$ has F-products, which for any family of $\mathcal{S}^\sharp$ Σ-models $\{(M_i, w_i) \mid M_i \in |Mod\Sigma|, w_i \in [\![M_i]\!]_\Sigma, i \in I\}$ may be defined by*

$$\{(\mu_J, w_J) : (M_J, w_J) \to (M_F, [\![\mu_I]\!] w_I) \mid J \in F\}, \tag{12.9}$$

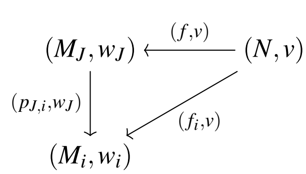
where $\{\mu_J : M_J \to M_F \mid J \in F\}$ is an F-product in $Mod\Sigma$ and w_J is the unique element of $[\![M_J]\!]$ such that for each $i \in J$, $[\![p_{J,i}]\!] w_J = w_i$.

Proof. Let $(M_i)_{i\in I}$ be a family in $|Mod\Sigma|$ and F be a filter over I. We first show that for each $J \in F$,

$$\{(p_{J,i}, w_J) : (M_J, w_J) \to (M_i, w_i) \mid i \in J\} \tag{12.10}$$

is a direct product in $Mod^\sharp\Sigma$. By the definition of w_J, each $(p_{J,i}, w_J)$ is well defined, i.e. $[\![p_{J,i}]\!] w_J = w_i$.

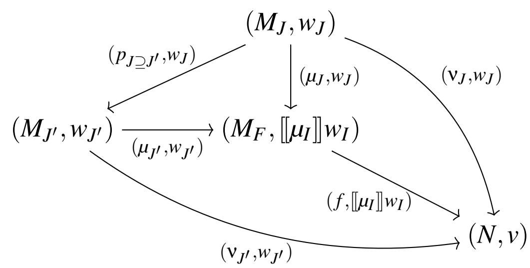
For any family of $\mathcal{S}^\sharp$ Σ-models $\{(f_i, v) : (N, v) \to (M_i, w_i) \mid i \in J\}$, by the universal property of the direct products in $Mod(\Sigma)$ there exists an unique $f : N \to M_J$ such that for each $i \in J$, $f; p_{J,i} = f_i$.



Hence, for each $i \in J$, $[\![p_{J,i}]\!]([\![f]\!] v) = [\![f_i]\!] v = w_i$. Since $[\![p_{J,i}]\!]$ are product projections, it follows that $[\![f]\!] v = w_J$. This completes the proof of the universal property of the direct product (12.10).

It follows immediately that for each $J' \subset J \in F$, $(p_{J \supseteq J'}, w_J) : (M_J, w_J) \to (M_{J'}, w_{J'})$ is a corresponding canonical projection in $Mod^\sharp\Sigma$. Let us show that (12.9) is a co-limit in

$Mod^\sharp\Sigma$.



First, note that each (μ_J, w_J) is well defined, i.e. that $[\![\mu_J]\!]w_J = [\![\mu_I]\!]w_I$, which is given by the following calculation:

$$[\![\mu_I]\!]w_I = [\![p_{I\supset J};\mu_J]\!]w_I = [\![\mu_J]\!]([\![p_{I\supset J}]\!]w_I) = [\![\mu_J]\!]w_J.$$

For establishing the universal property of the co-cone $(\mu_J, w_J)_{J\in F}$ let us consider another co-cone $(\nu_J, w_J)_{J\in F}$ over $(p_{J\supset J'}, w_J)_{J\supset J'\in F}$. Let (N, v) denote it vertex. By the universal property of $(\mu_J)_{J\in F}$ in $Mod\Sigma$ there exists an unique $f:\ M_F \to N$ such that for each $J \in F$, $\mu_J; f = \nu_J$. The argument is completed if we showed that $[\![f]\!]([\![\mu_I]\!]w_I) = v$. This holds by the following calculation:

$$[\![f]\!]([\![\mu_I]\!]w_I) = [\![\mu_I; f]\!]w_I = [\![\nu_I]\!]w_I = v.$$

□

Łoś theorem for stratified institutions

Sentences preserved by $\mathcal{F}$-products / factors. We want to extend preservation concepts of Chap. 6, that constitute the core concept of the method of ultraproducts, to the more general setup of stratified institutions. Let Σ be a signature of a stratified institution $\mathcal{S}$, F be a filter over a set I and $\{\mu_J :\ M_J \to M_F \mid J \in F\}$ be an F-product of a family of $(M_j)_{j\in I}$ of Σ-models. For any Σ sentence ρ we introduce the following notation:

$$A_\mu\rho = \bigcup_{J\in F}[\![\mu_J]\!]\bigcap_{j\in J}[\![p_{J,j}]\!]^{-1}[\![M_j,\rho]\!].$$

Let $\mathcal{F}$ be a class of filters and let ρ be a Σ-sentence. Then ρ

- is *preserved by $\mathcal{F}$-products* when $A_\mu\rho \subseteq [\![M_F, \rho]\!]$, and
- is *preserved by $\mathcal{F}$-factors* when $[\![M_F, \rho]\!] \subseteq A_\mu\rho$,

for each filter $F \in \mathcal{F}$ over a set I and for each F-product $\{\mu_J :\ M_J \to M_F \mid J \in F\}$ of a family $(M_j)_{j\in I}$ of Σ-models.

The preservation by $\mathcal{F}$-products / factors in ordinary institutions as defined in Chap. 6 is an instance of the above definition when $\mathcal{S}$ is trivially stratified (i.e. each $[\![M]\!]$ is a singleton set $\{*\}$). The other way around, the following result shows that under some conditions, stratified preservations by $\mathcal{F}$-products / factors can be explained as ordinary preservations.

Proposition 12.13. *For any stratified institution $\mathcal{S}$ with concrete $\mathcal{F}$-products the following are equivalent for any Σ-sentence ρ:*

1. *ρ is preserved by $\mathcal{F}$-products / factors in $\mathcal{S}$; and*
2. *ρ is preserved by $\mathcal{F}$-products / factors in $\mathcal{S}^\sharp$.*

Proof. In this proof we use the notations of Prop. 12.12. First note that since $\mathcal{S}$ has $\mathcal{F}$-products, by Prop. 12.12 $\mathcal{S}^\sharp$ has $\mathcal{F}$-products too. Moreover, by the assumption of preservation of satisfaction by model isomorphisms, without any loss of generality, we may consider only the F-products given by (12.9) of Prop. 12.12.

When the filtered products are concrete then $([\![M_F]\!], A_\mu\rho)$ represents an F-product of $([\![M_i]\!], [\![M_i,\rho]\!])_{i\in I})$ in the category of the $\mathcal{FOL}$ Σ-models where Σ is a signature with one sort and one unary relation symbol. This means that for each $(w_i \in [\![M_i]\!])_{i\in I}$

$$[\![\mu_I]\!]w_I \in A_\mu\rho \ \text{ if and only if there exists } \ J\in F \ \text{ such that for each } \ j\in J,\ w_j \in [\![M_j,\rho]\!].$$

It follows that ρ is preserved by F-products / factors in $\mathcal{S}^\sharp$ means $A_\mu\rho \subseteq [\![M_F,\rho]\!]$ / $[\![M_F,\rho]\!] \subseteq A_\mu\rho$. □

The conceptual equivalence between stratified preservation in $\mathcal{S}$ and ordinary preservation in $\mathcal{S}^\sharp$ is explained by the fact that the flattening to $\mathcal{S}^\sharp$ is structurally faithful to $\mathcal{S}$. This is not the case when considering the institution $\mathcal{S}^*$ of the global satisfaction. Consequently, we have a much weaker relationship between preservations in $\mathcal{S}$ and in $\mathcal{S}^*$. Nevertheless, this is still useful for establishing the compactness of $\mathcal{S}^*$.

Proposition 12.14. *In any stratified institution $\mathcal{S}$ with concrete F-products, if a sentence ρ is preserved by F-products in $\mathcal{S}$ then it is preserved by F-products in $\mathcal{S}^*$ too.*

Proof. For $\{\mu_J : M_J \to M_F \mid J \in F\}$ an F-product of a family $(M_j)_{j\in I}$ of Σ-models let us assume that $J' = \{j \in I \mid [\![M_j,\rho]\!] = [\![M_j]\!]\} \in F$. It follows that

$$[\![\mu_{J'}]\!] \bigcap_{j\in J'} [\![p_{J,j}]\!]^{-1}[\![M_j,\rho]\!] = [\![\mu_{J'}]\!] \bigcap_{j\in J'} [\![p_{J,j}]\!]^{-1}[\![M_j]\!] = [\![\mu_{J'}]\!][\![M_{J'}]\!] = [\![M_F]\!].$$

The latter two equalities hold because of the surjectivity of each $[\![p_{J,j}]\!]$ and of $[\![\mu_{J'}]\!]$ which are both consequences of the concreteness hypothesis. It follows that $A_\mu\rho = [\![M_F]\!]$. Hence $[\![M_F,\rho]\!] = [\![M_F]\!]$ because the preservation hypothesis means $A_\mu\rho \subseteq [\![M_F,\rho]\!]$. □

Compactness consequences. Arguably, compactness is the most important application of Łoś theorem. In the case of the stratified institutions there is an obvious obstacle to formulate compactness directly, which has to do with the stratified satisfaction relation being ternary. The solution is to go to $\mathcal{S}^\sharp$ and $\mathcal{S}^*$. From these two, the compactness of $\mathcal{S}$ can be appropriately defined as the compactness of $\mathcal{S}^\sharp$. Nevertheless, the compactness of the global institution $\mathcal{S}^*$ is still interesting.

According to Cor. 6.21, any institution in which all its sentences are preserved by ultraproducts is m-compact. Hence from Propositions 12.14 and 12.13 we get the following consequence.

Corollary 12.15 (Compactness of stratified institutions)**.** *Let $\mathcal{S}$ be a stratified institution with concrete ultraproducts such that each of its sentences are preserved by ultraproducts. Then both $\mathcal{S}^\sharp$ and $\mathcal{S}^*$ are m-compact.*

Preservation for Boolean connectives. Now we start to build a replica of the Łoś-style Thm. 6.6 in stratified institutions. Besides preservation results for Boolean connectives and quantifiers (like in Thm. 6.6), we will study also preservations of modalities and nominal related sentences. Unlike the former connectives, the latter two require a conceptual infrastructure that is possible only in stratified institutions.

The preservation results corresponding to Boolean connectives can be transfered directly from ordinary institutions (the corresponding parts of Thm. 6.6) to stratified institutions as a consequence of the transfer of preservation properties given by Prop. 12.13.

Corollary 12.16. *In any stratified institution $\mathcal{S}$ with concrete $\mathcal{F}$-products*

1. *both the sentences preserved by $\mathcal{F}$-products and those preserved by $\mathcal{F}$-factors are closed under conjunctions;*
2. *if ρ is preserved by $\mathcal{F}$-products then any semantic negation $\neg\rho$ of ρ is preserved by $\mathcal{F}$-factors; and*
3. *if ρ is preserved by $\mathcal{F}$-factors and $\mathcal{F}$ contains only ultrafilters then $\neg\rho$ is preserved by $\mathcal{F}$-products.*

Proof. By Fact 12.6, the conjunction and negation coincide in $\mathcal{S}$ and $\mathcal{S}^\sharp$. By Prop. 12.13, preservation by $\mathcal{F}$-products / factors also coincides in $\mathcal{S}$ and $\mathcal{S}^\sharp$. The conclusions for 1., 2., 3. follow because, by Thm. 6.6, the considered preservation properties hold in general in any ordinary institution and in particular in $\mathcal{S}^\sharp$. □

Some of the conclusions of Cor. 12.16 may be obtained under the slightly milder condition that does not require the $\mathcal{F}$-products to be concrete. However this generality is largely meaningless in the applications because the $\mathcal{F}$-products are commonly concrete.

Preservation for quantifiers. The invariance under quantifications of the preservation by F-products / factors parallels the corresponding parts of Thm. 6.6 but is more intricate that in the case of the Boolean connectives. Preservation by F-products requires some technical conditions that follows from the concreteness assumption. In this situation the best solution is to follow the route of Cor. 12.16, which in this case requires a transfer result about model reducts (Cor. 12.17 below). The case of preservation by F-factors is different as it is not related to the concreteness assumption, which means that a presumptive invention of F-products result of the kind of Cor. 12.17 is not needed.

Corollary 12.17. *For any signature morphism χ in any stratified institution $\mathcal{S}$ with concrete F-products, if $\mathrm{Mod}\chi$ preserves F-products in $\mathcal{S}$ then $\mathrm{Mod}^\sharp\chi$ preserves F-products in $\mathcal{S}^\sharp$.*

Proof. In this proof we will skip some of the details as these are rather straightforward. Indeed, we will focus on the important ideas. Let $\chi : \Sigma \to \Sigma'$ be signature morphism such that $Mod\chi$ preserves F-products and let

$$\{(\mu'_J, w_J) : (M'_J, w_J) \to (M'_F, [\![\mu'_I]\!] w_I) \mid J \in F\}$$

be an F-product in $Mod^\sharp\Sigma'$ like in Prop. 12.12. We denote $(Mod\chi)M'_i = M_i$, $(Mod\chi)M'_J = M_J$, $(Mod\chi)M'_F = M_F$, and $(Mod\chi)\mu'_J = \mu_J$. We have to show that

$$\{(\mu_J, [\![M'_J]\!]_\chi w_J) : (M_J, [\![M'_J]\!]_\chi w_J) \to (M_F, [\![M'_F]\!]_\chi([\![\mu'_I]\!] w_I)) \mid J \in F\} \tag{12.11}$$

is an F-product in $Mod^\sharp(\Sigma)$.

- First we establish that for each $J \in F$

  $$\{((Mod\chi)p_{J,i}, [\![M'_J]\!]_\chi w_J) : (M_J, [\![M'_J]\!]_\chi w_J) \to (M_i, [\![M'_i]\!]_\chi w_i) \mid i \in J\} \tag{12.12}$$

  is a direct product. This follows by checking directly the universal property of the direct product by using the fact that $((Mod\chi)p_{J,i})_{i\in J}$ is direct product plus calculations on the internal states also using that $(Mod\chi)$ and $[\![\_]\!]_\Sigma$ preserves direct products.
- Then it follows immediately that $\{((Mod\chi)p_{J\supseteq J'}, [\![M'_J]\!]_\chi w_J) \mid J' \subseteq J \in F\}$ is a diagram of projections.
- Finally, we establish that (12.11) is a co-limit from the co-limit property of μ in a similar way to how we established that (12.12) is a direct product.

□

Proposition 12.18. *If $\mathcal{F}$ is closed under reductions, $Mod\chi$ preserves $\mathcal{F}$-products, and ρ is preserved by $\mathcal{F}$-products, then any semantic existential χ-quantification $(\exists\chi)\rho$ of ρ is preserved by $\mathcal{F}$-products.*

Proof.
- By Prop. 12.13 ρ is preserved by $\mathcal{F}$-products in $\mathcal{S}^\sharp$.
- By Cor. 12.17 it follows that $Mod^\sharp\chi$ preserves $\mathcal{F}$-products.
- From Thm. 6.6 we know that in general, in any (ordinary) institution, from such conditions it follows that $(\exists\chi)\rho$ is preserved by $\mathcal{F}$-products. We apply this conclusion within $\mathcal{S}^\sharp$.
- By Fact 12.6 (existential quantification coincide in $\mathcal{S}$ and in $\mathcal{S}^\sharp$) and by Prop. 12.13 it now follows that $(\exists\chi)\rho$ is preserved by $\mathcal{F}$-products in $\mathcal{S}$.

□

Proposition 12.19. *In any stratified institution $\mathcal{S}$ with $\mathcal{F}$-products, if $\mathcal{F}$ is closed under reductions, $Mod\chi$ invents $\mathcal{F}$-products, and ρ' is preserved by $\mathcal{F}$-factors then any semantic existential χ-quantification $(\exists\chi)\rho'$ is preserved by $\mathcal{F}$-factors.*

Proof. Let $\chi\colon \Sigma \to \Sigma'$ be signature morphism, let $F \in \mathcal{F}$, and let $\{\mu_J\colon M_J \to M_F \mid J \in F\}$ be an F-product of a family $(M_i)_{i\in I}$ of Σ-models. The invention of $\mathcal{F}$-products hypothesis gives for each χ-expansion M' of M_F a $J \in F$, and an $F|_J$-product $\{\mu'_{J'}\colon M'_{J'} \to M' \mid J' \in F|_J\}$ of a family $(M'_j)_{j\in J}$ of χ-expansions of $(M_j)_{j\in J}$ such that $(Mod\chi)p' = p$ and $(Mod\chi)\mu' = \mu$. Under these notations we have that

1	$[\![M'_{J'}]\!]_\chi [\![p'_{J',j}]\!]^{-1}[\![M'_j,\rho']\!] \subseteq [\![p_{J',j}]\!]^{-1}[\![M'_j]\!]_\chi [\![M'_j,\rho']\!]$	$[\![\_]\!]_\chi$ natural
2	$[\![M'_{J'}]\!]_\chi \bigcap_{j\in J'}[\![p'_{J',j}]\!]^{-1}[\![M'_j,\rho']\!] \subseteq \bigcap_{j\in J'}[\![M'_j]\!]_\chi [\![p'_{J',j}]\!]^{-1}[\![M'_j,\rho']\!]$	$f\cap_i X_i \subseteq \cap_i f X_i$
3	$[\![M_j,(\exists\chi)\rho']\!] = \bigcup_{N'\upharpoonright_\chi = M_j}[\![N']\!]_\chi [\![N',\rho']\!]$	definition of semantic existential quantification
4	$[\![M'_j]\!]_\chi [\![M'_j,\rho']\!] \subseteq [\![M_j,(\exists\chi)\rho']\!]$	3
5	$[\![M']\!]_\chi \circ [\![\mu'_{J'}]\!] = [\![\mu_{J'}]\!] \circ [\![M'_{J'}]\!]_\chi$	$[\![\_]\!]_\chi$ natural
6	$[\![M']\!]_\chi A_{\mu'}(\rho') \subseteq A_\mu((\exists\chi)\rho')$	5, 2, 1, 4
7	$[\![M',\rho']\!] \subseteq A_{\mu'}\rho'$	ρ' preserved by F-factors
8	$[\![M_F,(\exists\chi)\rho']\!] = \bigcup_{M'\upharpoonright_\chi = M_F}[\![M']\!]_\chi [\![M',\rho']\!]$	definition of semantic existential quantification
9	$[\![M_F,(\exists\chi)\rho']\!] \subseteq A_\mu((\exists\chi)\rho')$	8, 7, 6.

□

Preservation for modalities. The following result about preservation of semantic possibility is formulated for binary possibility. The sole reason for that is simplicity of presentation, as it holds in the more general form for polyadic possibility, the proof in the general case being essentially the same as for the binary case.

Proposition 12.20. *Let $\mathcal{S}$ be a stratified institution endowed with a binary frame extraction $(L,\emptyset,Fr)\colon Mod^{\mathcal{S}} \Rightarrow Mod^{\mathcal{REL}^1}$. Assume that $\mathcal{S}$ has F-products for a filter F over a set I. Let ρ be a Σ-sentence and $\Diamond\rho$ a semantic possibility of ρ.*

1. *If Fr_Σ preserves direct products and ρ is preserved by F-products then $\Diamond\rho$ is also preserved by F-products.*
2. *If Fr_Σ preserves F-products and ρ is preserved by F-factors then $\Diamond\rho$ is also preserved by F-factors.*

Proof. We consider an F-product $\{\mu_J\colon M_J \to M_F \mid J \in F\}$ for a family $(M_i)_{i\in I}$ of Σ-models.

1.

1	$A_\mu\rho \subseteq [\![M_F,\rho]\!]$	ρ preserved by F-products
2	$[\![\mu_J]\!]\bigcap_{j\in J}[\![p_{J,j}]\!]^{-1}[\![M_j,\rho]\!] \subseteq [\![M_F,\rho]\!]$	1
3	$\bigcap_{j\in J}[\![p_{J,j}]\!]^{-1}(Fr_\Sigma M_j)^{-1}_\lambda B_j = (Fr_\Sigma M_J)^{-1}_\lambda \bigcap_{j\in J}[\![p_{J,j}]\!]^{-1}B_j$	Fr_Σ preserves direct products
4	$[\![\mu_J]\!](Fr_\Sigma M_J)^{-1}_\lambda B \subseteq (Fr_\Sigma M_F)^{-1}_\lambda [\![\mu_J]\!]B$	$[\![\mu_J]\!]\colon (Fr_\Sigma M_J)_\lambda \to (Fr_\Sigma M_F)_\lambda$ homomorphism

5	$[\![M_j, \Diamond\rho]\!] = (Fr_\Sigma M_j)_\lambda^{-1}[\![M_j, \rho]\!]$	definition of semantic possibility
6	$[\![M_F, \Diamond\rho]\!] = (Fr_\Sigma M_F)_\lambda^{-1}[\![M_F, \rho]\!]$	definition of semantic possibility
7	$A_\mu(\Diamond\rho) \subseteq [\![M_F, \Diamond\rho]\!]$	5, 3, 4, 2, 6.

2. Let $(B_J \subseteq [\![M_J]\!])_{J\in F}$ be a family of sets such that for all $J \supseteq J' \in F$ we have that $[\![p_{J\supseteq J'}]\!]B_J \subseteq B_{J'}$. Then

8	$(Fr_\Sigma M_F)_\lambda^{-1} \bigcup_{J\in F}[\![\mu_J]\!]B_J \subseteq \bigcup_{J\in F}[\![\mu_J]\!](Fr_\Sigma M_J)_\lambda^{-1}B_J$	Fr_Σ preserves F-products
9	$(Fr_\Sigma M_F)_\lambda^{-1}(A_\mu\rho) \subseteq A_\mu(\Diamond\rho)$	5, 3, 8 for $B_J = \bigcap_{j\in J}[\![p_{J,j}]\!]^{-1}[\![M_j, \rho]\!]$
10	$[\![M_F, \rho]\!] \subseteq A_\mu\rho$	ρ preserved by F-factors
11	$[\![M_F, \Diamond\rho]\!] \subseteq A_\mu(\Diamond\rho)$	6, 10, 9.

□

Preservation for hybrid features.

Proposition 12.21. *Let $\mathcal{S}$ be a stratified institution endowed with a nominals extraction N, Nm. Assume that $\mathcal{S}$ has F-products for a filter F over a set I. For $i \in N\Sigma$ we let ι be a semantic i-sentence and $@_i\rho$ a semantic satisfaction of a sentence ρ at i.*

1. *If Nm_Σ preserves direct products then ι is preserved by F-products.*
2. *ι is preserved by F-factors.*
3. *If ρ is preserved by F-products then $@_i\rho$ is preserved by F-products too.*
4. *If Nm_Σ preserves F-products and ρ is preserved by F-factors then $@_i\rho$ is preserved by F-factors too.*

Proof. We consider $\{\mu_J : M_J \to M_F \mid J \in F\}$ an F-product a family $(M_j)_{j\in I}$ in $Mod\Sigma$.

1.

1	$\bigcap_{j\in J}[\![p_{J,j}]\!]^{-1}(Nm_\Sigma M_j)_i = (Nm_\Sigma M_J)_i$	Nm_Σ preserves direct products
2	$[\![\mu_J]\!](Nm_\Sigma M_J)_i = (Nm_\Sigma M_F)_i$	$[\![\mu_J]\!] : Nm_\Sigma M_j \to Nm_\Sigma M_F$ homomorphism
3	$[\![\mu_J]\!]\bigcap_{j\in J}[\![p_{J,j}]\!]^{-1}(Nm_\Sigma M_j)_i = (Nm_\Sigma M_F)_i$	1, 2
4	$[\![M_j, \iota]\!] = \{(Nm_\Sigma M_j)_i\}$	definition of semantic nominals
5	$[\![M_F, \iota]\!] = \{(Nm_\Sigma M_F)_i\}$	definition of semantic nominals
6	$A_\mu\iota = [\![M_F, \iota]\!]$	3, 4, 5.

2.

7	$(Nm_\Sigma M_I)_i \in \bigcap_{j\in I}[\![p_{I,j}]\!]^{-1}(Nm_\Sigma M_j)_i$	$[\![p_{I,j}]\!] : Nm_\Sigma M_I \to Nm_\Sigma M_j$ homomorphism
8	$[\![M_F, \iota]\!] \subseteq A_\mu\iota$	7, 2, 4, 5.

3. We may assume that $(Nm_\Sigma M_F)_i \notin [\![M_F, \rho]\!]$ as the other case is trivial. Then

1	$A_\mu\rho \subseteq [\![M_F, \rho]\!]$	ρ preserved by F-factors

2	$(Nm_\Sigma M_F)_i \notin A_\mu \rho$	1, $(Nm_\Sigma M_F)_i \notin [\![M_F, \rho]\!]$
3	$(Nm_\Sigma M_F)_i \notin [\![\mu_J]\!] \bigcap_{j\in J} [\![p_{J,j}]\!]^{-1} [\![M_j, \rho]\!]$	2
4	$[\![\mu_J]\!](Nm_\Sigma M_J)_i = (Nm_\Sigma M_F)_i$	$[\![\mu_J]\!] : Nm_\Sigma M_J \to Nm_\Sigma M_F$ homomorphism
5	$(Nm_\Sigma M_J)_i \notin \bigcap_{j\in J} [\![p_{J,j}]\!]^{-1} [\![M_j, \rho]\!]$	2, 4
6	$\exists j \in J \ (Nm_\Sigma M_j)_i = [\![p_{J,j}]\!](Nm_\Sigma M_j)_i \notin [\![M_j, \rho]\!]$	5
7	$\exists j \in J \ [\![M_j, @_i \rho]\!] = \emptyset$	6, definition of $[\![M_j, @_i \rho]\!]$
8	$[\![\mu_J]\!] \bigcap_{j\in J} [\![p_{J,j}]\!]^{-1} [\![M_j, @_i \rho]\!] = \emptyset$	7
9	$A_\mu(@_i \rho) = \emptyset$	8
10	$A_\mu(@_i \rho) \subseteq [\![M_F, @_i \rho]\!]$	9.

4. We have to prove that $[\![M_F, @_i \rho]\!] \subseteq A_\mu(@_i \rho)$. We may assume $(Nm_\Sigma M_F)_i \in [\![M_F, \rho]\!]$ since the other case leads to a trivial situation. It follows that:

1	$[\![M_F, @_i \rho]\!] = [\![M_F]\!]$	definition and assumption
2	$[\![M_F, \rho]\!] \subseteq A_\mu \rho$	ρ preserved by F-products
3	$(Nm_\Sigma M_F)_i \in A_\mu \rho$	1, 2
4	$\exists J_1 \in F \ (Nm_\Sigma M_F)_i \in [\![\mu_{J_1}]\!] \bigcap_{j\in J_1} [\![p_{J_1,j}]\!]^{-1} [\![M_j, \rho]\!]$	3
5	$\exists J_2 \ (Nm_\Sigma M_F)_i \in [\![\mu_{J_2}]\!](Nm_\Sigma M_{J_2})_i$	Nm_Σ preserves F-product μ
6	$\exists J' \subseteq J_1 \cap J_2 \ (Nm_\Sigma M_{J'})_i \in \bigcap_{j\in J'} [\![p_{J',j}]\!]^{-1} [\![M_j, \rho]\!]$	4, 5, $[\![\mu_J]\!] : Nm_\Sigma M_j \to Nm_\Sigma M_F$ directed co-limit
7	$\forall j \in J' \ (Nm_\Sigma M_j)_i \in [\![M_j, \rho]\!]$	6, $[\![p_{J',j}]\!](Nm_\Sigma M_{J'})_i = (Nm_\Sigma M_F)_i$
8	$\forall j \in J' \ [\![M_j @_i \rho]\!] = [\![M_j]\!]$	7, definition of $[\![M_j @_i \rho]\!]$
9	$\bigcap_{j\in J'} [\![p_{J',j}]\!]^{-1} [\![M_j, @_i \rho]\!] = [\![M_{J'}]\!]$	8
10	$[\![\mu_{J'}]\!] \bigcap_{j\in J'} [\![p_{J',j}]\!]^{-1} [\![M_j, @_i \rho]\!] = [\![M_{J'}]\!]$	9, $[\![\mu_{J'}]\!]$ surjective
11	$[\![M_F @_i \rho]\!] = A_\mu(@_i \rho)$	1, 10.

□

The base case. The preservation results of Cor. 12.16 and of Prop.s 12.19–12.21 may be applied for lifting preservation properties from simpler to more complex sentences. They can be used at the induction step when establishing preservation properties by induction on the structure of the sentences. In the concrete situations the base case of such inductive processes may correspond to atomic (in the sense of uncompounded) sentences, that in the case of ordinary institutions are covered by the abstract concept of basic sentences (see Thm. 6.6). Unfortunately this does not apply in the case of stratified institutions, as in most concrete situations the models of stratified institutions (such as various kinds of Kripke models) do not support properly the concept of basic sentence. Moreover, in the modalised institutions $\mathcal{K}(\mathcal{S})$ the 'atomic' sentences are uncompounded sentences only from the perspective of $\mathcal{K}(\mathcal{S})$, as in $\mathcal{S}$ they may be compounded. A general solution to this problem is to borrow the corresponding preservation property from institutions where it can be established by other general means such Thm. 6.6.

Lemma 12.22. *Let* $(\Phi,\alpha,\beta)\colon\ \mathcal{B}'\to\mathcal{B}$ *be an institution morphism such that each* β_Σ *preserves* F*-products. Then for any* $\Phi\Sigma$*-sentence* ρ *that is preserved by* F*-products / factors, the* Σ*-sentence* $\alpha_\Sigma\rho$ *is preserved by* F*-products / factors too.*

Proof. Let us assume an F-product $\{\mu'_J\colon\ M'_J\to M'_F \mid J\in F\}$ of a family $(M'_i)_{i\in I}$ of Σ-models for a $\mathcal{B}'$-signature Σ. Then

1 $\{\beta_\Sigma\mu'_J \mid J\in F\}$ is F-product — β_Σ preserves F-products

2 for $J\in F$: $\forall i\in J\ (M'_i\models_\Sigma \alpha_\Sigma\rho)=(\beta_\Sigma M'_i\models_{\Phi\Sigma}\rho)$ — Satisfaction Condition of (Φ,α,β)

3 $(M'_F\models_\Sigma \alpha_\Sigma\rho)=(\beta_\Sigma M'_F\models_{\Phi\Sigma}\rho)$ — Satisfaction Condition of (Φ,α,β)

If ρ is preserved by F-products then

4 $(\exists J\in F\ \forall i\in J\ \beta_\Sigma M'_i\models_{\Phi\Sigma}\rho)$ implies $\beta_\Sigma M'_F\models_{\Phi\Sigma}\rho$ — 1

5 $(\exists J\in F\ \forall i\in J\ M'_i\models_\Sigma \alpha_\Sigma\rho)$ implies $M'_F\models_\Sigma\rho$ — 4, 2, 3.

Hence $\alpha_\Sigma\rho$ is preserved by F-products. If ρ is preserved by F-factors then

6 $\beta_\Sigma M'_F\models_{\Phi\Sigma}\rho$ implies $(\exists J\in F\ \forall i\in J\ M'_i\models_{\Phi\Sigma}\rho)$ — 1

7 $M'_F\models\alpha_\Sigma\rho$ implies $(\exists J\in F\ \forall i\in J\ M'_i\models_\Sigma\alpha_\Sigma\rho)$ — 6, 2, 3.

Hence $\alpha_\Sigma\rho$ is preserved by F-factors too. □

The way to apply Lemma 12.22 is for a base $(\Phi,\alpha,\beta)\colon\ \mathcal{S}^\sharp\to\mathcal{B}$ for a stratified institution $\mathcal{S}$. In the context of a decomposition of $\mathcal{S}$ with such a base under mild conditions we have that each β_Σ preserves F-products (see Exercise 12.22). Then by Prop. 12.13 we can further transfer the respective preservation properties to $\mathcal{S}$ itself.

On concrete applications. The collected conditions underlying the preservation results of Cor. 12.16 and of Prop.s 12.19–12.21 amount to the following maximal list:

1. $\mathcal{S}$ has concrete $\mathcal{F}$-products;
2. *Mod*χ preserves $\mathcal{F}$-products;
3. *Mod*χ invents $\mathcal{F}$-products;
4. Fr_Σ, Nm_Σ preserve $\mathcal{F}$-products.

There are three ways to establish them in concrete situations: directly or by using general results at the level of $\mathcal{K}(\mathcal{S})$ or at the level of decomposed stratified institutions. Exercises 12.24, 12.22, 12.28 provide such general results. The direct way is necessary when neither of the latter two methods are applicable.

Exercises

12.20. $\mathcal{SAUT}$ has all filtered products of models, and they are concrete.

12.21. $\mathcal{OFOL}$ has all filtered products of models, and they are concrete.

12.22. Filtered products by decomposition
Consider a decomposition of a stratified institution like in Section 12.3.

1. [85] If $Mod^{\mathcal{B}}\Sigma$ has small products then $Mod^{\widetilde{\mathcal{B}}}\Sigma$ has small products too.
2. [85] If for each signature Σ, $Mod^{\mathcal{B}}\Sigma$ has small products, $Mod^{C}(\Phi\Sigma)$ has small products that are preserved by the sub-category inclusion $Mod^{C}(\Phi\Sigma) \to Mod^{\widetilde{\mathcal{B}}}(\Phi\Sigma)$, and $[\![\_]\!]_{\Phi^0\Sigma}$ creates small products, then $\mathcal{S}$ has concrete small products of models.

12.23. By the decomposition technique, develop preservation / invention of filtered products by $Mod\chi$.

12.24. Filtered products in $\mathcal{K}(\mathcal{S})$
Consider that $Mod^{\mathcal{K}(\mathcal{S})} \subseteq KMod^{\mathcal{S}}$ is the sub-functor of the Δ-rigid models for some Δ. We assume that for each signature Σ the category $Mod^{\Delta}\Sigma$ has small products and directed co-limits of diagrams of projections and β^{Δ}_{Σ} lifts these. Then $\mathcal{K}(\mathcal{S})$ has concrete filtered products of models. Moreover, if a signature morphism χ preserves F-products in $\mathcal{S}$ then it preserves them in $\mathcal{K}(\mathcal{S})$ too.

12.25. Show that in $\mathcal{MFOL}$ we can establish concrete filtered products through either the results of Exercises 12.22 or 12.24.

12.26. Develop all details of the proof of Cor. 12.17.

12.27. Which of the conclusions of Cor. 12.16 can be obtained without the assumption that the $\mathcal{F}$-products are concrete?

12.28. Invention of filtered products in $\mathcal{K}(\mathcal{S})$
Consider a sub-functor $Mod^{\mathcal{K}(\mathcal{S})} \subseteq KMod^{\mathcal{S}}$ of Δ-rigid Kripke models. A signature morphism $\chi : \Sigma \to \Sigma'$ is Δ-*exact* when the square of the naturality of β^{Δ} for χ (as shown below) is a pullback.

$$
\begin{array}{ccccc}
\Sigma & \quad & Mod^{\Delta}(\Phi^{\Delta}\Sigma) & \xleftarrow{\ \beta^{\Delta}_{\Sigma}\ } & Mod\Sigma \\
\chi\downarrow & & \uparrow Mod(\Phi^{\Delta}\chi) & & \uparrow Mod(\chi) \\
\Sigma' & & Mod^{\Delta}(\Phi^{\Delta}\Sigma') & \xleftarrow[\ \beta^{\Delta}_{\Sigma'}\ ]{} & Mod\Sigma'
\end{array}
$$

Then for any class $\mathcal{F}$ of filters closed under reduction, the property that χ invents strongly (and completely) $\mathcal{F}$-products transfers from $\mathcal{S}$ to $\mathcal{K}(\mathcal{S})$. Furthermore, note that Prop. 6.11 gives sufficient quite effective conditions for inventions of $\mathcal{F}$-products applicable at the level of the base institution.

12.29. Preservation in $\mathcal{MFOL}$, $\mathcal{HFOL}$
Show that in $\mathcal{MFOL}$ each atomic sentence is preserved by F-products and by F-factors. Establish that in $\mathcal{MFOL}$ each sentence is preserved by ultraproducts. Extend this to $\mathcal{HFOL}$. Consequently, $\mathcal{MFOL}$ and $\mathcal{HFOL}$ are m-compact and compact.

12.30. Develop a direct proof of Prop. 12.18 with minimal technical assumptions replacing the concreteness assumption.

Notes. The theory of stratified institutions started as an axiomatic institution-theoretic study of modal logics and Kripke semantics. Stratified institutions have been introduced in [99, 5] and their definition slightly upgraded in [82]. The basic modal logic examples together with a form of $\mathcal{OFOL}$ and with the abstract connectives example have been discussed in [5], while in [82] more sophisticated modal logic examples have been introduced. The internal logic of stratified institutions, including institution-independent semantics of modalities and of hybrid features has been developed in [82]. The decomposition technique was introduced in [88] where it had been used for developing results on model amalgamation and on existence of diagrams in stratified institutions and in [85] for developing a study of quasi-varieties in stratified institutions. The method to 'modalise' ordinary institutions had been introduced in [100] and has been later on refined in [170, 81, 97] etc. In this chapter the 'modalisation' has been extended to stratified bases. The extension of the method of ultraproducts to stratified institutions was developed in [82], being essentially based on the earlier development of ultraproducts in modalised institutions from [100].

In [5] a stratified institution-theoretic approach to Tarski's Elementary Chain theorem had been developed for abstract connectives ($\mathcal{CON}$). Other model theory works with stratified institutions include [4, 138].

In [97] the authors had extended van Bentham's translation of modal logic into $\mathcal{FOL}$ [238] in its hybrid variant of [30] to a general encoding of 'hybridised institutions' in $\mathcal{FOL}$. This was later on taken as foundations for the institution-independent specification and verification language *H* [83].

In [86] the *3/2-institutions* of [84] are represented as stratified institutions. The theory of 3/2-institutions is an extension of ordinary institution theory that supports implicit partiality of the signature morphisms and which has been originally motivated by the institution theoretic modelling of the conceptual blending from the works of Fauconnier and Turner [106] on the one hand, and of Goguen [121, 131] on the other hand. This general representation provides a class of examples of non-strict stratification and consequently of proper stratified model amalgamation. In general, it would be interesting to apply the general stratified institution developments to types of examples that have not been explored yet in an general axiomatic manner.

Chapter 13

Many-valued Truth Institutions

So far, the satisfaction relation between models and sentences has been considered to be binary, $M \models \rho$ either holds true or it doesn't. In this chapter we explore a generalisation of ordinary institution theory where $M \models \rho$ is not necessarily binary. We will see how such a generalisation can be achieved and that basic concepts such as semantic consequence, the Galois connection between syntax and semantics, internal logic, but also more advanced concepts such as filtered products, preservation, interpolation, definability, logic translation, etc. do "survive" it but in a subtler form. From a pure theoretical standpoint (there are also more practical motivations that we skip here) this generalisation brings further clarifications to the complex network of causal relationships underlying model theory. This has to do with binary truth being a collapsed form of truth where many things happen somehow "by accident". Much institution-independent model theory may be developed in the many-valued truth fashion, but in this chapter we will only give the reader a taste of what this means. The interested reader may embark himself in the endeavour of further reshaping institution-independent model theory along the lines suggested in this chapter. Now we do the following things.

1. We introduce the many-valued truth generalisation of the concept of institution. Besides the definition, which is quite obvious, and a list of examples, we also study the concept of 'graded semantic consequence' which is the many-valued concept of semantic consequence. Arguably, this is the core concept of the chapter, in the same way semantic consequence is the core concept of binary model theory.
2. Next, we see what many-valued theories mean, which we call by the name of 'fuzzy theories'. A good understanding of this involves the study of closure of fuzzy theories and of concepts of consistency and compactness that are associated to fuzzy theories. Already at the level of the closure operators we have a diversity that is missing in binary truth institutions where the closure determined by the Galois connection between the syntax and the semantics is singular. This diversity justifies the abstract / axiomatic approach to theory closures in many-valued truth institutions.
3. The important issue of the semantics of the Boolean and the quantification connec-

tives is studied both from the consequence and the model theoretic perspectives. We establish conditions on the space of the truth values such that the model-theoretic connectives correspond to the consequence-theoretic ones for the graded semantic consequence. Besides connectives we will also see how the concept of basic sets of sentences gets a proper many-valued truth version.

4. A section is dedicated to the many-valued truth method of ultraproducts. Because of the non-collapsed nature of many-valued truth, the development of results of Łoś theorem kind in this context is more difficult than in the binary case. The scope of the results is also narrower. However, they are still powerful enough for supporting compactness properties.

5. Interpolation is the next and the last topic of mainstream institution-independent model theory that we study in the many-valued truth context. Here we do only definitions, examples, and establish its expected general connections to Beth definability and Robinson consistency, that we are already familiar with from Chapters 9 and 10. Of course, this is much less than the developments in the respective chapters. The main reason for this is that the theory of interpolation for the graded semantic consequence is in its infancy at the moment of writing this second edition of the book. Already the many-valued concept of interpolation is much more subtle than its binary instance, and the same is true for definability and Robinson consistency. Consequently, basic general causality relationships between these properties, in the many-valued context involve a much higher mathematical sophistication than in the binary case. But the developments in this section constitute a solid basis for a more comprehensive general theory of interpolation for the graded semantic consequence.

6. The final section of this chapter is dedicated to translation structures. Like in the binary case, we consider these at two different levels:

   - The ‘internal’ translations, where we define and study general categories of morphisms of fuzzy theories, together with compositionality properties of the kind we studied in Sections 4.2 and 4.3. These are motivated by computing science applications, especially in the area of the aggregation of programming / specification modules.

   - The ‘external’ translations, where we extend the concept of comorphism as the fundamental mathematical structure for doing logic-by-translation, from the binary to many-valued truth institutions. Many-valued truth comorphisms include also the capability to translate between institutions that are based on different spaces of truth values.

Sections 13.1, 13.2, 13.3 and 13.6 require only knowledge of some material from the first part of the book (until Chap. 5 included). Sec. 13.4 is related to Chap. 6 while Sec. 13.5 to Chap. 9, although both sections are relatively self-contained.

13.1 $\mathcal{L}$-institutions

We begin this section with the introduction of order-theoretic structures necessary to structure the spaces of truth values. Then we come up with the definition of $\mathcal{L}$-institutions, which are the many-valued generalisation of the ordinary concept of institution, discuss concrete examples, and introduce the concept of graded semantic consequence. Finally, we prove its fundamental consequence-theoretic properties that parallel those of ordinary / binary semantic consequence.

Order-theoretic structures for many-valued truth

The extension of the concept of institution from binary to many-valued truth is only about truth values. There are several structural levels for the space of truth values. The most primitive level is to consider a plain set of truth values, either in general or in some particular form. At higher levels we may consider various order-theoretic structures. Traditionally, the binary situation is treated as a Boolean algebra in order to support in a classical way the semantics of the common logical connectives such as $\wedge, \vee, \neg$, etc. The many-valued approach treats the structure of truth values rather axiomatically, so we can consider order-theoretic structures of various degrees of complexity. At the end, the most constrained such structure is the binary Boolean algebra.

Lattices. Lattices in various forms play an important role for structuring truth values. Recall that a *completely distributive lattice* is a complete lattice in which arbitrary joins distribute over arbitrary meets. Residuated lattices have already been discussed to a limited extent in Sec. 3.2 in the context of the $\mathcal{MVL}^\sharp$ example. Here, let us recall it once again. A *residuated lattice* $\mathcal{L} = (L, \leq, *, \Rightarrow)$ is a bounded lattice (with $\leq$ denoting the underlying partial order that has infimum (meets) $\wedge$, supremum (joins) $\vee$, greatest 1 and lowest 0 elements) and which comes equipped with an additional commutative and associative binary operation $*$ which has 1 as identity and such that for all elements x, y and z

– $(x * y) \leq (x * z)$ if $y \leq z$, and

– there exists an element $x \Rightarrow z$ such that $y \leq (x \Rightarrow z)$ if and only if $y * x \leq z$.

The operation $*$ is called the *residual conjunction* while $\Rightarrow$ is called *residual implication*. The following relations are known to hold in any residuated lattice:

$$(x \Rightarrow x) \ = \ 1 \tag{13.1}$$

$$x' \leq x \ \text{ implies } \ (x \Rightarrow y) \leq (x' \Rightarrow y). \tag{13.2}$$

$$\textstyle\bigwedge_i x_i \leq \bigwedge_i y_i \text{ when } x_i \leq y_i \text{ for each } i \in I. \tag{13.3}$$

$$\textstyle\bigwedge_i (x \Rightarrow y_i) \ = \ (x \Rightarrow \bigwedge_i y_i) \tag{13.4}$$

$$(x \Rightarrow y) * (y \Rightarrow z) \ \leq \ (x \Rightarrow z). \tag{13.5}$$

$$(\bigvee_i x_i) \Rightarrow y \;=\; \bigwedge_i (x_i \Rightarrow y). \tag{13.6}$$

$$(0 \Rightarrow x) \;=\; 1. \tag{13.7}$$

$$(1 \Rightarrow x) \;=\; x. \tag{13.8}$$

In addition, the following relations hold in any Heyting algebra (which is a residuated lattice with $*$ being $\wedge$):

$$\big((x \wedge y) \Rightarrow z\big) \;=\; \big(x \Rightarrow (y \Rightarrow z)\big). \tag{13.9}$$

$$x \wedge (x \Rightarrow 0) \;=\; 0. \tag{13.10}$$

Furthermore, the following relation holds in any Boolean algebra:

$$x \Rightarrow (y \vee z) \;=\; (x \Rightarrow y) \vee (x \Rightarrow z). \tag{13.11}$$

Examples of residuated lattices. There are some examples of residuated lattices that are really famous in the literature. They are famous because they are invoked and used a lot.

1. One of them is the class of Łukasiewicz lattices. We already presented them in Sec. 3.2 (for the $\mathcal{MVL}^\sharp$ example).
2. Another example is the Goguen / product residuated lattice on the interval $[0,1]$. The residual conjunction $x * y$ is defined as the numerical product $x \cdot y$. Then $(x \Rightarrow y) = 1$ when $x \leq y$ and $(x \Rightarrow y) = y/x$ otherwise.
3. The Gödel residuated lattice on $[0,1]$ is a Heyting algebra, where $(x \Rightarrow y) = 1$ when $x \leq y$ and $(x \Rightarrow y) = y$ otherwise.

Homomorphisms of residuated lattices. Sometimes we may change or translate the space of the truth values. This is achieved through adequate concepts of homomorphisms between the respective structures of truth values. As residuated lattices are the most prominent such structures, we discuss homomorphisms of residuated lattices. The best way to define it is to rely on residuated lattices being varieties of one-sorted (universal) algebras. Indeed, we can express the axioms of residuated lattices as equations (do Ex. 13.2). Then a *homomorphism of residuated lattices* $h\colon \; \mathcal{L} \to \mathcal{L}'$ is just a function $h\colon \; L \to L'$ that preserves the interpretations of $0, 1, \wedge, \vee, *, \Rightarrow$.

Finiteness and compactness. In a partially ordered set $(L, \leq)$ an element x is called *finite* if for every directed subset D of L, if D has a (join) supremum $\bigvee D$ and $x \leq \bigvee D$ then $x \leq d$ for some $d \in D$. If the lattice $(L, \leq)$ is complete, then x is *compact* when $x \leq \bigvee_{j \in J} x_j$ implies that $x \leq x_{j_1} \vee \ldots \vee x_{j_k}$ for some finite subset $\{x_1, \ldots, x_k\} \subseteq J$; when always $k = 1$, x is called *completely join-prime*. $(L, \leq)$ is *compact* when all its elements are compact. Any finite partially ordered set is trivially compact. The totally ordered set $\{0\} \cup \{\frac{1}{n} \mid n \in \omega\}$ is an example of an infinite compact partial order; it is also a complete Heyting algebra.

Continuity. In any complete lattice $\mathcal{L}$ a function $f: L \to L$ is *meet-continuous* when for any non-empty family $(x_i)_{i \in I}$, $f(\bigwedge_i x_i) = \bigwedge_i f x_i$, it is *join-continuous* when $f \bigvee_i x_i = \bigvee_i f x_i$, and it is *continuous* when it is both meet- and join-continuous. Note that any meet- or join-continuous function f is increasing monotonic, i.e. $x \leq y$ implies $fx \leq fy$.

$\mathcal{L}$-institutions: definition and examples

Like stratified institutions, $\mathcal{L}$-institutions also represent a 'non-classical' extension of the ordinary concept of institution. But unlike stratified institutions, $\mathcal{L}$-institutions represent a truly straightforward extension.

The definition. Given a set L, called the *space of the truth values*, an *L-institution*

$$I = \left(Sig^I, Sen^I, Mod^I, \models^I\right)$$

consists of

- a category Sig^I whose objects are called *signatures* and whose arrows are called *signature morphisms*,
- a functor $Sen^I: Sig^I \to \mathbb{S}et$ giving for each signature a set whose elements are called *sentences* over that signature,
- a functor $Mod^I: (Sig^I)^{op} \to \mathbb{C}at$, giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-*(model) homomorphisms*, and
- a family $\models^I$ of L-valued relations, indexed by the class of the signatures, i.e. $\models^I_\Sigma : |Mod^I\Sigma| \times Sen^I\Sigma \to L$ for each $\Sigma \in |Sig^I|$, called the *satisfaction relation*,

such that for each morphism $(\varphi: \Sigma \to \Sigma') \in Sig^I$, the *Satisfaction Condition*

$$M' \models^I_{\Sigma'} (Sen^I\varphi)\rho \quad = \quad (Mod^I\varphi)M' \models^I_\Sigma \rho \tag{13.12}$$

holds for each $M' \in |Mod^I\Sigma'|$ and $\rho \in Sen^I\Sigma$.

That was the most general definition of $\mathcal{L}$-institutions, when $\mathcal{L}$ is just a bare set L. The theory of $\mathcal{L}$-institutions require more structure on $\mathcal{L}$, such as being a partial order, i.e., $\mathcal{L} = (L, \leq)$, or more than that, being a more structured partial order such as a lattice or a residuated lattice. In such situations, an *$\mathcal{L}$-institution* means just an L-institution. With respect to abbreviations of notations, for $\mathcal{L}$-institutions we apply the same conventions as for ordinary institutions.

The Satisfaction Condition says that the *truth degree is an invariant with respect to change of notation*. Note that when presenting the satisfaction relation $\models$ as a natural transformation $Sen \Rightarrow [|Mod(-)| \to L]$, where $[|Mod\Sigma| \to L]$ denotes the set of the functions from $|Mod\Sigma|$ to L (which can also be denoted $L^{|Mod\Sigma|}$) and for any signature morphism $\varphi : \Sigma \to \Sigma'$, $[|Mod\varphi| \to L]f = f \circ |Mod\varphi|$, the Satisfaction Condition (13.12) arises just as the naturality property of $\models$:

$$
\begin{array}{ccc}
\Sigma & \qquad & Sen\Sigma \xrightarrow{\models_\Sigma} [|Mod\Sigma| \to L] \\
\varphi \big\downarrow & & Sen\varphi \big\downarrow \qquad\qquad \big\downarrow [|Mod\varphi| \to L] \\
\Sigma' & & Sen\Sigma' \xrightarrow[\models_{\Sigma'}]{} [|Mod\Sigma'| \to L]
\end{array}
$$

Evidently, the ordinary institutions are just $\mathcal{L}$-institutions for which $\mathcal{L}$ is the binary Boolean algebra. For this reason, in the context of the theory of $\mathcal{L}$-institutions, ordinary institution can be referred to as *binary institutions*.

Like in the case of the binary institutions, the following property, although highly expected both from the general and the concrete applications perspectives, does not follow from the axioms of $\mathcal{L}$-institutions. So, it has to be assumed.

Assumption: The satisfaction is preserved by model isomorphisms, i.e. for each Σ-model isomorphism $h : M \to N$ and each Σ-sentence ρ,

$$(M \models_\Sigma \rho) = (N \models_\Sigma \rho).$$

Semantically equivalent sentences. This concept extends the related concept from binary institution theory as follows. We say that two Σ-sentences ρ and ρ' are *semantically equivalent*, which we also denote as $\rho \models\!\!\!\mid \rho'$, when for each Σ-model M we have that $(M \models \rho) = (M \models \rho')$.

The concept of $\mathcal{L}$-institution may accomodate situations in which the significance of many-valued truth is rather diverse. While some examples may arise as natural many-valued truth generalisations of established binary institutions, in other situations many-valued truth is motivated directly by applications without a previous binary version. The next examples illustrate this situation.

Many-valued first order logic ($\mathcal{MVL}$). We have introduced it as a binary institution in Sec. 3.2 ($\mathcal{MVL}^\sharp$), now we re-introduce it as an $\mathcal{L}$-institution. $\mathcal{MVL}$ shares with $\mathcal{MVL}^\sharp$ the signatures category and the model functor, while the $\mathcal{MVL}$ sentences are precisely the $\mathcal{MVL}^\sharp$ pre-sentences. The many-valued satisfaction relation $\models_\Sigma$ is the function $\models_\Sigma$: $|Mod\Sigma| \times Sen\Sigma \to L$ defined within the $\mathcal{MVL}^\sharp$ example.

Fuzzy equational logic ($\mathcal{FEL}$). This is a ‘fuzzyfication’ of $\mathcal{EQL}$, the result being a logic of similarity. Its main idea is to replace the ordinary crips equality $=$ with a *fuzzy equality* denoted $\approx$. $\mathcal{FEL}$ and $\mathcal{EQL}$ share their syntax, the same signatures and virtually the same sentences (with $=$ replaced by $\approx$ in the case of $\mathcal{FEL}$). The real difference between them occurs at the level of the semantics. Assuming that $\mathcal{L}$ is a complete residuated lattice, a *fuzzy* (S,F)*-algebra* is just an (S,F)-algebra A endowed with a designated fuzzy equality relation $(\approx_A)_s : A_s \times A_s \to L$, for each $s \in S$, such that the following axioms hold:

$$x \approx_A x = 1 \quad \text{fuzzy reflexivity}$$
$$x \approx_A y \leq y \approx_A x \quad \text{fuzzy symmetry}$$
$$x \approx_A y * y \approx_A z \leq x \approx_A z \quad \text{fuzzy transitivity}$$
$$x \approx_A y = 1 \text{ implies } x = y \quad \text{reverse of fuzzy reflexivity.}$$

Moreover the interpretation of the operation symbols preserve the fuzzy equality as follows:

$$(a_1(\approx_A)_{s_1}b_1) * \ldots * (a_n(\approx_A)_{s_n}b_n) \leq A_\sigma(a_1,\ldots,a_n)(\approx_A)_s A_\sigma(b_1,\ldots,b_n)$$

for each $\sigma \in F_{s_1\ldots s_n \to s}$. Then $A \models t \approx t' = A_t \approx_A A_{t'}$ for atomic fuzzy equations and this extends to universally quantified sentences by $A \models (\forall X)t = t' = \bigwedge_{A'\upharpoonright_\chi = A} A'_t \approx_A A'_{t'}$, where χ is the signature extension $(S,F) \subseteq (S,F+X)$.

Temporal logic ($\mathcal{TL}$). We introduce this $\mathcal{L}$-institution in a propositional form, which can be upgraded easily to a first-order version. We fix a complete total order $\mathcal{L} = (L,\leq)$, that models the 'time'.

- Like in $\mathcal{PL}$, the signatures of this $\mathcal{L}$-institution are the sets.
- For any set P, the P-sentences are formed by the grammar

  $$S ::= P \mid S \wedge S \mid \neg S \mid S\,\mathcal{U}\,S.$$

- A P-model M consists of an interpretation $M_\pi \subseteq L$ for each $\pi \in P$. For any function $\varphi : P \to P'$, the φ-reduct $M'\upharpoonright_\varphi$ of a P'-model M' is defined by $(M'\upharpoonright_\varphi)_\pi = M'_{\varphi(\pi)}$.
- For each P-model M, each $w \in L$, and each P-sentences ρ we define $(M \models^w \rho) \in \{0,1\}$ by induction on the structure of ρ as follows:
  - for each $\pi \in P$, $(M \models^w \pi) = 1$ if and only if $w \in M_\pi$;
  - $(M \models^w \rho_1 \wedge \rho_2) = 1$ if and only if $(M \models^w \rho_1) = 1$ and $(M \models^w \rho_2) = 1$;
  - $(M \models^w \neg\rho) = 1 - (M \models^w \rho)$;
  - $(M \models^w \rho_1\,\mathcal{U}\,\rho_2) = 1$ if and only if there exists $w_2 \geq w$ such that $(M \models^{w_2}) = 1$ and for each $w_1 \in [w,w_2)$, $(M \models^{w_1} \rho_1) = 1$.
- For any P-model M and any P-sentence ρ we then define

  $$(M \models_P \rho) = \bigvee\{w \in L \mid \forall w' \leq w,\ M \models^{w'} \rho\}.$$

  The Satisfaction Condition (13.12) follows swiftly from the relation

  $$M' \models^w \varphi\rho = M'\upharpoonright_\varphi \models^w \rho,$$

  which gets a straightforward proof by induction on the structure of ρ.

- Another way to define $(M \models_P \rho)$, that fits the way satisfaction is commonly considered in *linear temporal logic*, is given by the formula

$$(M \models_P \rho) \;=\; \bigwedge\{w \in L \mid M \models^w \rho\}.$$

This represents a kind of inverse degree of satisfaction, that gives the first 'moment' when ρ holds.

Fuzzy multi-algebras ($\mathcal{FMA}$). We fix a residuated lattice $\mathcal{L}$.

- The signatures are triples (S, F, C) where
  - S is a set (of sort symbols),
  - F is an indexed family $\{F_{w\to s} \mid w \in S^*, s \in S\}$ of sets (of operation symbols), and
  - C is an indexed family $\{C_s \mid s \in S\}$ (of deterministic constants).
- Signature morphisms map the three components in a compatible way like in $\mathcal{FOL}$ or $\mathcal{MVL}$. An (S, F, C)*-model* M consists of
  - for each sort $s \in S$, a set M_s,
  - for each operation symbol $\sigma \in F_{w\to s}$, a function $M_\sigma : \; M_w \times M_s \to L$, and
  - for each deterministic constant $c \in C_s$, an element $M_c \in M_s$.
- The (S, F, C)*-sentences* are formed from atoms $t \prec t'$ (with t and t' being $(S, F + C)$-terms of the same sort) by iterative applications of connectives $(\wedge, \vee, \Rightarrow, *)$ and quantifications with blocks of first order variables considered as (new) deterministic constants.[1]
- For defining the satisfaction between models and sentences we first define for each (S, F, C)-model M a *term evaluation* function $M[\_,\_] : \; T_{(S,F+C)} \times M \to L$ by the following recursive formula:

$M[\sigma(t_1, \ldots, t_n), a] =$

$$\begin{cases} 1, & \sigma \in C_s, M_\sigma = a, \\ 0, & \sigma \in C_s, M_\sigma \neq a, \\ \bigvee\{M_\sigma(b_1, \ldots, b_n, a) \wedge \bigwedge_{1\le i\le n} M[t_i, b_i] \mid (b_1, \ldots, b_n) \in M_w\}, & \sigma \in F_{w\to s}. \end{cases}$$

Then $M \models \rho$ is defined by induction on the structure of ρ as follows:
  - $(M \models t \prec t') = \bigwedge\{M[t, a] \Rightarrow M[t', a] \mid a \in M\}$,
  - $(M \models \rho_1 \otimes \rho_2) = (M \models \rho_1) \otimes (M \models \rho_2)$ for $\otimes \in \{\wedge, \vee, *, \Rightarrow\}$,
  - $(M \models (\forall X)\rho) = \bigwedge\{M' \models \rho \mid M'\!\upharpoonright_{(S,F,C)} = M\}$, and
  - $(M \models (\exists X)\rho) = \bigvee\{M' \models \rho \mid M'\!\upharpoonright_{(S,F,C)} = M\}$.

[1] Works such as [159] employ also the deterministic equality $t \doteq t'$ as atomic sentence, however we omit this here since it may be derived from the current syntax.

Flattening $\mathcal{L}$-institutions to binary institutions. In Chap. 12 we introduced two reductions of stratified institutions to ordinary institutions, with the general aim to import as much as possible from ordinary institution theory to stratified institutions. This saved us a significant amount of development effort, reducing it to the aspects that are truly characteristic to stratified institutions. Now we do the same with $\mathcal{L}$-institutions by the general reduction of many-valued truth to binary truth which is advocated by the skeptics of many-valued truth. It works as follows. Given any $\mathcal{L}$-institution $I = (Sig, Sen, Mod, \models)$ we define the binary institution $I^\sharp = (Sig^\sharp, Sen^\sharp, Mod^\sharp, \models^\sharp)$:

- $Sig^\sharp = Sig$, $Mod^\sharp = Mod$;
- $Sen^\sharp\Sigma = Sen\Sigma \times L$;
- $M \models^\sharp_\Sigma (\rho, \kappa)$ if and only if $(M \models_\Sigma \rho) \geq \kappa$.

We have already met with an instance of this flattening in Sec. 3.2 when we introduced many-valued first order logic as the binary institution. $\mathcal{MVL}^\sharp$ is obtained precisely by the general flattening construction on $\mathcal{L}$-institutions applied to $\mathcal{MVL}$. The notation $(\_)^\sharp$ used in Sec. 3.2 anticipated the general construction above.

While the flattening of $\mathcal{L}$-institutions to binary institutions has the advantage of reducing things to a well studied and matured framework and functions well in some aspects, it falls short in several areas that involve some fine grained aspects of multiple truth values. One example of this situation is the concept of graded semantic consequence that we discuss immediately below.

Graded semantic consequence

Given an $\mathcal{L}$-institution there are two ways to extend the satisfaction relation to a semantic consequence relation between sets of sentences and single sentences, both of them generalising the semantic consequence relation of binary institution theory.

1. The *crisp semantic consequence*, defined by $E \models e$ if and only if for each model M, $(M \models E) = 1$ implies $(M \models e) = 1$.
2. Given an $\mathcal{L}$ institution such that $\mathcal{L}$ is a complete meet-semilattice, for each Σ-model M and each set E of Σ-sentences we define

$$(M \models_\Sigma E) = \bigwedge\{M \models_\Sigma \rho \mid \rho \in E\}. \tag{13.13}$$

   Then we define the *graded semantic consequence*, by

$$(E \models_\Sigma e) = \bigwedge\{(M \models_\Sigma E) \Rightarrow (M \models_\Sigma e) \mid M \in |Mod\Sigma|\}.$$

The graded semantic consequence is more subtle and more in the spirit of many-valued truth than the crisp one, though the definition of the latter requires more infrastructure on the space of the truth values, namely that *$\mathcal{L}$ is a complete residuated lattice*. This difference in subtlety may be traced to the fact that while the crisp semantic consequence can be derived from the semantic consequence of the binary flattening $I^\sharp$ of the $\mathcal{L}$-institution I (that $E \models e$ holds in I means $\{(\rho,1) \mid \rho \in E\} \models (e,1)$ in $I^\sharp$), the graded semantic consequence is a concept beyond $I^\sharp$.

One of the important properties of the semantic consequence in binary institution theory is that it satisfies the axioms of entailment systems. The graded semantic consequence enjoys the same properties but in a many-valued form, which is called 'graded entailment'.

Graded entailment. Let $\mathcal{L} = (L, \leq, *)$ such that $(L, \leq)$ is a complete meet-semilattice (with 1 denoting its upper bound) and $*$ is a binary operation on L. By being more abstract, this framework for truth values covers the complete residuated lattices. An *$\mathcal{L}$-entailment system* $(Sig, Sen, \vdash)$ consists of

- a functor $Sen : \ Sig \to \mathbb{S}et$, and
- a family $\vdash \ = \ (\vdash_\Sigma : \ \mathcal{P}Sen\Sigma \times Sen\Sigma \to L)_{\Sigma \in |Sig|}$ such that the following axioms hold:

$$\{\gamma\} \vdash_\Sigma \gamma \ = \ 1 \qquad \textit{reflexivity}$$

$$(E \vdash_\Sigma \gamma) \ \leq \ (E' \vdash_\Sigma \gamma) \text{ when } E \subseteq E' \qquad \textit{monotonicity}$$

$$(E \vdash_\Sigma \Gamma) * (\Gamma \vdash_\Sigma \rho) \ \leq \ (E \vdash_\Sigma \rho) \quad (\text{where } (E \vdash \Gamma) = \textstyle\bigwedge_{\gamma \in \Gamma} (E \vdash \gamma).) \qquad \textit{transitivity}$$

$$(E \vdash_\Sigma \gamma) \ \leq \ (\varphi E \vdash_{\Sigma'} \varphi\gamma) \text{ for any signature morphism } \varphi : \ \Sigma \to \Sigma' \qquad \textit{translation.}$$

Graded entailments may be intuitively interpreted in various ways, as provability degree, as degree of confidence in proofs, or even as a(n inverse) measure for the complexity of a proof. The more complex a proof, the lower its truth value in the lattice $\mathcal{L}$. In other words, the degree of confidence in a proof is decreasing monotonic with respect to the complexity of the respective proof. Other interesting interpretations of graded entailment are possible either from a proof-theoretic or a semantic perspective.

An important technical argument supporting the use of $*$ rather than $\wedge$ in *transitivity* comes from the semantics; in Thm. 13.1 below we will see that in general the many-valued semantic consequence satisfies *transitivity* when formulated using $*$. A version of *transitivity* with $\wedge$ instead of $*$ may be too strong to hold in general. With $*$, it does hold because of the adjunction between $*$ and $\Rightarrow$, while $\Rightarrow$ plays the core role in the definition of the graded semantic consequence.

When we instantiate this definition of graded entailment to binary, we obtain a slightly diferent concept of entailment system than that from Chap. 11 as emerged from the more general concept of proof system. On the one hand, the fuzzy relations $\vdash_\Sigma$ are defined as relations between sets of sentences and single sentences, while in Chap. 11 they were relations between sets of sentences. This suggests that the binary concept is stronger than the graded / many-valued one. One the other hand, the abbreviation $E \vdash \Gamma$ introduced in the axiom *transitivity* allows infinite Γs which not only extends $\vdash$ to fuzzy relations between sets of sentences, but when considered in the binary case allows for infinite unions.

The graded semantic entailment system. Thm. 13.1 below represents the counterpart of the properties stated in Prop. 3.7. But while that result was so straightforward that we did not even bother to include an explicit proof, the proof of Thm. 13.1 has much more substance.

Theorem 13.1 (Semantic entailment)**.** *Let $\mathcal{L}$ be a complete residuated lattice. The semantic consequence of an $\mathcal{L}$-institution is an $\mathcal{L}$-entailment system, called the* semantic entailment system of I.

Proof. We check one by one the axioms of $\mathcal{L}$-entailment systems for $\models$. Let $\varphi : \Sigma \to \Sigma'$ denote a morphism of signatures and M and M' denote variable models in $Mod\Sigma$ and $Mod\Sigma'$, respectively.

– *reflexivity* – For each model M by (13.1) we have that $(M \models \gamma) \Rightarrow (M \models \gamma) = 1$. Hence $(\{\gamma\} \models \gamma) = 1$.

– *monotonicity* – Let $E \subseteq E' \subseteq Sen\Sigma$. For each Σ-model M, because $E \subseteq E'$ from (13.13) it follows that

$$M \models E' \;\leq\; M \models E. \tag{13.14}$$

From (13.14) and (13.2) it follows that

$$(M \models E') \Rightarrow (M \models \gamma) \;\leq\; (M \models E) \Rightarrow (M \models \gamma)$$

hence by (13.3) it follows that $E' \models \gamma \;\leq\; E \models \gamma$.

– *transitivity* – Let $E, \Gamma \subseteq Sen\Sigma$ and $\rho \in Sen\Sigma$. We have that:

1 $(E \models \Gamma) \;=\; \bigwedge_{\gamma\in\Gamma}(E \models \gamma)$ — definition of $E \models \Gamma$

2 $(E \models \gamma) \;=\; \bigwedge_M \big((M \models E) \Rightarrow (M \models \gamma)\big)$ — definition of graded semantic consequence

3 $\bigwedge_{\gamma\in\Gamma} \big((M \models E) \Rightarrow (M \models \gamma)\big) \;=\; \big((M \models E) \Rightarrow \bigwedge_{\gamma\in\Gamma}(M \models \gamma)\big)$ — (13.4)

4 $(E \models \Gamma) \;=\; \bigwedge_M \big((M \models E) \Rightarrow (M \models \Gamma)\big)$ — 1, 2, 3, (13.13)

Then we have that
$(E \models \Gamma) * (\Gamma \models \rho) =$

5 $= \bigwedge_M \big((M \models E) \Rightarrow (M \models \Gamma)\big) * \bigwedge_M \big((M \models \Gamma) \Rightarrow (M \models \rho)\big)$ — 4

6 $\leq \bigwedge_M \big(((M \models E) \Rightarrow (M \models \Gamma)) * ((M \models \Gamma) \Rightarrow (M \models \rho))\big)$ — 5, $*$ monotone

7 $\leq \bigwedge_M \big((M \models E) \Rightarrow (M \models \rho)\big)$ — 7, (13.5)

8 $= (E \models \rho)$ — 7, definition of graded semantic consequence.

– *translation* – Let $E \subseteq Sen\Sigma$, $\gamma \in Sen\Sigma$.

1 $(E \models_\Sigma \gamma) \;=\; \bigwedge_M \big((M \models_\Sigma E) \Rightarrow (M \models_\Sigma \gamma)\big)$ — definition of graded semantic consequence

2 $(\varphi E \models_{\Sigma'} \varphi\gamma) \;=\; \bigwedge_{M'} \big((M' \models_{\Sigma'} \varphi E) \Rightarrow (M' \models_{\Sigma'} \varphi\gamma)\big)$ — def. of graded semantic consequence

3 $(M' \models_{\Sigma'} \varphi\rho) \;=\; (M'{\restriction_\varphi} \models_\Sigma \rho)$ — Satisfaction Condition

4 $(M' \models_{\Sigma'} \varphi E) \;=\; \bigwedge_{e\in E}(M' \models_{\Sigma'} \varphi e)$ — (13.13) (definition of $M' \models_{\Sigma'} \varphi E$)

5 $(M'{\restriction_\varphi} \models_\Sigma E) \;=\; \bigwedge_{e\in E}(M'{\restriction_\varphi} \models_\Sigma e)$ — (13.13) (definition of $M'{\restriction_\varphi} \models_\Sigma E$)

6	$(M' \models_{\Sigma'} \varphi E) \ = \ (M' \restriction_\varphi \models_\Sigma E)$	3, 4, 5
7	$(E \models_\Sigma \gamma) \ \leq \ (\varphi E \models_{\Sigma'} \varphi\gamma)$	1, 2, 3, 6, $(Mod\varphi)\ Mod\Sigma' \subseteq Mod\Sigma$.

□

Then, how about the fact that the proof of *transitivity* relies on the adjunction property between $*$ and $\Rightarrow$ in residuated lattices? This is not apparent from the proof of Thm. 13.1. The explanation is that this is hidden in (the proof of) (13.5).

Exercises

13.1. Prove the relations (13.1) – (13.11).

13.2. Provide an equational axiomatisation for the class of residuated lattices.

13.3. Show that the following mappings are homomorphisms of residuated lattices.

1. $\{0, \frac{1}{2}, 1\} \subseteq \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ is a homomorphism of Łukasiewicz residuated lattices. Provide a generalisation.
2. $h\colon\ [0,1] \to \{0,1\}$ where $h0 = 0$, $hx = 1$ for each $x \in [0,1]$ is a homomorphism between the Goguen's residuated lattice and the binary residuated lattice.
3. $h\colon\ [0,1] \to \{0, \frac{1}{2}, 1\}$ defined by $h0 = 0$, $h1 = 1$, $hx = \frac{1}{2}$ for each $x \in (0,1)$ is a homomorphism between Goguen's residuated lattice and the Heyting algebra on $\{0, \frac{1}{2}, 1\}$.

13.4. Can $\mathcal{TL}$ be presented as a stratified institution?

13.5. Let $\mathcal{L}$ be a residuated lattice. Prove that in any $\mathcal{L}$-entailment system, for any set E of Σ-sentences and any Σ-sentences ρ_1, ρ_2, we have

$$E \cup \{\rho_1\} \vdash_\Sigma \rho_2 \ \leq \ (E \vdash_\Sigma \rho_1) \Rightarrow (E \vdash_\Sigma \rho_2).$$

13.6. [79] **$\mathcal{L}$-institutions determined by $\mathcal{L}$-entailments.**
Let $\mathcal{L}$ be a complete residuated lattice. For any $\mathcal{L}$-entailment system $(Sig, Sen, \vdash)$ there exists an $\mathcal{L}$-institution such that its semantic entailment system coincides with $(Sig, Sen, \vdash)$.

13.2 Fuzzy theories

In this section we address only three aspects of theories: the Galois connection between syntax and semantics, closure of theories, and consistency together with compactness. We will generalise these concepts from binary to many-valued. In the last section of this chapter we will address compositionality properties of theories.

The Galois connection between syntax and semantics

In binary institution theory, a Σ-theory is a set of Σ-sentences. Thus, any theory may be represented by its characteristic function $Sen\Sigma \to 2$, which for each sentence gives a truth value for its membership to the respective theory. This new perspective on theories is the basis for the generalisation of the concept of theory to many-valued truth.

Fuzzy theories. For any fixed set L and for any functor $Sen : Sig \to \mathbb{S}et$, a *fuzzy* Σ-*theory* is just a function $X : Sen\Sigma \to L$. When $\mathcal{L}$ is a complete lattice, for any Σ-theory $X : Sen(\Sigma) \to L$ and for any $E \subseteq Sen(\Sigma)$ we denote

$$X(E) = \bigwedge \{Xe \mid e \in E\}. \tag{13.15}$$

Note that a fuzzy theory in an $\mathcal{L}$-institution I corresponds exactly to an ordinary theory in its binary flattening $I^\sharp$ by representing any function $X : Sen\Sigma \to L$ as the set $\{(\rho, X\rho) \mid \rho \in Sen\Sigma,\ X\rho \neq 0\}$.

The Galois connection. The concept of Galois connection between syntax and semantics introduced in Sec. 4.1 for binary institutions admits the following natural extension to many-valued truth. Let $\mathcal{L}$ be a complete lattice. In any $\mathcal{L}$-institution:

- For any Σ-model M we let the fuzzy theory M^* such that $M^*\rho \ = \ M \models \rho$. For any class of models $\mathcal{M} \subseteq |Mod\Sigma|$ we let $\mathcal{M}^* = \bigwedge_{M \in \mathcal{M}} M^*$.
- For any fuzzy Σ-theory X we let $X^* = \{M \in |Mod\Sigma| \mid X \leq M^*\}$. The elements of X^* are called the *models of* X, or alternatively (Σ, X)-*models*.

Fact 13.2. *For each signature* Σ*, the mappings* $(\_)^*$ *defined above represent a Galois connection between* $(\mathcal{P}|Mod\Sigma|, \supseteq)$ *and* $(L^{Sen\Sigma}, \leq)$.

Closure systems

In Sec. 4.1, for binary institutions we have considered theories that are closed under semantic consequence. Concepts of closures of theories can be regarded as axiomatic treatments of consequence relations. Here we study them in the many-valued truth context.

$\mathcal{L}$-closure systems. Given a partial order $\mathcal{L} = (L, \leq)$, an $\mathcal{L}$-*closure system* is a tuple $(Sig, Sen, \mathcal{C})$ where

– $Sen : Sig \to \mathbb{S}et$ is a functor, and

– $\mathcal{C}$ is a Sig-indexed family of functions $\mathcal{C}_\Sigma : L^{Sen\Sigma} \to L^{Sen\Sigma}$ satisfying the following axioms (for $\varphi : \Sigma \to \Sigma'$ any signature morphism):

$X \leq \mathcal{C}_\Sigma X$ for each X	*C-reflexivity*
$\mathcal{C}_\Sigma X \ \leq \ \mathcal{C}_\Sigma Y$ when $X \leq Y$	*C-monotonicity*
$\mathcal{C}_\Sigma(\mathcal{C}_\Sigma X) \ = \ \mathcal{C}_\Sigma X$	*C-transitivity*
$\mathcal{C}_\Sigma(Sen\varphi\ ;\ X') \ \leq \ Sen\varphi\ ;\ \mathcal{C}_{\Sigma'}(X')$	*C-translation.*

In the binary framework there is a straightforward equivalence between the concepts of entailment system and that of closure system:

$$E \vdash_\Sigma e \ \text{ if and only if } \ e \in \mathcal{C}_\Sigma E.$$

One consequence of this is that to any given entailment system (such as the semantic entailment system) it corresponds only one closure system. But in the many-valued framework, the relationship between the two concepts is much more interesting. Below we will see that fuzzy theories, unlike in the binary case, may admit several different meaningful closures associated to the same graded entailment system. Different closure systems represent different perspectives on the concept of consequence, a multiplicity that, due to its collapsed nature, is missing in the binary truth context.

Goguen closure. Provided some conditions on $\mathcal{L}$ are fulfilled this closure applies to any graded entailment system. Let $\mathcal{L} = (L, \leq, *)$ be a complete lattice with a binary operation $*$ and let $(Sig, Sen\vdash)$ be an $\mathcal{L}$-entailment system. A fuzzy theory $X : Sen\Sigma \to L$ is *Goguen closed* with respect to the entailment system when for each entailment $E \vdash_\Sigma \rho$,

$$X(E) * (E \vdash \rho) \leq X\rho.$$

With the following result we establish a sufficient condition on $\mathcal{L}$ such that the Goguen closure is an $\mathcal{L}$-closure operator. We assume a fixed $\mathcal{L}$-entailment system.

Proposition 13.3. *If $*$ is increasing monotone then the Goguen closed theories are closed under arbitrary meets.*

Proof. Let $(X_i)_{i\in I}$ be any family of Σ-theories. We check the closure condition for $\bigwedge_i X_i$:

$$\begin{aligned}
(\textstyle\bigwedge_i X_i)E \ * \ (E \vdash \rho) &= \big(\textstyle\bigwedge_i X_i(E)\big) * (E \vdash \rho) && (13.15)\\
&\leq \textstyle\bigwedge_i \big(X_i(E) * (E \vdash \rho)\big) && \text{monotone}\\
&\leq \textstyle\bigwedge_i X_i\rho && X_i \text{ weakly closed}\\
&= (\textstyle\bigwedge_i X_i)\rho.
\end{aligned}$$

□

Prop. 13.3 allows the following definition: for any fuzzy theory X let X°, called the *Goguen closure* of X, denote the least Goguen closed fuzzy theory greater than X.

Proposition 13.4. *For any signature morphism $\varphi : \Sigma \to \Sigma'$, if X' is a Goguen closed fuzzy Σ'-theory then $(Sen\varphi);X'$ is a Goguen closed fuzzy Σ-theory.*

Proof. We check the closure condition for $(Sen\varphi);X'$.

$$\begin{aligned}
((Sen\varphi);X')E \ * \ (E \vdash_\Sigma \rho) &= X'(\varphi E) * (E \vdash_\Sigma \rho)\\
&\leq X'(\varphi E) * (\varphi E \vdash_{\Sigma'} \varphi\rho) && \textit{translation}, * \text{ monotone}\\
&\leq X'(\varphi\rho) && X' \text{ is Goguen closed.}
\end{aligned}$$

□

Corollary 13.5. *The Goguen closure $(\_)^\circ$ defines an $\mathcal{L}$-closure system.*

Proof. The first three axioms of $\mathcal{L}$-closure systems follow immediately from the definition of X°. *C-translation* follows from Prop. 13.4 by noting that $(Sen\varphi);X' \leq (Sen\varphi);X'^\circ$.

□

Galois connection closure. As expected, the Galois connection between the syntax and the semantics in $\mathcal{L}$-institutions define an $\mathcal{L}$-closure system on fuzzy theories. However, because it has a semantic nature, the basic framework is now stronger than in the case of the previous closure system.

Proposition 13.6. *In any $\mathcal{L}$-institution, the Galois connection between $(\mathcal{P}|Mod(\Sigma)|, \supseteq)$ and $(L^{Sen\Sigma}, \leq)$ determines an $\mathcal{L}$-closure system $(Sig, Sen, (\_)^{**})$.*

Proof. The first three properties of closure systems follow from the general properties of a Galois connection. Let us now prove *C-translation*. Let $\varphi : \ \Sigma \to \Sigma'$ be a signature morphism and X' be any fuzzy Σ'-theory.

$$
\begin{aligned}
((Sen\varphi);X')^{**} &\leq ((Sen\varphi);(X'^{**}))^{**} && \textit{C-reflexivity}\\
&= ((Mod\varphi)(X'^{*}))^{***} && \text{Satisfaction Condition}\\
&= ((Mod\varphi)(X'^{*}))^{*} && \text{general Galois connection properties}\\
&= (Sen\varphi);(X'^{**}) && \text{Satisfaction Condition.}
\end{aligned}
$$

□

In any $\mathcal{L}$-institution, for any fuzzy Σ-theory X, X^{**} is called the *Galois closure of X*.

Proposition 13.7. *Let $\mathcal{L}$ be a complete residuated lattice. Consider any $\mathcal{L}$-institution and any fuzzy Σ-theory X. Then $X^{\circ} \leq X^{**}$.*

Proof. Since $X \leq X^{**}$, the conclusion followed if we proved that X^{**} is also Goguen closed. This goes as follows:

1 $(E \models \rho) \ = \ \bigwedge_M \big((M \models E) \Rightarrow (M \models \rho)\big)$ — definition of graded semantic consequence

2 $\bigwedge_M \big((M \models E) \Rightarrow (M \models \rho)\big) \ \leq \ \bigwedge_{X \leq M^*} \big((M \models E) \Rightarrow (M \models \rho)\big)$

3 $X^{**}E \ = \ \bigwedge_{X \leq M^*} (M \models E)$ — definition of X^{**}

4 $(M \models E) * ((M \models E) \Rightarrow (M \models \rho)) \ \leq \ (M \models \rho)$ — adjointness of $*$

5 $X^{**}E * (E \models \rho) \ \leq \ \bigwedge_{X \leq M^*} M \models \rho$ — 1, 2, 3, 4, $*$ monotone

6 $X^{**}\rho \ = \ \bigwedge_{X \leq M^*} (M \models \rho)$ — definition of X^{**}

7 $X^{**}E * (E \models \rho) \ \leq \ X^{**}\rho$ — 5, 6.

□

Thus we have presented two meaningful general examples of $\mathcal{L}$-closure systems. Other examples are the subjects of Exercises 13.10 and 13.12. Others are possible both in general and particular situations.

Consistency and Compactness

We have seen previously in the book that model-theoretic consistency is one of the most important property that we consider for theories, and that model-theoretic compactness is about how / when consistency can be extended from finite to infinite. The concepts of consistency and compactness can be generalised from binary truth to many-valued truth in natural ways. And this can also be done in multiple ways.

Consistency. The following is a straightforward generalisation of the concept of consistent theory from binary institution theory to $\mathcal{L}$-institutions. In any $\mathcal{L}$-institution, a fuzzy Σ-theory T is *consistent* when there exists a Σ-model M such that $T \leq M^*$.

Now, we introduce a related concept of consistency that is relative to a fixed truth value. First we prepare some notations.

- For any truth value $\kappa \in L$, let T_κ denote the *constant theory* defined by $T_\kappa \rho = \kappa$ for each sentence ρ.
- For any Σ-theory T and $\Gamma \subseteq Sen\Sigma$ the theory $T|\Gamma$ is defined for each $\rho \in Sen\Sigma$ by

$$(T|\Gamma)\rho = \begin{cases} T\rho, & \rho \in \Gamma \\ 0, & \text{otherwise.} \end{cases}$$

In any $\mathcal{L}$-institution, for any truth value κ, a set E of Σ-sentences is κ-*consistent* when $T_\kappa|E$ is consistent. E is *consistent* when there exists $\kappa > 0$ such that E is κ-consistent, otherwise it is *inconsistent*. Note that κ-consistency can be explained as binary consistency as follows.

Fact 13.8. *In any $\mathcal{L}$-institution I, E is κ-consistent if and only if $(E,\kappa) = \{(e,\kappa) \mid e \in E\}$ is consistent in $I^\sharp$, the binary flattening of I.*

Note also that in the binary case both concepts of consistency discussed above collapse to a single concept.

Compactness, model theoretically. An $\mathcal{L}$-institution I is *m-compact* when its binary flattening $I^\sharp$ is m-compact. This means that for each fuzzy Σ-theory T if $T|\Gamma$ is consistent for each finite $\Gamma \subseteq Sen\Sigma$ then T is consistent too. This concept of compactness involves potentially all truth values. The following concept of compactness refers to an arbitrarily fixed truth value.

Let $\kappa \in L$ be any truth value. Then I is κ-*m-compact* when each set E of Σ-sentences is κ-consistent if E_0 is κ-consistent for each finite $E_0 \subseteq E$.

Whilst in the binary case the two concepts of compactness defined above collapse to the same concept, this is not the case in a proper many-valued context. However we can establish that the former is stronger than the latter.

Proposition 13.9 (κ-m-compactness by m-compactness)**.** *Any m-compact $\mathcal{L}$-institution is κ-m-compact for each truth value κ.*

Proof. Let $\Gamma \subseteq Sen\Sigma$ be finite. We have that

$$(T_\kappa|E)|\Gamma = T_\kappa|(E \cap \Gamma). \tag{13.16}$$

Since Γ is finite it follows that $E \cap \Gamma$ is finite too. Hence, by the hypothesis of the κ-m-compactness implication, $T_\kappa|(E \cap \Gamma)$ is consistent. By (13.16) it follows that $(T_\kappa|E)|\Gamma$ is consistent. By the m-compactness assumption it follows that $T_\kappa|E$ is consistent. □

Compactness, consequence theoretically. We can approach compactness at an even more general level. An $\mathcal{L}$-entailment system $(Sig, Sen, \vdash)$ is *compact* when for any entailment $E \vdash_\Sigma \gamma$ we have

$$E \vdash \gamma = \bigvee \{E_0 \vdash \gamma \mid E_0 \ \text{finite} \subseteq E\}$$

When instantiated to the binary case, this concept of compactness yields the binary consequence-theoretic compactness because the join / supremum of a set is 1 only when the set contains at least one value 1. This argument is almost trivial. The following result is based on a many-valued replica of this argument, and this is more refined then its binary instance.

Let us say that an $\mathcal{L}$-entailment system $(Sig, Sen, \vdash)$ is κ-*compact*, for $\kappa \in L$, when for each entailment $E \vdash \gamma$, if $E \vdash \gamma \geq \kappa$ then there exists $E_0 \subseteq E$ finite such that $E_0 \vdash \gamma \geq \kappa$.

Proposition 13.10. *Any compact $\mathcal{L}$-entailment system $(Sig, Sen, \vdash)$ such that the meet operation $\wedge$ is join-continuous is κ-compact for any finite $\kappa \in L$.*

Proof. Let $\kappa \in L$ be a finite element and $E \vdash \gamma$. We thus have the following:

$$\begin{aligned}
\kappa &= \kappa \wedge (E \vdash \gamma) & \\
&= \kappa \wedge \bigvee \{E_0 \vdash \gamma \mid E_0 \subseteq E \ \text{finite}\} & \text{compactness hypothesis} \\
&= \bigvee \{\kappa \wedge (E_0 \vdash \gamma) \mid E_0 \subseteq E \ \text{finite}\} & \text{continuity of } \wedge.
\end{aligned}$$

The set $\{\kappa \wedge (E_0 \vdash \gamma) \mid E_0 \subseteq E \ \text{finite}\}$ is directed since for any finite $E_0, E_0' \subseteq E$ by *monotonicity* we have that $(E_0 \vdash \gamma), (E_0' \vdash \gamma) \leq (E_0 \cup E_0' \vdash \gamma)$. Hence by the finiteness of κ there exists finite $E_0 \subseteq E$ such that $\kappa = \kappa \wedge (E_0 \vdash \gamma)$, which means $\kappa \leq (E_0 \vdash \gamma)$. □

Exercises

13.7. Can the canonical representation of fuzzy theories as theories in the binary flattening of the respective $\mathcal{L}$-institution be presented as an adjoint functor?

13.8. The following shows that the Goguen closures subsume the classical binary closure in entailment systems. If $\mathcal{L}$ is the binary Boolean algebra and $*$ is the Boolean conjunction, then a theory $X : \ Sen\Sigma \to 2$ is Goguen closed if and only if $\rho \in X^{-1}1$ whenever $X^{-1}1 \vdash \rho$.

13.9. Which of the results of Section 4.1, including also the exercises, is a binary instance of Prop. 13.4?

13.10. Let $\mathcal{L}$ be any complete residuated lattice. Show that for any signature of an $\mathcal{L}$-institution the following mappings define a Galois connection $(L^{|Mod\Sigma|}, \geq) \underset{+}{\overset{+}{\rightleftarrows}} (L^{Sen\Sigma}, \leq)$:

- for any fuzzy class of models $\mathbb{M} : \ |Mod\Sigma| \to L$, any $\rho \in Sen\Sigma$, $\mathbb{M}^+ = \bigwedge_{M \in |Mod\Sigma|} (\mathbb{M}(M) \Rightarrow \ M \models \rho)$,
- for any fuzzy Σ-theory T and any Σ-model M, $\ T^+ M = \bigwedge_{\rho \in Sen\Sigma} (T\rho \Rightarrow \ M \models \rho)$.

Is the closure system $(\_)^{++}$ lower or higher than the Galois closure system $(\_)^{**}$?

13.11. [79] **A representation of Goguen closures.**
If $\mathcal{L}$ be a completely distributive lattice and $*$ is associative and continuous then for each fuzzy theory X, $\ X^\circ \rho = \bigvee_E (X(E) * (E \vdash \rho))$.

13.12. [79] **Strong closures of $\mathcal{L}$-entailment systems.**
Let $\mathcal{L}$ be a complete residuated lattice and let $(Sig, Sen, \vdash)$ be an $\mathcal{L}$-entailment system. Then

1. Show that the following defines an $\mathcal{L}$-closure system:
$$X^{\bullet}\rho = \bigwedge\{E' \vdash \psi\rho \mid \psi : \ \Sigma \to \Sigma',\ E' \subseteq Sen\Sigma',\ X\gamma \leq E' \vdash \psi\gamma \text{ for each } \gamma \in Sen\Sigma'\}.$$
2. If $\mathcal{L}$ is the binary Boolean algebra, the entailment system has infinite unions, X is any Σ-theory and ρ is any Σ-sentence, then $\rho \in X^{\bullet}$ if and only if $X \vdash \rho$.
3. In any $\mathcal{L}$-entailment system, for any fuzzy Σ-theory X we have that $X^{\circ} \leq X^{\bullet}$.
4. In the case of the graded semantic entailment system of an $\mathcal{L}$-institution, which one of $(\_)^{\bullet}$ (as defined in this exercise) or the Galois closure $(\_)^{**}$ is higher?

13.13. [79] **The compact entailment sub-system**
Let $(Sig, Sen, \vdash)$ be a $\mathcal{L}$-entailment system such that $\mathcal{L}$ is a complete lattice with a join-continuous binary operation $*$ and such that the meet operation $\wedge$ is join-continuous too. Then
$$E \vdash^{\omega} \gamma = \bigvee\{E_0 \vdash \gamma \mid E_0 \ \text{ finite} \subseteq E\}$$
defines an $\mathcal{L}$-entailment system over the same sentence functor Sen.

13.14. Explore the relationship between m-compactness and entailment-theoretic compactness.

13.3 Internal logic

In this section we study the meaning of common logical connectives, first for $\mathcal{L}$-entailment systems, and then for $\mathcal{L}$-institutions. Then we will explore when the semantic connectives are expressible as entailment-theoretic connectives. The full development of $\mathcal{L}$-institution theoretic semantics of many-valued logical systems requires also a generalisation of the concept of basic sentence from binary to $\mathcal{L}$-institution theory; we will do this in the final part of this section.

Connectives

Entailment theoretic connectives. In an $\mathcal{L}$-entailment system, a Σ-sentence ρ

- is a *conjunction* of sentences ρ_1 and ρ_2 when for any set of sentences E,
$$E \vdash \rho \ = \ (E \vdash \rho_1) \wedge (E \vdash \rho_2);$$
- is a *residual conjunction* of sentences ρ_1 and ρ_2 when for any set of sentences E,
$$E \vdash \rho \ = \ (E \vdash \rho_1) * (E \vdash \rho_2);$$
- is an *implication* of sentences ρ_1 and ρ_2 when for any set of sentences E,
$$E \vdash \rho \ = \ E \cup \{\rho_1\} \vdash \rho_2;$$
- is a *disjunction* of sentences ρ_1 and ρ_2 when $\mathcal{L}$ has joins and for any set of sentences E,
$$E \vdash \rho \ = \ (E \vdash \rho_1) \vee (E \vdash \rho_2);$$

- is a *negation* of the sentence ρ' when for any sentence e,

$$\{\rho,\rho'\} \vdash e \;=\; 1;$$

- is a *universal* χ*-quantification* of a Σ'-sentence ρ' for $\chi: \;\Sigma \to \Sigma'$ signature morphism when for any set of Σ-sentences E

$$E \vdash_\Sigma \rho \;=\; \chi E \vdash_{\Sigma'} \rho';$$

- is an *existential* χ*-quantification* of a Σ'-sentence ρ' for $\chi: \;\Sigma \to \Sigma'$ signature morphism when for any Σ-sentence e

$$\rho \vdash_\Sigma e \;=\; \rho' \vdash_{\Sigma'} \chi e.$$

These definitions can be extended at the level of the whole $\mathcal{L}$-entailment system. For instance we say that the $\mathcal{L}$-entailment system *has conjunctions* when *any* two Σ-sentences have a conjunction. And similarly for the other connectives.

If we read the connectives defined for proof systems (in Sec. 11.4) in the simplified context of (binary) entailment systems then those definitions look identical to the corresponding ones from Sec. 11.4. The difference resides only in the interpretations of the respective entailments, as a binary or as a proper many-valued relation. In binary logic the inequalities that are implicit in the equation defining the entailment theoretic implication are known as *Modus Ponens* ($\leq$) and the *Deduction Theorem* ($\geq$). We may extend this terminology to graded entailments.

Like in the binary situation, we can consider the *least entailment system* that contains a given entailment system and that has some of the connectives defined above. This is supported by the following fact.

Fact 13.11. *Any intersection of entailment systems (that share the same sentence functor) is an entailment system. Moreover, the property of having a certain connective is invariant with respect with such intersections.*

Model-theoretic connectives. The many-valued semantic connectives mimic those defined for binary institutions in Sec. 5.1 and 5.2, but now their interpretation is in a many-valued truth context. A Σ-sentence ρ is an $\mathcal{L}$-institution

- is a *conjunction* of sentences ρ_1 and ρ_2 when $\mathcal{L}$ has meets and for each Σ-model M,

$$(M \models \rho) \;=\; (M \models \rho_1) \wedge (M \models \rho_2);$$

- is a *residual conjunction* of sentences ρ_1 and ρ_2 when $\mathcal{L}$ is a residuated lattice and for each Σ-model M,

$$(M \models \rho) \;=\; (M \models \rho_1) * (M \models \rho_2);$$

- is an *implication* of sentences ρ_1 and ρ_2 when $\mathcal{L}$ is a residuated lattice and for each Σ-model M,

$$(M \models \rho) \;=\; (M \models \rho_1) \Rightarrow (M \models \rho_2);$$

- is a *disjunction* of sentences ρ_1 and ρ_2 when $\mathcal{L}$ has joins and for each Σ-model M,

$$(M \models \rho) \;=\; (M \models \rho_1) \vee (M \models \rho_2);$$

- is a *negation* of a sentence ρ' when $\mathcal{L}$ is a residuated lattice for each Σ-model M,

$$(M \models \rho') \;=\; (M \models \rho) \Rightarrow 0;$$

- is a *universal* χ*-quantification* of a Σ'-sentence ρ' for $\chi : \Sigma \to \Sigma'$ signature morphism when $\mathcal{L}$ is a complete lattice and for each Σ-model M

$$(M \models_\Sigma \rho) = \bigwedge\{M' \models_{\Sigma'} \rho' \mid (Mod\chi)M' = M\};$$

- is an *existential* χ*-quantification* of a Σ'-sentence ρ' for $\chi : \Sigma \to \Sigma'$ signature morphism when $\mathcal{L}$ is a complete lattice and for each Σ-model M

$$(M \models_\Sigma \rho) \;=\; \bigvee\{M' \models_{\Sigma'} \rho' \mid (Mod\chi)M' = M\}.$$

These definitions can be extended at the level of the whole $\mathcal{L}$-institution. For instance we say that the $\mathcal{L}$-institution *has conjunctions* when any two Σ-sentences have a conjunction, etc.

Model-theoretic versus entailment-theoretic connectives. Given an $\mathcal{L}$-institution I, when $\mathcal{L}$ is a complete residuated lattice we thus have two different definitions for each connective, one in terms of satisfaction by models and another one in terms of the semantic $\mathcal{L}$-entailment system of I. It is important to establish the relationship between these two in order to be able to have an entailment-based calculus for the semantic consequence.

Proposition 13.12. *Consider the semantic $\mathcal{L}$-entailment system of an $\mathcal{L}$-institution such that $\mathcal{L}$ is a complete residuated lattice. Let ρ be a Σ-sentence and $\varphi : \Sigma \to \Sigma'$ be a signature morphism. Then*

1. *ρ is the entailment-theoretic conjunction of ρ_1 and ρ_2 if it is the model-theoretic conjunction of ρ_1 and ρ_2.*
2. *ρ is the entailment-theoretic universal / existential χ-quantification of ρ' if it is its model-theoretic universal / existential χ-quantification.*

Let us further assume that $\mathcal{L}$ is a Heyting algebra. Then

3. *ρ is the entailment-theoretic implication of ρ_1 and ρ_2 if it is the model-theoretic implication of ρ_1 and ρ_2.*
4. *ρ is the entailment-theoretic negation of ρ' if it is its model-theoretic negation.*

Let us further assume that $\mathcal{L}$ is a completely distributive Boolean algebra. Then

5. *ρ is the entailment-theoretic disjunction of ρ_1 and ρ_2 if it is the model-theoretic disjunction of ρ_1 and ρ_2.*

Proof. 1. We calculate as follows:

$$\begin{aligned} E \models \rho &= \bigwedge_M \big((M \models E) \Rightarrow (M \models \rho)\big) && \text{definition} \\ &= \bigwedge_M \big((M \models E) \Rightarrow (M \models \rho_1) \wedge (M \models \rho_2)\big) && \text{hypothesis} \\ &= \bigwedge_M \big(((M \models E) \Rightarrow (M \models \rho_1)) \wedge ((M \models E) \Rightarrow (M \models \rho_2))\big) && 13.4 \\ &= \bigwedge_M \big(((M \models E) \Rightarrow (M \models \rho_1))\big) \wedge \bigwedge_M \big(((M \models E) \Rightarrow (M \models \rho_2))\big) && \\ &= (E \models \rho_1) \wedge (E \models \rho_2) && \text{definition.} \end{aligned}$$

2. – *universal quantification* – On the one hand we have

1 $E \models_\Sigma \rho = \bigwedge_M (M \models E \Rightarrow M \models \rho)$ — definition

2 $\quad = \bigwedge_M (M \models E \Rightarrow \bigwedge_{Mod(\chi)N'=M} (N' \models \rho'))$ — 1, definition

3 $\quad = \bigwedge_M \bigwedge_{Mod(\chi)N'=M} (M \models E \Rightarrow N' \models \rho')$ — 2, (13.4).

On the other hand we have

4 $\chi E \models_{\Sigma'} \rho' = \bigwedge_{M'} (M' \models \chi E \;\Rightarrow\; M' \models \rho')$ — definition

5 $\quad = \bigwedge_{M'} ((Mod\chi)M' \models E \;\Rightarrow\; M' \models \rho')$ — 4, Satisfaction Condition.

It follows that

$$\begin{aligned} &E \models_\Sigma \rho \;\leq\; \chi E \models_{\Sigma'} \rho' && 3, 5, \text{ consider } M = (Mod\chi)M' \\ &\chi E \models_{\Sigma'} \rho' \;\leq\; E \models_\Sigma \rho && 3, 5, \text{ consider } M' = N'. \end{aligned}$$

– *existential quantification* – On the one hand we have

1 $\rho \models_\Sigma e = \bigwedge_M (M \models \rho \;\Rightarrow\; M \models e)$ — definition

2 $\quad = \bigwedge_M (\bigvee_{(Mod\chi)N'=M} (N' \models \rho') \;\Rightarrow\; M \models e)$ — 1, definition

3 $\quad = \bigwedge_M \bigwedge_{(Mod\chi)N'=M} (N' \models \rho' \;\Rightarrow\; M \models e)$ — 2, (13.6).

On the other hand we have

4 $\rho' \models_{\Sigma'} \chi e = \bigwedge_{M'} (M' \models \rho' \;\Rightarrow\; M' \models \chi e)$ — definition

5 $\quad = \bigwedge_{M'} (M' \models \rho' \;\Rightarrow\; (Mod\chi)M' \models e)$ — 4, Satisfaction Condition.

Then from 3 and 5, by considering $M' = N'$, we obtain

$$\rho \models_\Sigma e \;=\; \rho' \models_{\Sigma'} \chi e.$$

3. For any set E of sentences:

$$\begin{aligned}
E \models \rho' &= \textstyle\bigwedge_M \big((M \models E) \Rightarrow (M \models \rho')\big) && \text{definition}\\
&= \textstyle\bigwedge_M \big((M \models E) \Rightarrow ((M \models \rho_1) \Rightarrow (M \models \rho_2))\big) && \text{hypothesis}\\
&= \textstyle\bigwedge_M \big(((M \models E) \wedge (M \models \rho_1)) \Rightarrow (M \models \rho_2)\big) && (13.9)\\
&= \textstyle\bigwedge_M \big((M \models E \cup \{\rho_1\}) \Rightarrow (M \models \rho_2)\big) && (13.13)\\
&= E \cup \{\rho_1\} \models \rho_2 && \text{definition.}
\end{aligned}$$

4. For any sentence e:

$$\begin{aligned}
\{\rho, \rho'\} \models e &= \textstyle\bigwedge_M \big(M \models \{\rho, \rho'\} \Rightarrow M \models e\big) && \text{definition}\\
&= \textstyle\bigwedge_M \big(((M \models \rho) \wedge (M \models \rho')) \Rightarrow M \models e\big) && (13.13)\\
&= \textstyle\bigwedge_M \big(((M \models \rho) \wedge ((M \models \rho) \Rightarrow 0)) \Rightarrow M \models e\big) && \text{hypothesis}\\
&= \textstyle\bigwedge_M \big(0 \Rightarrow M \models e\big) && (13.10)\\
&= 1 && (13.7).
\end{aligned}$$

5. For any set of sentences E:

$$\begin{aligned}
E \models \rho' &= \textstyle\bigwedge_M \big((M \models E) \Rightarrow (M \models \rho')\big) && \text{definition}\\
&= \textstyle\bigwedge_M \big((M \models E) \Rightarrow ((M \models \rho_1) \vee (M \models \rho_2))\big) && \text{hypothesis}\\
&= \textstyle\bigwedge_M \big((M \models E \Rightarrow M \models \rho_1) \vee (M \models E \Rightarrow M \models \rho_2)\big) && (13.11)\\
&= \textstyle\bigwedge_M \big(M \models E \Rightarrow M \models \rho_1\big) \vee \textstyle\bigwedge_M \big(M \models E \Rightarrow M \models \rho_2\big) && \text{distributivity}\\
&= (E \models \rho_1) \vee (E \models \rho_2) && \text{definition.}
\end{aligned}$$

□

The model-theoretic connectives represent yet another situation when the binary flattening diverges from the respective $\mathcal{L}$-institution. In general it is not possible to establish a general causality relationship between the model-theoretic connectives in the $\mathcal{L}$-institution and in its binary flattening.

Basic sentences

The extension of the concept of basic sentence of Sec. 5.5 to $\mathcal{L}$-institutions constitutes the last bit in the definition of internal logic for many-valued truth institutions. Let $\kappa \in L$. A set E of Σ-sentences is κ-*basic* if and only if there exists a Σ-model $M_{E,\kappa}$ such that for each Σ-model M

$$(M \models E) \geq \kappa \ \text{ if and only if there exists a homomorphism } \ h\colon\ M_{E,\kappa} \to M.$$

Let $\mathcal{F}$ be a family of filters. When $M_{E,\kappa}$ is ($\mathcal{F}$-)finitely presented in the category of the Σ-models then we say that E is *($\mathcal{F}$-)finitary* κ-*basic*. Moreover, E is *(($\mathcal{F}$-)finitary) basic* when it is (($\mathcal{F}$-)finitary) κ-basic for each $\kappa \in L$. When $E = \{e\}$ is a singleton set, we say that e is a *(($\mathcal{F}$-)finitary)* κ-*basic sentence*.

Fact 13.13. *Let I be an $\mathcal{L}$-institution. Then a set E of Σ-sentences is κ-basic if and only if the set $(E,\kappa) = \{(e,\kappa) \mid e \in E\}$ is basic in $I^\sharp$.*

But the following concept is genuinely a many-valued theoretic one as it cannot be reduced to a concept from binary institution theory. A sentence is *($\mathcal{F}$-)finitary basic* when it is ($\mathcal{F}$-)finitary κ-basic for each truth value κ. Like with basic sentences in binary institution theory, in concrete $\mathcal{L}$-institutions sets of atoms tend to be basic. These $\mathcal{L}$-institutions include $\mathcal{MVL}$, $\mathcal{TL}$ but not $\mathcal{FMA}$.

Beyond atoms. Like in the case of binary institutions, in concrete $\mathcal{L}$-institutions the concept of basic sentence can be significantly wider than just sets of atoms. In general, we can extend Fact 5.22 to $\mathcal{L}$-institution as follows.

Proposition 13.14. *In an $\mathcal{L}$-institution let $\kappa \in \mathcal{L}$ be a completely join-prime truth value, let $\chi\colon\ \Sigma \to \Sigma'$ be a signature morphism, let ρ be a Σ-sentence and ρ' be a Σ' sentence. If χ is (finitary) quasi-representable, ρ' is a (finitary) κ-basic, and ρ is an existential χ-quantification of ρ', then ρ' is (finitary) κ-basic.*

Exercises

13.15. In any $\mathcal{L}$-institution with negations and conjunctions, for any finite set E of Σ-sentences by $\overline{E}$ we denote $\neg \bigwedge_{e\in E} e$. Then for any finite sets E and Γ of Σ-sentences we have that

$$E \models \overline{\Gamma} \ = \ \Gamma \models \overline{E}.$$

13.16. [79] The compact entailment sub-system of Exercise 13.13 inherits whatever connectives the original entailment system has.

13.17. [79] Let $(Sig, Sen, \vdash)$ be a compact entailment system. For any connectives we let $(Sig, Sen, \vdash'$) be the entailment system which has the respective connectives and is generated by $(Sig, Sen, \vdash)$. Then $(Sig, Sen, \vdash')$ is compact too.

13.18. Let I be an $\mathcal{L}$-institution. If I has universal χ-quantifications then $I^\sharp$ has them too. However, by means of counterexamples show that in general this type of causality does not hold for any other model theoretic connectives discussed in this section.

13.19. [90] **Basic sentences in $\mathcal{MVL}$**
In $\mathcal{MVL}$ and set E of atomic sentences is basic. Moreover if E and κ are finite then E is finitary κ-basic.

13.20. [90] **Basic sentences in $\mathcal{TL}$**
In $\mathcal{TL}$, for any finite $\kappa \in \mathcal{L}$, each set E of atoms for a signature P is finitary κ-basic.

13.21. Basic sentences in $\mathcal{FEL}$
In $\mathcal{FEL}$, are the sets of atoms $t \approx t'$ (κ-) basic?

13.22. Show that in $\mathcal{FMA}$ the atomic sentences $t \prec t'$ are not necessarily basic.

13.23. [90] **Unions of sets of basic sentences**
If the $\mathcal{L}$-institution has coproducts of models then the set of the (κ-)basic sets of sentences is closed under unions. Moreover, when κ is finite, this property extends to finitary κ-basic sets of sentences.

13.24. [90] **Coproducts of $\mathcal{MVL}$-models**
Show that $\mathcal{MVL}$ has coproducts of models.

13.25. [89] **Inter-compactness**
In any $\mathcal{L}$-institution I, for any $\kappa \in L$, two sets of Σ-sentences Γ_1, Γ_2 are κ-*inter-consistent* when $(M \models \Gamma_1) * (M \models \Gamma_2) \geq \kappa$ for some Σ-model M. They are just *inter-consistent* when they are κ-inter-consistent for some $\kappa > 0$, otherwise they are *inter-inconsistent*. I is *inter-compact* when for any two inter-inconsistent sets E_1, E_2 of Σ-sentences there are finite subsets $\Gamma_i \subseteq E_i$, $i = 1, 2$, such that Γ_1 and Γ_2 are inter-inconsistent. Show that on the one hand, if I is inter-compact then it is "compact", in the sense that any inconsistent set of sentences admits a finite subset that is inconsistent too, and on the other hand the reverse is also true in any of the following two situations:

1. $\mathcal{L}$ is a Heyting algebra, or
2. $\mathcal{L}$ is a finite total order and I has semantic residual conjunctions.

13.4 Filtered products

In Chap. 6 we developed the method of ultraproducts in binary institutions. Then in Sec. 12.5 we extended that to stratified institutions. In this section we re-develop the method of ultraproducts for $\mathcal{L}$-institutions, by following the same methodology that we used in Chap. 6 and Sec. 12.5 and which is implicit in the standard proofs of the classical Łoś theorem (in first-order model theory).

Here, the structure of this section follows this methodology. We do the following:

1. We discuss filtered products in $\mathcal{L}$-institutions.
2. We define the concept of preservation of sentences by $\mathcal{F}$-products / factors for $\mathcal{L}$-institutions and develop invariance results for preservation under Boolean and quantification connectives. A big difference with respect to the corresponding invariance results developed in binary ordinary or in stratified institutions, is the absence of results about negation. The reason for this is that, unless $\mathcal{L}$ is a Boolean algebra, negation is technically problematic for ultraproducts. And many-valued model theory is interesting when $\mathcal{L}$ is less constrained than Boolean algebras. The consequence of this gap is that the general method of ultraproducts in $\mathcal{L}$-institutions is weaker in the applications. For instance, in general it cannot be applied to $\mathcal{MVL}$, but it can be applied to various interesting negation-free sub-institutions of $\mathcal{MVL}$.
3. Finally, we derive model-theoretic compactness consequences in $\mathcal{L}$-institutions in the standard way.

With respect to $\mathcal{L}$, we tacitly assume for each concept of result the minimal structure on $\mathcal{L}$ which supports the formulation of the respective concept of result. For instance, a result about implication requires a residuated lattice structure on $\mathcal{L}$. The "common denominator" for the developments in this section is $\mathcal{L}$ being a complete residuated lattice.

Filtered products in $\mathcal{L}$-institutions

At the abstract level, filtered products of models in $\mathcal{L}$-institutions are no different from those in binary institutions because of their mere categorical nature. Otherwise said, filtered products as colimits of diagrams of projections is a concept that also works in $\mathcal{L}$-institutions. The same holds for the preservation, lifting and invention of filtered products by model reducts. However, in concrete $\mathcal{L}$-institutions the existence of filtered products has to be established. We do this for $\mathcal{MVL}$, and it is useful to compare the construction of the filtered products to that in $\mathcal{FOL}$ from Sec. 6.1 in order to understand the impact of many-valued truth on the existence and the form of filtered products of models.

Filtered products of $\mathcal{MVL}$ models. The construction of filtered products of $\mathcal{MVL}$ models follows the same general steps like for $\mathcal{FOL}$ models in Sec. 6.1. However in $\mathcal{MVL}$ this depends on some conditions on the truth values that in the collapsed binary context hold trivially.

Proposition 13.15 (F-products in $\mathcal{MVL}$)**.** *If $\mathcal{L}$ has all meets and directed joins then $\mathcal{MVL}$ has F-products of models for each filter F.*

Proof. Let $(M_i)_{i\in I}$ be a family of $\mathcal{MVL}$ models for a fixed signature and let $(p_i : M_I \to M_i)_{i\in I}$ be a direct product. This always exists and may be obtained as follows:

- For each sort s, $((M_I)_s \xrightarrow{(p_i)_s} (M_i)_s)_{i\in I}$ is the cartesian product of sets.
- For each symbol of constant σ, $(M_I)_\sigma = ((M_i)_\sigma)_{i\in I}$.
- For each relation symbol π, $(M_I)_\pi(m^1,\dots,m^n) = \bigwedge_{i\in I}(M_i)_\sigma(m_i^1,\dots,m_i^n)$, where for each $i \in I$, m_i is a short notation for $p_i m$.

By defining the following equivalence on each $(M_I)_s$:

$$m \sim_F m' \text{ if and only if } \{i \mid m_i = m_i'\} \in F.$$

we then construct the F-product M_F of $(M_i)_{i\in I}$ by

- For each sort s, $(M_F)_s = (M_I)_s/_{\sim_F}$.
- For each symbol of constants σ, $(M_F)_\sigma = (M_I)_\sigma/_{\sim_F}$.
- Let π be any relation symbol. As a matter of notation, for any m in M_I and any $J \in F$ by m_J we abbreviate $p_{I\supseteq J}m$, where $p_{I\supseteq J} : M_I \to M_J$ is the canonical projection. Moreover we extend this notation from elements to strings of elements.
  - Let m be an argument for $(M_I)_\pi$. We define
  $$(M_F)_\pi(m/_{\sim_F}) = \bigvee D_m \tag{13.17}$$
  where $D_m = \{(M_J)_\pi m_J \mid J \in F\}$. Note that here we use the assumption on the existence of directed joins as D_m is directed. Indeed, for any $J_1, J_2 \in F$, $(M_{J_k})_\pi m_{J_k} \leq (M_{J_1\cap J_2})_\pi m_{J_1\cap J_2}$ since $m_{J_1\cap J_2} = p_{J_k\supseteq J_1\cap J_2} m_{J_k}$ and $p_{J_k\supseteq J_1\cap J_2} m_{J_k}$ is a model homomorphism.

– The correctness of definition (13.17) follows from the fact that $m \sim_F m'$ implies $\bigvee D_m = \bigvee D_{m'}$. In order to establish this, by symmetry it is enough to prove that $\bigvee D_m \leq \bigvee D_{m'}$. Let $J \in F$. We prove that $(M_J)_\pi m_J \leq \bigvee D_{m'}$. Let
$$J' = \{i \mid m_i = m'_i\} \in F \text{ and } J'' = J \cap J' \in F.$$
By the homomorphism property of $p_{J \supseteq J''}$ it follows that
$$(M_J)_\pi m_J \leq (M_{J''})_\pi (p_{J \supseteq J''} m_J) = (M_{J''})_\pi m_{J''} = (M_{J''})_\pi m'_{J''} \in D_{m'}.$$

- Now it follows that the co-cone $(\mu_J : M_J \to M_F)_{J \in F}$ defined by $\mu_J m_J = m/_{\sim_F}$ is a co-limit in the respective category of models; we skip the straightforward proof of this.

□

When $\mathcal{L}$ is the binary Boolean algebra, that version of $\mathcal{MVL}$ is just $\mathcal{FOL}$ and then the construction of filtered products in Prop. 13.15 coincides with the construction of $\mathcal{FOL}$ filtered products in Sec. 6.1.

Preservation by filtered products / factors in general

The development of a modular body of results on preservation by filtered products for $\mathcal{L}$-institutions follow the general schema of Thm. 6.6. But in the many-valued truth context of $\mathcal{L}$-institutions some of the pieces of the puzzle of Thm. 6.6 become problematic, or at least more difficult to prove. Although the preservation concepts in an $\mathcal{L}$-institution I can be reduced to corresponding preservation concepts in its binary flattening $I^\sharp$, this does not help with transferring most of the preservation results of Thm. 6.6 because there is almost no technical relationship between the model-theoretic connectives in I and in $I^\sharp$.

Definition of preservation. The following definition extends the corresponding concept of preservation from binary institution theory (in Sec. 6.2) to $\mathcal{L}$-institutions. In any $\mathcal{L}$-institution, let Σ be any signature and e be any Σ-sentence. Also let $\mathcal{F}$ be any class of filters and κ be any value in $\mathcal{L}$. Then

- e is κ-*preserved by $\mathcal{F}$-products* when for each $F \in \mathcal{F}$ and each F-product $(\mu_J : M_J \to M_F)_{J \in F}$ (where F is a filter over I)

$$\{i \in I \mid M_i \models e \geq \kappa\} \in F \text{ implies } M_F \models e \geq \kappa \tag{13.18}$$

- e is κ-*preserved by $\mathcal{F}$-factors* when for each F-product as above we have the reverse of (13.18).

As a matter of terminology, when $\mathcal{F}$ is the class of all ultrafilters we rather say directly "κ-preserved by ultraproducts / ultrafactors". When $\mathcal{F}$ is the class of all singleton filters we rather say "κ-preserved by direct products / factors". Also, when do not specify the truth value κ and we just say "preserved by $\mathcal{F}$-products / factors", we mean that the sentence is κ-preserved for *all* truth values κ.

Note that whilst κ-preservation represents just a rephrasing of the preservation concepts from binary institution theory because "ρ is κ-preserved by ..." is technically the same with "(ρ,κ) is preserved by ..." in the binary flattening, this is not the case for preservation for *all* truth values. In other words "ρ is preserved by ..." in an $\mathcal{L}$-institution cannot be reduced to preservation in its binary flattening of a single sentence.

The following technical lemma, which will be used in some invariance-of-preservation proofs below, gives a succint characterisation of the 'global' preservation by filtered factors.

Lemma 13.16 (Preservation by F-factors)**.** *For any filter F and any Σ-sentence ρ the following are equivalent:*

1. *ρ is preserved by F-factors.*
2. *For each F-product $(\mu_J : M_J \to M_F)_{J\in F}$: $\{i \mid M_F \models \rho \ \leq \ M_i \models \rho\} \in F$.*

Proof. (1) $\Rightarrow$ (2): We take $\kappa = (M_F \models \rho)$ and apply the κ-preservation by F-factors property.
(2) $\Rightarrow$ (1): Let us consider any truth value κ and assume that $\kappa \leq (M_F \models \rho)$. Then

$$\{i \mid k \leq \ M_i \models \rho\} \supseteq \{i \mid M_F \models \rho \ \leq \ M_i \models \rho\}.$$

Since F is filter it follows that $\{i \mid k \leq \ M_i \models \rho\} \in F$. □

Preservation of basic sentences. By reducing κ-basic sentences to ordinary basic sentences in binary institution theory (Fact 13.13), from Thm. 6.6 we obtain immediately the preservation of basic sentences in $\mathcal{L}$-institutions.

Corollary 13.17 (Preservation of basic sentences by filtered products / factors)**.** *In any $\mathcal{L}$-institution, for any $\kappa \in L$,*

1. *Each κ-basic sentence is κ-preserved by all filtered products.*
2. *Each F-finitary κ-basic sentence is κ-preserved by F-factors.*

Consequently

3. *Each basic sentence is preserved by all filtered products.*
4. *Each F-finitary basic sentence is preserved by F-factors.*

Invariance of preservation under propositional connectives. In what follows we will use explicit notations for the model-theoretic connectives. For instance $\rho_1 \wedge \rho_2$ denotes a sentence which is the model-theoretic conjunction of sentences ρ_1 and ρ_2. Of course there may be several of those, so $\rho_1 \wedge \rho_2$ represents *any* of them. Likewise for the other connectives $\vee$, $*$, $\Rightarrow$, or quantifications.

Conjunction is the only many-valued model-theoretic connective, from all Boolean / propositional and quantification connectives, that determine an invariance of preservation that can be reduced to the corresponding property from binary institution theory. This is because for any sentences ρ_1 and ρ_2, in the binary flattening we always have that

$$M \models (\rho_1 \wedge \rho_2, \kappa) \quad \text{if and only if} \quad M \models (\rho_1, \kappa) \ \text{ and } \ M \models (\rho_2, \kappa).$$

So, from Thm. 6.6 we have:

Corollary 13.18 (Invariance of preservation under conjunction)**.** *The set of the sentences that are κ-preserved by F-products / F-factors is closed under conjunctions.*

For basic sentences and conjunctions we have invariance of preservation by both filtered products and filtered factors. This fortunate situation is not enjoyed by any of the other connectives, propositional or quantification.

Proposition 13.19 (Invariance of preservation by factors under other propositional connectives)**.** *The set of the sentences that are preserved by F-factors are closed under* $\wedge$, $\vee$, $*$.

Proof. Let ρ, ρ' be sentences preserved by F-factors and $\otimes$ being one of $\wedge$, $\vee$, $*$. Then for any F-product $(\mu_J : \ M_J \to M_F)_{J \in F}$ we have that:

1	$J = \{i \mid M_F \models \rho \ \leq \ M_i \models \rho\} \in F$	Lemma 13.16
2	$J' = \{i \mid M_F \models \rho' \ \leq \ M_i \models \rho')\} \in F$	Lemma 13.16
3	$\forall i \in J \cap J' \ \ M_F \models \rho \otimes \rho' \ \leq \ M_i \models \rho \otimes \rho'$	1, 2, $\otimes$ monotone, model-theoretic propositional connectives
4	$\{i \mid M_F \models \rho \otimes \rho' \ \leq \ M_i \models \rho \otimes \rho'\} \in F$	1, 2, 3, F filter.

From 4 by Prop. 13.16 it follows that $\rho \otimes \rho'$ is preserved by F-factors. □

Proposition 13.20 (Invariance of preservation under implication)**.** *If ρ is preserved by F-factors and ρ' is preserved by F-products then $\rho \Rightarrow \rho'$ is preserved by F-products.*

Proof. Let $(\mu_J : \ M_J \to M_F)_{J \in F}$ be any F-product of models. Let us assume

1 $\quad J = \{i \mid \kappa \leq \ M_i \models \rho \Rightarrow \rho'\} \in F.$

Then

2	$J = \{i \mid \kappa \leq \ M_i \models \rho \Rightarrow M_i \models \rho'\} \in F$	1, model-theoretic implication
3	$J = \{i \mid \kappa * M_i \models \rho \ \leq \ M_i \models \rho'\} \in F$	2, adjunction property of $*$
4	$J' = \{i \mid M_F \models \rho \ \leq \ M_i \models \rho\} \in F$	Lemma 13.16
5	$J' \subseteq J'' = \{i \mid \kappa * M_F \models \rho \ \leq \ \kappa * M_i \models \rho\} \in F$	4, monotonicity of $*$, F filter
6	$J \cap J'' \subseteq \{i \mid \kappa * M_F \models \rho \ \leq \ M_i \models \rho'\} \in F$	3, 5, F filter
7	$\kappa * M_F \models \rho \ \leq \ M_F \models \rho'$	6, ρ' preserved by F-products
8	$\kappa \leq \ M_F \models \rho \Rightarrow M_F \models \rho'$	7, adjunction property of $*$
9	$\kappa \leq \ M_F \models \rho \Rightarrow \rho'$	8, model-theoretic implication.

□

Invariance of preservations under quantification connectives. The results on preservation by quantifications rely primarily on various conditions on the model reduct functor involved, such as preservation, lifting, invention, of filtered products. In Sec. 6.2 we have developed general results supporting these properties in concrete situations. All of them are also valid for the many-valued context as this does not bring anything really new with respect to these properties. In general the *representability* property of the signature morphism is sufficient, and, as we know very well, in concrete situations this covers the first-order quantifications.

Proposition 13.21 (Invariance of preservation under quantifications (I))**.** *Let $\mathcal{F}$ be a class of filters that is closed under reductions and let $\chi : \Sigma \to \Sigma'$ be a signature morphism that invents $\mathcal{F}$-products. Let ρ be a Σ'-sentence.*

1. *If ρ that is κ-preserved by $\mathcal{F}$-products then $(\forall\chi)\rho$ is κ-preserved by $\mathcal{F}$-products too.*
2. *If ρ is κ-preserved by $\mathcal{F}$-factors and κ is completely prime-join then $(\exists\chi)\rho$ is κ-preserved by $\mathcal{F}$-factors too.*

Proof. Consider $F \in \mathcal{F}$ and an F-product $(\mu_J : M_J \to M_F)_{J \in F}$ of a family $(M_i)_{i \in I}$ of models.

1. Let us assume that

1 $J'' = \{i \mid M_i \models (\forall\chi)\rho \ \geq \kappa\} \in F.$

Let M' be any χ-expansion of M_F. We prove that $M' \models \rho \ \geq \kappa$. We have that:

2 $\exists J \in F \ \forall i \in J \ \exists M'_i$ χ-expansion of M_i — χ invents $\mathcal{F}$-products
$\exists F|_J$-product $(\mu'_{J'} : M'_{J'} \to M')_{J' \in F|_J}$ such that $(Mod\chi)\mu'_{J'} = \mu_{J'}$

3 $\forall i \in J \cap J'' \ M'_i \models \rho \ \geq \kappa$ — 1

4 $J \cap J'' \subseteq \{i \mid M'_i \models \rho \ \geq \kappa\}$ — 3, F filter

5 $\{i \in J \mid M'_i \models \rho \ \geq \kappa\} \in F|_J$ — 4

6 $M' \models \rho \ \geq \kappa$ — 5, $F|_J \in \mathcal{F}$, ρ is κ-preserved by $\mathcal{F}$-products.

2. Let us assume that

1 $M_F \models (\exists\chi)\rho \ \geq \kappa.$

We have that:

2 $M_F \models \bigvee_{(Mod\chi)M' = M_F} M' \models \rho \ \geq \kappa$ — 1, model-theoretic existential quantification

3 $\exists M' \ (Mod\chi)M' = M_F, \ M' \models \rho \ \geq \kappa$ — 2, κ completely join-prime

4 $\exists J \in F \ \forall i \in J \ \exists M'_i$ χ-expansion of M_i — 3, χ invents $\mathcal{F}$-products
$\exists F|_J$-product $(\mu'_{J'} : M'_{J'} \to M')_{J' \in F|_J}$ such that $(Mod\chi)\mu'_{J'} = \mu_{J'}$

5	$\{i \mid M'_i \models \rho \ \geq \kappa\} \in F\|_J$	4, $F\|_J \in \mathcal{F}$, ρ κ-preserved by $\mathcal{F}$-factors
6	$\{i \mid M'_i \models \rho \ \geq \kappa\} \subseteq \{i \mid (M_i \models (\exists\chi)\rho \ \geq \kappa\}$	4, model-theoretic existential quantification
7	$F\|_J \subseteq F$	F filter
8	$\{i \mid M_i \models (\exists\chi)\rho \ \geq \kappa\} \in F$	5, 6, 7.

□

Proposition 13.22 (Invariance of preservation under quantifications (II))**.** *Let $\mathcal{F}$ be a family of filters that is closed under reductions and let $\chi : \ \Sigma \to \Sigma'$ be a signature morphism that preserves $\mathcal{F}$-products. Let ρ be any Σ'-sentence. If ρ is κ-preserved by $\mathcal{F}$-products and κ is completely join-prime then $(\exists\chi)\rho$ is κ-preserved by $\mathcal{F}$-products too.*

Proof. We consider $F \in \mathcal{F}$ and an F-product $(\mu_J : \ M_J \to M_F)_{J\in F}$ and we assume

$$J'' = \{i \mid M_i \models (\exists\chi)\rho \ \geq \kappa\} \in F.$$

Hence

$$J'' = \{i \mid \bigvee_{(Mod\chi)M'_i = M_i} M'_i \models \rho \ \geq \kappa\} \in F.$$

Since κ is completely join-prime we have that

$$J'' = \{i \mid \exists M'_i \ (Mod\chi)M'_i = M_i, \ M'_i \models \rho \ \geq \kappa\} \in F.$$

Let $(\mu'_J : \ M'_J \to M'_{F|_{J''}})_{J\in F|_{J''}}$ be an $F|_{J''}$-product of $(M'_i)_{i\in J''}$. Because $Mod\chi$ preserves F-products

$$((Mod\chi)\mu_J : \ (Mod\chi)M'_J \to (Mod\chi)M'_{F|_{J''}})_{J\in F|_{J''}}$$

is an $F|_{J''}$-product of $(M_i)_{i\in J''}$. Since $F|_{J''}$ is a final sub-poset of F it follows that

$$(Mod\chi)M'_{F|_{J''}} \text{ is isomorphic to } M_F. \tag{13.19}$$

Since for each $i \in J''$, $M'_i \models \rho \ \geq \kappa$ and since ρ is κ-preserved by $\mathcal{F}$-products it follows that $M'_{F|_{J''}} \models \rho \ \geq \kappa$. This implies $(Mod\chi)M'_{F|_{J''}} \models (\exists\chi)\rho \ \geq \kappa$ and by (13.19) and by the fundamental assumption that model isomorphisms preserve satisfaction we finally obtain that $M_F \models (\exists\chi)\rho \ \geq \kappa$. □

We still miss an invariance under universal quantification of preservation by $\mathcal{F}$-factors. This requires an additional condition as follows.

Proposition 13.23. *Let F be a filter over a set I such that F is closed under arbitrary intersections. Let $\chi : \ \Sigma \to \Sigma'$ be a signature morphism that lifts F-products and let ρ be any Σ'-sentence. If ρ is κ-preserved by F-factors then $(\forall\chi)\rho$ is κ-preserved by F-factors too.*

Proof. We consider an F-product $(\mu_J : \ M_J \to M_F)_{J\in F}$ of a family $(M_i)_{i\in F}$ of models such that

1 $M_F \models (\forall\chi)\rho \ \geq \kappa$.

This means:

2 $\forall M'$ such that $(Mod\chi)M' = M_F$, $M' \models \rho \ \geq \kappa$ — model-theoretic existential quantification.

For each $i \in I$, let M'_i be any χ-expansion of M_i. We have that:

3 $\exists$ F-product $(\mu'_J : \ M'_J \to M'_F)_{J \in F}$ such that $(Mod\chi)M'_F = M_F$ — lifting assumption

4 $M'_F \models \rho \ \geq \kappa$ — 3, 2

5 $\{i \mid M'_i \models \rho \ \geq \kappa\} \in F$ — 3, 4, ρ is κ-preserved by F-factors

Let $J = \bigcap\{\{i \mid M'_i \models \rho \ \geq \kappa\} \mid \forall i, (Mod\chi)M'_i = M_i\}$. Then:

6 $J \in F$ — 5, F closed under arbitrary intersections

7 $J \subseteq \{i \mid M_i \models (\forall\chi)\rho \ \geq \kappa\}$ — definition of J, model-theoretic existential quantification

8 $\{i \mid M_i \models (\forall\chi)\rho \ \geq \kappa\} \in F$ — 6, 7.

□

The condition that F is closed under arbitrary intersections is rather restrictive because if we consider $X = \bigcap_{J \in F} J$ then $F = \{J \subseteq I \mid X \subseteq J\}$. Then the F-product is a direct product indexed by X. However this still includes the important case of the singleton filters, which leads to the following important consequence:

Corollary 13.24. *Let χ be a signature morphism that lifts direct products and let ρ be a Σ'-sentence. If ρ is κ-preserved by direct factors then $(\forall\chi)\rho$ is κ-preserved by direct factors too.*

A sample application. We cannot obtain a preservation-by-ultraproducts result for $\mathcal{MVL}$ by applying the invariance-of-preservation results developed above. The reason is the absence of such results for negation. However, we can obtain preservation-by-ultraproducts for some interesting classes of sentences. The following is such an example. In $\mathcal{MVL}$, let us say that $(\forall X)(H \Rightarrow e)$ is an *extended Horn sentence* when

- H is formed from existentially quantified atoms (including trivial quantifications) by iterations of $\wedge, \vee, *$.
- e is an existentially quantified atoms.

Proposition 13.25. *Under appropriate conditions for $\mathcal{L}$ and $\kappa \in L$, any extended Horn sentence is preserved by ultraproducts.*

Proof. Consider $(\forall X)(H \Rightarrow e)$ an extended Horn sentence.

- By Prop. 13.14, the existentially quantified atoms are finitary basic, hence by Cor. 13.17 they are κ-preserved by ultrafactors and ultraproducts.
- Hence H is preserved by ultrafactors (cf. Prop. 13.19).

- Then, by Prop. 13.20, $H \Rightarrow e$ is preserved by ultraproducts.
- Finally, by Prop. 13.21, $(\forall X)H \Rightarrow e$ is preserved by ultraproducts.

□

We formulated Prop. 13.25 above for ultraproducts, but the result holds more generally for any class $\mathcal{F}$ of filtered products that is closed under reductions.

Semantic compactness consequences

In Sec. 6.4 we saw how in binary institution theoretic setting model-theoretic compactness follows by preservation-by-ultraproducts. In Sec. 12.5 this was extended to stratified institutions. Manifold concrete compactness properties follow from such general compactness results. In the many-valued context this works like for binary institutions.

Corollary 13.26 (m-compactness by ultraproducts)**.** *Any $\mathcal{L}$-institution with ultraproducts of models such that each sentence is preserved by ultraproducts is m-compact.*

Proof. Let I be the $\mathcal{L}$-institution. Each sentence is preserved by ultraproducts in I implies that each sentence of $I^\sharp$ is preserved by ultraproducts too. By Cor. 6.21 $I^\sharp$ is m-compact, which means that I is m-compact too. □

From Cor. 13.26 and Prop. 13.9 we obtain immediately a κ-m-compactness consequence of preservation by ultraproducts. However we may strengthen that by weakening the preservation hypothesis to κ-preservation-by-ultraproducts.

Corollary 13.27 (κ-m-compactness by ultraproducts)**.** *In any $\mathcal{L}$-institution, if each sentence is κ-preserved by ultraproducts then the $\mathcal{L}$-institution is κ-m-compact.*

Proof. Let E be a set of sentences such that for each $E_0 \subseteq E$ finite E_0 is κ-consistent, which means that (E_0, κ) is consistent in the binary flattening. Then (E, κ) is consistent too by applying Cor. 6.21 only to the sentences of the form (ρ, κ), which we know that are preserved by ultraproducts from the κ-preservation condition in the $\mathcal{L}$-institution. Hence E is κ-consistent. □

Compared to Cor. 13.26, Cor. 13.27 assumes less and achieves less. However in the applications we guess it is rather difficult to meet with meaningful situations when κ-preservation by ultraproducts holds only for certain κs but not for all (excluding the trivial case $\kappa = 0$).

Exercises

13.26. Filtered products in concrete $\mathcal{L}$-institutions
Establish the existence of filtered products in categories of models in $\mathcal{FEL}$, $\mathcal{TL}$, and $\mathcal{FMA}$. (*Hint:* Can you find a canonical isomorphism between categories of $\mathcal{FMA}$ models and categories of $\mathcal{MVL}$ models?)

13.27. [90] **Preservation in $\mathcal{FMA}$**
In $\mathcal{FMA}$ let us assume that each truth value is completely prime-join. Then each atomic sentence $t \prec t'$ is preserved by filtered products. (*Hint:* Consider "phantom" sentences $t \sim c$, where t is a term and c is a constant, with satisfaction defined by $(M \models t \sim c) = M[t, M_c]$. By induction on the structure of t prove that $t \sim c$ are preserved by filtered products and factors. Then use that $t \prec t'$ is semantically equivalent to $(\forall x)(t \sim x) \Rightarrow (t' \sim x)$.)

13.28. In $\mathcal{MVL}$, can you find a set of extended Horn sentences that is inconsistent?

13.29. [90] **Compactness in $\mathcal{FMA}$**
In $\mathcal{FMA}$ let us assume that each truth value is completely prime-join. Let us consider the sub-institution $\mathcal{FMA}'$ of $\mathcal{FMA}$ obtained by restricting the sentences to those formed from the atoms $t \prec t'$ by iterations of conjunctions and universal and existential quantifications with deterministic variables. Then $\mathcal{FMA}'$ is m-compact.

13.5 Graded interpolation and definability

In Chapters 9 and 10 we have studied interpolation and definability, respectively, in the context of binary institutions. There both interpolation and definability were considered as properties of the semantic consequence relation. But in Sec. 11.4 (at the exercises) we saw how both interpolation and definability can be defined more abstractly in proof or in entailment systems. Some interesting results could and can be obtained at the consequence-theoretic level, although most of the important results require the model-theoretic conceptual infrastructure.

In this section we extend both concepts to graded consequences in general (in $\mathcal{L}$-entailment systems) and in particular to semantic graded consequences in $\mathcal{L}$-institutions. The material of this section consists only of basic concepts and properties, both topics deserving a much more extensive development. The structure of this section is as follows:

1. We define 'graded interpolation' as an interpolation property specific to graded entailment, in particular to graded semantic consequence. We discuss an example that illustrates how much more refined gets interpolation when we go from binary to the many-valued context. What can be an interpolation non-problem in the former context may be an interesting problem in the latter.

2. Then we define 'graded definability' which is the many-valued generalisation of the institution-independent binary truth concept of definability of Chap. 10. On the one hand, for the implicit definability we have a consequence-theoretic and a semantic concept, and an expected relationship between them. On the other hand, explicit definability can also be defined at both levels, but the relationship between the two definitions is trivial.

3. We establish the 'graded definability property', which is the equivalence between implicit and explicit graded definability at the consequence-theoretic level. Like in binary institutions, that implicit implies explicit definability gets caused by Craig-Robinson interpolation in its graded form.

4. The general relationship between Craig-Robinson and Craig interpolation, and between interpolation and Robinson consistency, do survive the many-valued context, but in a more sophisticated mathematical form. We address all these in the exercises part of the section, through a stepwise approach.

The definition of graded interpolation

In the definition of Craig-Robinson interpolation (CRi, from Sec. **??**) we can get more abstract by replacing the semantic consequence relation $\models$ by any entailment system. Thus, given a binary entailment system $(Sig, Sen, \vdash)$, a commutative square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is a CRi square when for each $E_1 \subseteq Sen\Sigma_1$, $E_2, \Gamma_2 \subseteq Sen\Sigma_2$, if $\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash_{\Sigma'} \theta_2 E_2$ then there exists $E \subseteq Sen\Sigma$ such that

$$E_1 \vdash_{\Sigma_1} \varphi_1 E \;\text{ and }\; \Gamma_2 \cup \varphi_2 E \vdash_{\Sigma_2} E_2.$$

Now let us write $E \vdash \Gamma$ as $E \vdash \Gamma = 1$ and $E \not\vdash \Gamma$ as $E \vdash \Gamma = 0$. Then the CRi implication above can be written as an inequality in the binary Boolean algebra:

$$\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2 \;\leq\; (E_1 \vdash \varphi_1 E) \wedge (\varphi_2 E \cup \Gamma_2 \vdash E_2). \tag{13.20}$$

But from the axioms of entailment systems we can easily get the reverse inequality, so we can rephrase the above inequality by strengthening it to an equality:

$$\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2 \;=\; (E_1 \vdash \varphi_1 E) \wedge (\varphi_2 E \cup \Gamma_2 \vdash E_2). \tag{13.21}$$

Then the idea of the concept of graded interpolation is to interpret (13.21) in a many-valued truth context. But at the general level this should be done with the residual conjunction $*$ instead of the ordinary lattice theoretic conjunction $\wedge$ in order to achieve coherence with the *transitivity* axiom. Of course, in some concrete situations – the binary context included –, the operation $*$ is the meet $\wedge$. The many-valued generalisation of the reverse of (13.20) holds under a minimal requirement on $*$, without any other conditions, as shown by the following result.

Proposition 13.28. *Consider an $\mathcal{L}$-entailment system such that the binary operation $*$ is monotone and consider any commutative square of signature morphisms*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

Then for any $E \subseteq Sen\Sigma$, $E_1 \subseteq Sen\Sigma_1$ *and* $E_2, \Gamma_2 \subseteq Sen\Sigma_2$

$$(E_1 \vdash \varphi_1 E) * (\varphi_2 E \cup \Gamma_2 \vdash E_2) \ \leq \ \theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2. \tag{13.22}$$

Proof. We have that:

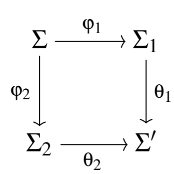
1	$\theta_1 E_1 \vdash \theta_1(\varphi_1 E) \ \leq \ \theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_1(\varphi_1 E)$	*monotonicity*
2	$\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2 \Gamma_2$	*reflexivity*, *monotonicity*
3	$\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_1(\varphi_1 E) \cup \theta_2 \Gamma_2 \ = \ (\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_1(\varphi_1 E)) \wedge (\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2 \Gamma_2))$	definition
4	$\theta_1 E_1 \cup \theta_1(\varphi_1 E) \ \leq \ \theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_1(\varphi_1 E) \cup \theta_2 \Gamma_2$	1, 2, 3.

Then (13.22) is obtained by the following calculations:

$$
\begin{array}{ll}
(E_1 \vdash \varphi_1 E) * (\varphi_2 E \cup \Gamma_2 \vdash E_2) \leq & \\
\quad \leq (\theta_1 E_1 \vdash \theta_1(\varphi_1 E)) * (\theta_2(\varphi_2 E) \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2) & \textit{translation}, * \text{ monotone} \\
\quad \leq (\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_1(\varphi_1 E) \cup \theta_2 \Gamma_2) * (\theta_2(\varphi_2 E) \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2) & 4, * \text{ monotone} \\
\quad = (\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2(\varphi_2 E) \cup \theta_2 \Gamma_2) * (\theta_2(\varphi_2 E) \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2) & \varphi_1;\theta_1 = \varphi_2;\theta_2 \\
\quad \leq \theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2 & \textit{transitivity}.
\end{array}
$$

□

Like in the binary case, in the light of Prop. 13.28 the interpolation property is represented by the reverse of (13.22). In any $\mathcal{L}$-entailment system, given a commutative square of signature morphisms



and $E_1 \subseteq Sen\Sigma_1$ and $E_2, \Gamma_2 \subseteq Sen\Sigma_2$ we say that $E \subseteq Sen\Sigma$ is a *Craig-Robinson interpolant of* E_1, E_2 *and* Γ_2 when

$$\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2 \ \leq \ (E_1 \vdash \varphi_1 E) * (\varphi_2 E \cup \Gamma_2 \vdash E_2). \tag{13.23}$$

Of course, under the monotonicity hypothesis of Prop. 13.28 we can replace the inequality 13.23 with the equality

$$\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2 \ = \ (E_1 \vdash \varphi_1 E) * (\varphi_2 E \cup \Gamma_2 \vdash E_2).$$

When Γ_2 is empty then E is called a *Craig interpolant* (of E_1 and E_2). When interpolants exist for all E_1, E_2 (and eventually Γ_2) the respective commutative square of signature morphisms is called a *Craig(-Robinson) interpolation square* (abbr. C(R)i square). When $\mathcal{L}$ is a residuated lattice, the concepts introduced in this definition extend also to $\mathcal{L}$-institutions by considering the semantic $\mathcal{L}$-entailment system given by Prop. 13.1.

An $\mathcal{L}$-entailment system (or $\mathcal{L}$-institution) has $\langle \mathfrak{L}, \mathfrak{R} \rangle$-CRi / Ci for $\mathfrak{L}, \mathfrak{R} \subseteq Sig$ classes of signature morphisms, when each pushout square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

with $\varphi_1 \in \mathfrak{L}$ and $\varphi_2 \in \mathfrak{R}$ is a CRi / Ci square.

An example

We can illustrate an important feature of graded interpolation that distinguishes it sharply from binary / crisp interpolation. With crisp interpolation, when the premise of interpolation is false then the presumptive interpolation problem is dead. However if we replace 'false' by 'not true' and then through a many-valued strike replace 'not true' by 'not *entirely* true', we may get graded interpolation problems that would not occur in the binary truth context.

Let us consider $\mathcal{L}$ to be the residuated lattice determined by the Goguen / product. We consider the following commutative square of inclusions of propositional logic signatures:

$$\begin{array}{ccc} p & \xrightarrow{\subseteq} & p, q_1 \\ {\scriptstyle \subseteq}\downarrow & & \downarrow{\scriptstyle \subseteq} \\ p, q_2 & \xrightarrow[\subseteq]{} & p, q_1, q_2 \end{array}$$

We also consider two $\mathcal{MVL}$ models, M and N, for p, q_1, q_2 as defined by the following truth table:

	p	q_1	q_2
M	$1/2$	$1/2$	$1/2$
N	$1/2$	1	$1/4$

We let the $\mathcal{L}$-institution be the fragment of $\mathcal{MVL}$ determined only by this data. In this $\mathcal{L}$-institution we have that

$$p * q_1 \models \{p, q_2\} = 1/4; \quad p * q_1 \models p = 1; \quad p \models \{p, q_2\} = 1/4.$$

It follows that p is a graded Craig interpolant for $p * q_1$ and $\{p, q_2\}$.

Now let us have another look at this case. $p * q_1 \models p = 1$ holds in general because in any residuated lattice $x * y \leq x$. In binary propositional logic $p \models \{p, q_2\}$ is false unless we constrain the models only to those in which q_2 is true, which would make q_2 redundant. Similar considerations apply to the premise $p * q_1 \models \{p, q_2\}$, a situation that makes this a non-problem in classical propositional logic. However, in the many-valued version of this example we get a good graded interpolation problem that has a solution.

Other versions of graded interpolation

The following definition strengthen the concept of graded interpolation introduced previously. We formulate it only for the Craig interpolation case, the respective Craig-Robinson extension being straightforward. In any $\mathcal{L}$-entailment system, for any $\kappa, \kappa_1, \kappa_2 \in L \setminus \{0\}$ such that $\kappa \leq \kappa_1 * \kappa_2$, a commutative square of signature morphisms

$$\begin{CD} \Sigma @>{\varphi_1}>> \Sigma_1 \\ @V{\varphi_2}VV @VV{\theta_1}V \\ \Sigma_2 @>>{\theta_2}> \Sigma' \end{CD}$$

is a $(\kappa, \kappa_1, \kappa_2)$-*Ci square* when for any $E_1 \subseteq Sen\Sigma_1$ and $E_2 \subseteq Sen\Sigma_2$ such that

$$\theta_1 E_1 \vdash \theta_2 E_2 \ \geq \kappa$$

there exists $E \subseteq Sen\Sigma$, called the $(\kappa, \kappa_1, \kappa_2)$-*interpolant of* E_1, E_2, such that

$$E_1 \vdash \varphi_1 E \ \geq \kappa_1 \ \text{ and } \ \varphi_2 E \cup \Gamma_2 \vdash E_2 \ \geq \kappa_2.$$

The graded interpolation concept just defined is stronger than the graded interpolation previously introduced in the sense given by the following fact.

Fact 13.29. *A commutative square of signature morphisms which has the property that for each* $\kappa \neq 0$ *there exists* κ_1, κ_2 *such that it is a* $(\kappa, \kappa_1, \kappa_2)$-*Ci square, is a Ci square in the sense of the former definition of graded interpolation too.*

On the other hand, a weaker version of interpolation may be obtained by requiring that the values of the entailments involved in the interpolation relation are non-zero. In this way an inequality relation need not be considered anymore.

Graded definability

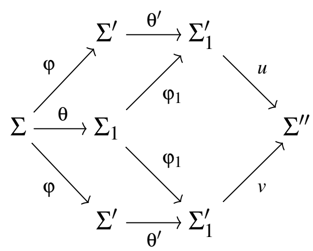
In what follows we refine to many-valued truth the concepts of implicit and of explicit definability of Chap. 10. We do this both at the semantic and at the more abstract consequence-theoretic level. Then, in this context, by assuming graded interpolation, we prove the 'definability property', that implicit implies explicit definability. Here we do not consider fully its reverse implication which in classical first-order model theory is considered not interesting. However, this can still be developed in the many-valued context of this section along corresponding ideas from Sec. 10.2.

Semantic implicit graded definability. In any $\mathcal{L}$-institution, for any $\kappa \in L$, a signature morphism $\varphi : \ \Sigma \to \Sigma'$ is *defined* κ-*implicitly* by $E' \subseteq Sen\Sigma'$ when for any Σ'-models M'_1 and M'_2 if

- $M'_1 \models E' \ \wedge \ M'_2 \models E' \ \geq \kappa$ and
- $(Mod\varphi)M'_1 = (Mod\varphi)M'_2$

then $M'_1 = M'_2$.

Consequence-theoretic implicit graded definability. In any $\mathcal{L}$-entailment system, for any $\kappa \in L$, a signature morphism $\varphi : \Sigma \to \Sigma'$ is *defined* κ*-implicitly* by $E' \subseteq Sen\Sigma'$ when for any diagram of pushout squares like below



(13.24)

and for any Σ'_1-sentence ρ we have that

$$u(\theta' E') \cup v(\theta' E') \cup u\rho \ \vdash \ v\rho \geq \kappa.$$

In the many-valued context semantic implicit definability can be abstracted to consequence-theoretic implicit definability as shown by Prop. 13.30 below. But this abstraction is more subtle than in the binary truth case because the two implicit definability properties involved are relative to different truth values. These are related by the following order-theoretic concept. Let $\mathcal{L}$ be a partially ordered set with bottom (0) and top (1) elements and with a binary commutative monotone operation $*$ which admits 1 as identity. For any $\kappa, \ell \in L$ we say that ℓ is a *lower-companion* to κ when $\{x \mid x * \kappa \neq 0\} \subseteq \{x \mid \ell \leq x\}$. Note that when $\mathcal{L}$ is the binary Boolean algebra, 1 is a lower-companion to itself, which explains why the following result can be interpreted properly in the binary case.

Proposition 13.30. *In any semi-exact $\mathcal{L}$-institution let $\kappa, \ell \in L$ such that ℓ is a lower-companion to κ. Then a signature morphism $\varphi : \Sigma \to \Sigma'$ is defined κ-implicitly by $E' \subseteq Sen\Sigma'$ in the consequence-theoretic sense if it is defined ℓ-implicitly by E' in the semantic sense.*

Proof. We have to prove that $u(\theta' E') \cup v(\theta' E') \cup u\rho \models v\rho \ \geq \kappa$. which means that for any Σ''-model M'' we have to prove that

1 $\quad (M'' \models u(\theta' E') \ \wedge \ M'' \models v(\theta' E') \ \wedge \ M'' \models u\rho) \Rightarrow M'' \models v\rho \ \geq \kappa.$

We distinguish two cases:

1. When $M''\restriction_u\restriction_{\theta'} \models E' \ \wedge \ M''\restriction_v\restriction_{\theta'} \models E' \ \not\geq \ell$. It follows that

   2 $\quad M'' \models u(\theta' E') \ \wedge \ M'' \models v(\theta' E') \ \wedge \ M'' \models u\rho \ \not\geq \ell$ — Sat. Condition, *Reductio ad Absurdum*

   3 $\quad \kappa * (M'' \models u(\theta' E') \ \wedge \ M'' \models v(\theta' E') \ \wedge \ M'' \models u\rho) = 0$ — 2, ℓ lower-companion to κ, *Reductio ad Absurdum*

   4 $\quad \kappa * (M'' \models u(\theta' E') \ \wedge \ M'' \models v(\theta' E') \ \wedge \ M'' \models u\rho) \ \leq \ M'' \models v\rho$ — 3.

   By the residual adjunction property, from 4 we obtain 1.

2. When $M''\restriction_u\restriction_{\theta'} \models E' \ \wedge \ M''\restriction_v\restriction_{\theta'} \models E' \ \geq \ell$. Then:

5	$M''\restriction_u\restriction_{\theta'}\restriction_{\varphi} = M''\restriction_v\restriction_{\theta'}\restriction_{\varphi}$	diagram (13.24) commutes, *Mod* functor
6	$M''\restriction_u\restriction_{\theta'} = M''\restriction_v\restriction_{\theta'}$	5, ℓ-implicit definability by E'
7	$M''\restriction_u\restriction_{\varphi_1} = M''\restriction_v\restriction_{\varphi_1}$	$\varphi_1;u = \varphi_1;v$, *Mod* functor
8	$M''\restriction_u = M''\restriction_v$	6, 7, uniqueness property of model amalgamation in the square $(\theta,\varphi,\varphi_1,\theta')$
9	$M'' \models u\rho \; = \; M'' \models v\rho$	8, Satisfaction Condition
10	$M'' \models u(\theta'E') \wedge M'' \models v(\theta'E') \wedge M'' \models u\rho \; \leq M'' \models v\rho$	9
11	$(M'' \models u(\theta'E') \wedge M'' \models v(\theta'E') \wedge M'' \models u\rho) \Rightarrow M'' \models v\rho \; = 1 \geq \kappa$	10, residual adjunction property.

□

Explicit graded definability We may recall from Chap. 10 that, in binary institutions, a signature morphism $\varphi : \Sigma \to \Sigma'$ is *explicitly defined* by $E' \subseteq Sen\Sigma'$, when for each pushout square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ \Sigma_1 & \xrightarrow[\varphi_1]{} & \Sigma'_1 \end{array} \tag{13.25}$$

and each $\rho \in Sen\Sigma'_1$ there exists $E_\rho \in Sen\Sigma_1$ such that

$$\theta'E' \cup \rho \models \varphi_1 E_\rho \;\text{ and }\; \theta'E' \cup \varphi_1 E_\rho \models \rho. \tag{13.26}$$

This can be refined to the many-valued context just by fixing a truth value $\kappa \in L$ and then turning (13.26) to

$$(\theta'E' \cup \rho \models \varphi_1 E_\rho) * (\theta'E' \cup \varphi_1 E_\rho \models \rho) \; \geq \kappa.$$

Unlike semantic implicit definability, this relation can be expressed *directly* within the more abstract context on any $\mathcal{L}$-entailment system:

$$(\theta'E' \cup \rho \vdash \varphi_1 E_\rho) * (\theta'E' \cup \varphi_1 E_\rho \vdash \rho) \; \geq \kappa.$$

The definability property by interpolation in the graded context. The following result shows that at the general many-valued consequence-theoretic level, explicit definability implies implicit definability. Note the rather lax conditions of the result: no pushout property required, and any set Γ of sentences instead of E_ρ. This situation is consonant to the rather trivial nature of this part of the definability property; in the classical studies of definability this is often skipped. We do this property here only for the consequence-theoretic level; in order to have this at the proper semantic level we need also a kind of reversal of Prop. 13.30, which should involve concepts and ideas from Sec. 10.2. We will not do this here.

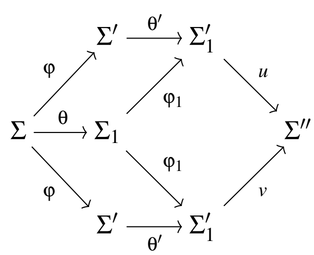
Proposition 13.31. *In any $\mathcal{L}$-entailment system with $*$ monotone, for any signature morphism $\varphi : \Sigma \to \Sigma'$ and any $E' \subseteq Sen(\Sigma')$, for any commuting diagram of signature morphisms like (13.24) and for all $\Gamma \subseteq Sen\Sigma_1$, $\rho \in Sen\Sigma'_1$ we have that*

$$(\theta' E' \cup \rho \vdash \varphi_1 \Gamma) * (\theta' E' \cup \varphi_1 \Gamma \vdash \rho) \;\; \leq \;\; (\theta'; u)E' \cup (\theta'; v)E' \cup u\rho \vdash v\rho.$$

Proof. This is an immediate consequence of Prop. 13.28 by setting φ_1 and φ_2 to φ_1, θ_1 to u, θ_2 to v, E to Γ, E_1 to $\theta' E' \cup \rho$, E_2 to ρ and Γ_2 to $\theta' E'$. □

The following couple of results represent many-valued replicas of the definability-by-interpolation result of Thm. 10.5. Thm. 13.32 below happens at the more abstract level of $\mathcal{L}$-entailment systems.

Theorem 13.32. *In any $\mathcal{L}$-entailment system that has $\langle \mathfrak{L}, \mathfrak{R} \rangle$-CRi for classes $\mathfrak{L}, \mathfrak{R}$ of signature morphisms that are stable under pushouts, for any signature morphism $\varphi : \Sigma \to \Sigma' \in \mathfrak{L} \cap \mathfrak{R}$ and any $E' \subseteq Sen\Sigma'$, for any diagram of pushout squares:*



(13.27)

for any $\rho \in Sen\Sigma'_1$ there exists $E_\rho \subseteq Sen\Sigma_1$ such that

$$(\theta' E' \cup \rho \vdash \varphi_1 E_\rho) * (\theta' E' \cup \varphi_1 E_\rho \vdash \rho) \;\geq\; (\theta'; u)E' \cup (\theta'; v)E' \cup u\rho \vdash v\rho.$$

Proof.

- By the stability under pushouts we get that $\varphi_1 \in \mathfrak{L} \cap \mathfrak{R}$.
- By the CRi property of the $\mathcal{L}$-entailment system we have that the right hand side pushout square of diagram (13.27) is a CRi square.
- Then E_ρ is obtained as the interpolant for the interpolation problem corresponding to setting φ_1 and φ_2 to φ_1, θ_1 to u, θ_2 to v, E_1 to $\theta' E' \cup \rho$, E_2 to ρ and Γ_2 to $\theta' E'$.

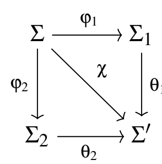
□

By putting together the results of Prop. 13.30 and of Thm. 13.32 we obtain the semantic graded definability property:

Corollary 13.33 (Semantic graded definability by interpolation)**.** *In any semi-exact $\mathcal{L}$-institution with $\langle \mathfrak{L}, \mathfrak{R} \rangle$-CRi we let $\kappa, \ell \in L$ such that ℓ is a lower-companion to κ. Then a signature morphism in $\mathfrak{L} \cap \mathfrak{R}$ is defined κ-explicitly when it is defined ℓ-implicitly.*

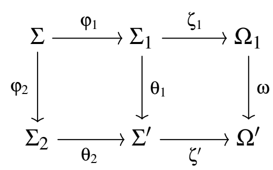
Exercises

13.30. [89] **Interpolation in localized $\mathcal{L}$-institutions**
Let $\mathcal{L}$ be the closed interval $[0,1]$ or a set $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ endowed with the structure of residuated lattice given by the Łukasiewicz arithmetic conjunction, and consider an $\mathcal{L}$-institution I that has designated model-theoretic disjunctions (denoted by $\vee$). Let M' be any Σ'-model. The construction of Exercise 3.10 can be applied directly without any changes to I. Any commutative square of signature morphisms in I like the following one



gets interpreted in the obvious manner as a commutative square of signature morphisms in I/M'. As a matter of notations let $I/M' = (Sig', Sen', Mod', \models)$. Let $\rho \in Sen\Sigma = Sen'\chi$, $\rho_1 \in Sen\Sigma_1 = Sen'\theta_1$ and $\rho_2 \in Sen\Sigma_2 = Sen'\theta_2$. Show that ρ is a Craig interpolant for $\varphi_1\rho \vee \rho_1$ and $\{\varphi_2\rho, \rho_2\}$ in I/M'.

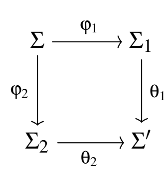
13.31. [89] **Compositionality of graded interpolation**
Consider an $\mathcal{L}$-entailment system such that $*$ is associative and monotone. If both the left-hand side and the right-hand side squares below are CRi squares then the outer square is a CRi square too.



This was a "horizontal" compositionality property. What about a "vertical" one? (*Hint:* It can be done even with slightly weaker hypotheses.)

13.32. [89] **Craig-Robinson interpolation via Craig interpolation**
Let $\mathcal{L}$ be a Heyting algebra. In any $\mathcal{L}$-institution I with model-theoretic implications any Ci square is a CRi square. (*Hint:* Draw inspiration from the proof of Prop. 9.24.)

13.33. [89] **Robinson consistency by interpolation**
In any $\mathcal{L}$-institution such that $\mathcal{L}$ is a residuated lattice, for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each set E' of Σ'-sentences and each $\kappa \in L$, we let $[\varphi^{-1}E']_\kappa = \{\rho \in Sen\Sigma \mid E' \models \varphi\rho \ \geq \kappa\}$. For any $\ell, \kappa_1, \kappa_2 \in L \setminus \{0\}$, a commutative square of signature morphisms



is a $(\ell, \kappa_1, \kappa_2)$-*Rc* square when for any finite sets E_i of Σ_i-sentences, $i = 1,2$, if $[\varphi_1^{-1}E_1]_{\kappa_1}$ and $[\varphi_2^{-1}E_2]_{\kappa_2}$ are inter-consistent (in the sense introduced in Ex. 13.25) then $\theta_1 E_1$ and $\theta_2 E_2$ are ℓ-inter-consistent. Prove that in any $\mathcal{L}$-institution with conjunctions and negations, if ℓ is a lower-companion to κ then any $(\kappa, \kappa_1, \kappa_2)$-Ci square is a $(\ell, \kappa_1, \kappa_2)$-Rc square.

13.34. [89] **Interpolation by Robinson consistency**
For any $\kappa, \ell \in L$, ℓ is an *upper-companion to* κ if and only if $\ell * \kappa \neq 0$. In any inter-compact $\mathcal{L}$-institution (in the sense introduced in Ex. 13.25) with conjunctions and negations, if ℓ is an upper-companion to κ then any $(\ell, \kappa_1, \kappa_2)$-Rc square is a $(\kappa, \kappa_1, \kappa_2)$-Ci square.

13.35. How do the results of Exercises 13.33 and 13.34 relate to the result of Thm. 9.15?

13.36. Formulate and prove a reversal of Prop. 13.30, when consequence-theoretic graded implicit definability determines semantic graded implicit definability. (*Hint:* Look at Prop. 10.4.)

13.6 Translation structures

In this section we extend the binary institution theory infrastructure of translations to $\mathcal{L}$-institutions. We do this on the expected two levels:

1. We introduce morphisms of fuzzy theories and study their co-limits and model amalgamation properties by generalising results of Sections 4.2 and 4.3 that allow for the lifting of these two properties from categories of signatures to categories of theories. The motivation for these developments is similar to that from binary institution theory, namely a modular approach to the existence of fuzzy theory co-limits and to model amalgamation that reduces both problems in a general manner to the level of the underlying categories of signatures. These results do extend their binary truth version to many-valued truth. We also extend to many-valued truth the concept of 'institution of theories', and exploit it here in a similar way we did in binary truth institution theory, such as for supporting the encodings between many-valued truth institutions.
2. We develop a concept of comorphism for many-valued truth institutions that generalises the binary concept of comorphism with the aim to provide a mathematical device for translating and encoding between many-valued truth institutions. We do this in such a way that allows for the lattice of truth values to vary between the source and the target of the comorphisms. This flexibility increases the applicability potential of this concept.

Categories of fuzzy theories

The generalisation of the concept of theory morphism introduced in Sec. 4.1 to fuzzy theories in $\mathcal{L}$-institutions faces the following specific hurdle: unlike in binary institutions, in $\mathcal{L}$-institutions there can be more than one concept of theory closure (as we have seen in Sec. 13.2). Here the solution is to go a bit more general and parameterise the concept of fuzzy theory morphism by a respective concept of $\mathcal{L}$-closure system. Thus consider an $\mathcal{L}$-institution equipped with an $\mathcal{L}$-closure system denoted by $(\_)^\bullet$. A *morphism of fuzzy theories* $\varphi : (\Sigma, T) \to (\Sigma', T')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $T \leq (Sen\varphi); T'^\bullet$.

It is interesting to note that this is yet another situation that illustrates the view of category theorists that a category is in fact defined by its arrows rather than by its objects as there can be several categories sharing the same class fuzzy theories.

Proposition 13.34. *For any choice of a closure system $(\_)^\bullet$, the respective fuzzy theory morphisms form a category under the composition inherited from the category of the signatures.*

Proof. We essentially have to show the composition of theory morphisms is still a theory morphism. Let $\varphi : (\Sigma, T) \to (\Sigma', T')$ and $\varphi' : (\Sigma', T') \to (\Sigma'', T'')$ be theory morphisms. We prove that $\varphi;\varphi' : (\Sigma, T) \to (\Sigma'', T'')$ is a theory morphism too. We have that

1	$T \leq (Sen\varphi); T'^\bullet$	φ theory morphism
2	$T' \leq (Sen\varphi'); T''^\bullet$	φ' theory morphism
3	$T'^\bullet \leq ((Sen\varphi'); T''^\bullet)^\bullet$	2, *C-monotonicity*
4	$((Sen\varphi'); T''^\bullet)^\bullet \leq (Sen\varphi'); T''^{\bullet\bullet}$	*C-translation*
5	$T''^{\bullet\bullet} = T''^\bullet$	*C-transitivity*
6	$T \leq (Sen\varphi); (Sen\varphi'); T''^\bullet$	1, 3, 4, 5
7	$(Sen\varphi); (Sen\varphi') = Sen(\varphi;\varphi')$	functoriality of *Sen*
8	$T \leq Sen(\varphi;\varphi'); T''^\bullet$	6, 7.

□

Fuzzy theory morphisms as binary theory morphisms. Concerning the concept of theory morphisms, the following question arises: to what extent is it possible to express fuzzy theory morphisms as binary institution-theoretic morphisms of theories in the binary flattening? In what follows we explore this issue by clarifying the relationship between the respective categories of theory morphisms.

For any $\mathcal{L}$-institution I we let $Th^\bullet$ denote the category of theory morphisms with respect to a closure operator $(\_)^\bullet$ and $Th^\sharp$ denote the category of theory morphisms in $I^\sharp$.

Proposition 13.35. *If $(\_)^\bullet$ is a closure system that is lower than the Galois connection closure system $(\_)^{**}$ then there exists an embedding (i.e. injective on objects and faithful) functor $\Phi^\bullet : Th^\bullet \to Th^\sharp$ defined for each fuzzy theory (Σ, T) by*

$$\Phi^\bullet(\Sigma, T) = (\Sigma, T^\sharp) \ \ where \ \ T^\sharp = \{(\rho, T\rho) \mid \rho \in Sen\Sigma, T\rho \neq 0\}.$$

Proof. Note that given an theory morphism $\varphi : (\Sigma, T) \to (\Sigma', T')$, in the realm of signature morphisms, $\Phi^\bullet\varphi = \varphi$. All we have to do is to prove that this is a binary theory morphism $(\Sigma, T^\sharp) \to (\Sigma', T'^\sharp)$, that is $T'^\sharp \models^\sharp \varphi T^\sharp$. We consider any Σ'-model M' such that $M' \models^\sharp T'^\sharp$. Then:

1	$M'^* \geq T'$	$M' \models^\sharp T'^\sharp$, definition of $T'^\sharp$
2	$M'^* = M'^{***} \geq T'^{**}$	1, general properties of Galois connections
3	$(Sen\varphi); M'^* \geq (Sen\varphi); T'^{**}$	2
4	$(Sen\varphi); T'^{**} \geq (Sen\varphi); T'^\bullet$	$(\_)^\bullet \leq (\_)^{**}$
5	$(Sen\varphi); T'^\bullet \geq T$	$\varphi : (\Sigma, T) \to (\Sigma', T')$ theory morphism
6	$(Sen\varphi); M'^* \geq T$	3, 4, 5

7	$(Sen\varphi);M'^* = ((Mod\varphi)M')^*$	Satisfaction Condition in I
8	$((Mod\varphi)M')^* \geq T$	6, 7
9	$(Mod\varphi)M' \models^\sharp T'^\sharp$	8
10	$M' \models^\sharp \varphi T^\sharp$	9, Satisfaction Condition in $I^\sharp$.

□

$\mathcal{L}$-institutions of fuzzy theories. Prop. 13.34 generalises the result of Prop. 4.1 that in Sec. 4.1 was the basis for the definition of the institution I^{th} of the 'theories of the (binary) institution I'. We have seen how the institution I^{th} is useful for importing concepts such as (co-)limits and model amalgamation from the categories of signatures to the categories of theories, but, more importantly, for a smooth formulation of institution encodings in terms of ordinary comorphisms. Similarly to I^{th}, Prop. 13.34 can be used as the basis for defining $\mathcal{L}$-institutions of fuzzy theories over a fixed $\mathcal{L}$-institution I. Because in the many-valued situation the concept of morphism of fuzzy theories is parameterised by an $\mathcal{L}$-closure system of choice, any I may determine several $\mathcal{L}$-institutions of fuzzy theories. Moreover, this construction requires a condition on the $\mathcal{L}$-closure system involved, which is necessary to get the mode reducts inherited from the $\mathcal{L}$-institution to the $\mathcal{L}$-institution of fuzzy theory morphisms.

Proposition 13.36. *Let I be an $\mathcal{L}$-institution endowed with a closure system $(\_)^\bullet$ that is lower than the Galois connection closure system $(\_)^{**}$. Let $\varphi\colon (\Sigma,T) \to (\Sigma',T')$ be any fuzzy theory morphism. Then, for any (Σ',T')-model M', its φ-reduct is a (Σ,T)-model.*

Proof. We have the following:

1	$M'^* \geq T'$	M' model of (Σ',T')
2	$M'^* \geq T'$ if and only if $M' \models^\sharp T'^\sharp$	definition of $I^\sharp$
3	$\varphi\colon (\Sigma,T^\sharp) \to (\Sigma',T'^\sharp) \in Th^\sharp$	$\varphi\colon (\Sigma,T) \to (\Sigma',T') \in Th^\bullet$, Prop. 13.35
4	$(Mod\varphi)M' \models^\sharp T^\sharp$	1, 2, 3
5	$((Mod\varphi)M')^* \geq T$ if and only if $(Mod\varphi)M' \models^\sharp T^\sharp$	definition of $I^\sharp$
6	$((Mod\varphi)M')^* \geq T$	4, 5.

□

Note that both concrete $\mathcal{L}$-closure systems discussed in Sec. 13.2 fullfil the conditions of Prop. 13.36.

Proposition 13.37 ($\mathcal{L}$-institutions of fuzzy theories)**.** *Let I be an $\mathcal{L}$-institution endowed with a closure system $(\_)^\bullet$ that is lower than the Galois connection closure system $(\_)^{**}$. Then*

$$I^\bullet = (Th^\bullet, Sen^\bullet, Mod^\bullet, \models^\bullet)$$

*is an $\mathcal{L}$-institution, called the $\mathcal{L}$-*institution of fuzzy theories with respect to *$(\_)^\bullet$, where*

– *Sen*$^\bullet = \Pi^\bullet; Sen$ *where* $\Pi^\bullet : Th^\bullet \to Sig$ *is the forgetful functor,*

– $Mod^\bullet(\Sigma, T)$ *is the full subcategory of* $Mod(\Sigma)$ *determined by the* (Σ, T)*-models (i.e., the* Σ*-models* M *such that* $T \leq M^*$*), and*

– *for each* (Σ, T)*-model* M *and each* $\rho \in Sen^\bullet(\Sigma, T) = Sen\Sigma$*,*

$$M \models^\bullet_{(\Sigma,T)} \rho \;=\; M \models_\Sigma \rho.$$

Proof. • That $Th^\bullet$ is a category is given by Prop. 13.34.

• That $Mod^\bullet$ is a functor $(Th^\bullet)^{op} \to \mathbb{C}at$ is given by the fact that Mod is a functor in conjunction with Prop. 13.36.

• $Sen^\bullet$ is a functor as composition of two functors.

• The Satisfaction Condition in $I^\bullet$ follows immediately from the Satisfaction Condition in I because the sentence translations and the model reducts of $I^\bullet$ are inherited from I.

□

Colimits of fuzzy theories and model amalgamation

Prop. 4.2 showed us how in binary institutions limit / colimits of theories exist in dependence of limits / colimits of signatures. Below we extend this to fuzzy theories. Since in the applications the importance of colimits surpasses that of limits of theories we will do only the colimits result, the limit result being proposed as an exercise. We fix an $\mathcal{L}$-institution such that $\mathcal{L}$ is a complete lattice and is endowed with a closure system $(\_)^\bullet$. For any signature morphism $\varphi : \Sigma \to \Sigma'$ and any theory (Σ, T) we let the theory $(\Sigma', \varphi T)$ be defined by

$$(\varphi T)\rho' = \bigvee_{\varphi\rho = \rho'} T\rho.$$

Proposition 13.38. *The forgetful functor* $\Pi^\bullet : Th^\bullet \to Sig$ *lifts colimits.*

Proof. Consider a functor $D : J \to Th^\bullet$ such that $D_j = (\Sigma_j, T_j)$ for each $j \in |J|$ and let $\mu : D; \Pi^\bullet \Rightarrow \Sigma$ be a colimit cocone in *Sig*. Let

$$T = \bigvee_{j \in |J|} \mu_j T_j.$$

We prove that $\mu : D \Rightarrow (\Sigma, T)$ is a colimit cocone in $Th^\bullet$.

• First we show that for each $j \in |J|$, $\mu_j : (\Sigma_j, T_j) \to (\Sigma, T)$ is a theory morphism.

1	$\forall \rho_j \;\; T_j\rho_j \leq (\mu_j T_j)(\mu_j \rho_j)$	definition of $\mu_j T_j$
2	$\mu_j T_j \leq T$	definition of T
3	$\forall \rho_j \;\; T_j\rho_j \leq T(\mu_j \rho_j)$	1, 2

4 $T(\mu_j \rho_j) \leq T^\bullet(\mu_j \rho_j)$ *C-reflexivity*
5 $\forall \rho_j \; T_j \rho_j \leq T^\bullet(\mu_j \rho_j)$ 3, 4
6 $T_j \leq (Sen\mu_j); T^\bullet$ 5.

By Prop. 13.34 we get that $\mu : D \Rightarrow (\Sigma, T)$ is a cocone in $Th^\bullet$.

- Now let $\mu' : D \Rightarrow (\Sigma', T')$ be any cocone in $Th^\bullet$. By the colimit property of $\mu : D; \Pi^\bullet \Rightarrow \Sigma$ there exists a unique $\varphi : \Sigma \to \Sigma'$ such that $\mu; \varphi = \mu'$ (in *Sig*). It remains to prove that $\varphi : (\Sigma, T) \to (\Sigma', T')$ is a theory morphism, ie. that $T \leq (Sen\varphi); T'^\bullet$. Since $T = \bigvee_{j \in |J|} \mu_j T_j$ it is enough to prove that for each $j \in |J|$, $\mu_j T_j \leq (Sen\varphi); T'^\bullet$. This goes as follows:

1 $\forall \rho \;\; (\mu_j T_j)\rho = \bigvee_{\mu_j \rho_j = \rho} T_j \rho_j$ definition of $\mu_j T_j$
2 $\forall \rho_j \;\; T_j \rho_j \leq T'^\bullet(\mu'_j \rho_j)$ $\mu'_j : (\Sigma_j, T_j) \to (\Sigma', T')$ fuzzy theory morphism
3 $\forall \rho \;\; (\mu_j T_j)\rho \leq \bigvee_{\mu_j \rho_j = \rho} T'^\bullet(\mu'_j \rho_j)$ 1, 2
4 $\forall \rho_j$ such that $\mu_j \rho_j = \rho \quad T'^\bullet(\mu'_j \rho_j) = T'^\bullet(\varphi\rho)$ $\mu'_j = \mu_j; \varphi$
5 $\forall \rho \;\; (\mu_j T_j)\rho \leq T'^\bullet(\varphi\rho)$ 3, 4.

□

Model amalgamation. Now we establish model amalgamation properties for fuzzy theories in dependence of corresponding model amalgamation properties for the underlying signatures. The concept of model amalgamation in $\mathcal{L}$-institutions is the same as model amalgamation in binary institutions because the satisfaction relation does not play any role in this. In spite of the fact that model amalgamation at the level of fuzzy theories does involve the satisfaction relation, model amalgamation still can be reduced to ordinary model amalgamation by considering $I^\bullet$. The following result constitutes an extension of the result of Thm. 4.8 to many-valued truth.

Proposition 13.39. *Let I be an $\mathcal{L}$-institution endowed with a closure system $(\_)^\bullet$ that is lower than the Galois connection closure system. If I has (unique) J-model amalgamation then $I^\bullet$ has (unique) J-model amalgamation too.*

Proof. Let $D : J \to Th^\bullet$ be a functor. For each $j \in |J|$ let $D_j = (\Sigma_j, T_j)$. Let $\mu' : D \Rightarrow (\Sigma', T')$ be a colimit cocone. Let $\{M_j \mid j \in |J|\}$ be a D-model.

- Then $\{M_j \mid j \in |J|\}$ is a $D; \Pi^\bullet$-model too.
- Let $\mu : D; \Pi^\bullet \Rightarrow \Sigma$ be a colimit cocone. By Prop. 13.38 we lift μ to a colimit cocone $\mu : D \Rightarrow (\Sigma, T)$. Let M be the amalgamation in I of $\{M_j \mid j \in |J|\}$ with respect to μ. Then for each $j \in |J|$

1 $\forall \rho_j \;\; T_j \rho_j \leq M_j^* \rho_j$ M_j is (Σ_j, T_j)-model
2 $M_j = (Mod\mu_j)M$ M is the amalgamation of $\{M_j \mid j \in |J|\}$ with respect to μ

3	$((Mod\mu_j)M)^*\rho_j = M^*(\mu_j\rho_j)$	Satisfaction Condition
4	$T_j \leq (Sen\mu_j);M^*$	1, 2, 3
5	$T_j \leq (Sen\mu_j);(M^*)^\bullet$	4, *C-reflexivity*.

- Hence, for each $j \in |J|$, $\mu_j: (\Sigma_j, T_j) \to (\Sigma, M^*)$ is a fuzzy theory morphism, thus $\mu: D \Rightarrow (\Sigma, M^*)$ is a cocone.
- By the colimit property of $\mu: D \Rightarrow (\Sigma, T)$ it follows that there exists an unique fuzzy theory morphism $\theta: (\Sigma, T) \to (\Sigma, M^*)$ such that $\mu;\theta = \mu$. By the colimit property of μ in *Sig* it follows that $\theta = 1_\Sigma$. Hence $T \leq (M^*)^\bullet$. Since $(M^*)^\bullet \leq (M^*)^{**} = M^*$ it follows

$$T \leq M^*. \tag{13.28}$$

- Let $\varphi: (\Sigma', T') \to (\Sigma, T)$ be the isomorphism given by the colimit properties of $\mu: D \Rightarrow (\Sigma, T)$ and of $\mu': D \Rightarrow (\Sigma', T')$. Then we consider $M' = (Mod\varphi)M$. We have that

1	$T' \leq (Sen\varphi);T^\bullet$	$\varphi: (\Sigma', T') \to (\Sigma, T)$ fuzzy theory morphism
2	$T^\bullet \leq T^{**}$	$(\_)^\bullet \leq (\_)^{**}$
3	$T \leq M^*$	(13.28)
4	$T^{**} \leq M^{***} = M^*$	3, Galois connection properties of $(\_)^*$
5	$T' \leq (Sen\varphi);M^*$	1, 2, 4
6	$(Sen\varphi);M^* = M'^*$	Satisfaction Condition
7	$T' \leq M'^*$	5, 6.

- Hence M' is a (Σ', T')-model and moreover for each $j \in |J|$,

$$(Mod\mu'_j)M' = Mod(\mu_j;\varphi^{-1})M' = (Mod\mu_j)M = M_j.$$

  This shows that M' is a model amalgamation of $\{M_j \mid j \in |J|\}$.
- For the unique model amalgamation variant of this result it is enough to note that the uniqueness of M' follows from the uniqueness of M which holds by the uniqueness assumption of the model amalgamation property in $\mathcal{L}$.

□

MV-comorphisms

The extension of the institution-theoretic method of logic-by-translation to many-valued truth institutions relies on the generalisation of the binary concept of comorphism to $\mathcal{L}$-institutions. An important aspect of this extension is the possibility to change the set of truth values across $\mathcal{L}$-institutions, in other words the source and the target of a comorphism may have different $\mathcal{L}$s. In this respect the following notation will be useful: by $I[\mathcal{L}]$ we mean that I is an $\mathcal{L}$-institution.

Let $\mathcal{L}$ and $\mathcal{L}'$ be partially ordered sets and let $I[\mathcal{L}] = (Sig, Sen, Mod, \models)$ and $I'[\mathcal{L}'] = (Sig', Sen', Mod', \models')$. Then $(\Phi, \alpha, \beta, \lambda) : I[\mathcal{L}] \to I'[\mathcal{L}']$ is an *MV-comorphism* when

- $\Phi : Sig \to Sig'$ is functor (called the *signature translation functor*),
- $\alpha : Sen \Rightarrow \Phi; Sen'$ is a natural transformation (called the *sentence translation*),
- $\beta : \Phi^{\mathrm{op}}; Mod' \Rightarrow Mod$ is a natural transformation (called the *model translation*), and
- $\lambda : \mathcal{L}' \to \mathcal{L}$ is a monotone function

such that for each I-signature Σ, each Σ-sentence ρ and each $\Phi\Sigma$-model M' the following Satisfaction Condition holds:

$$\beta_\Sigma M' \models_\Sigma \rho \;=\; \lambda(M' \models'_{\Phi\Sigma} \alpha_\Sigma \rho).$$

When λ is an identity it may be omitted and then the MV-comorphism may be called an $\mathcal{L}$*-comorphism*. At the abstract level, the components Φ, α, β of an MV-comorphism do not differ from those in the definition of binary institution-theoretic comorphisms because the many-valued truth aspect does not have a presence at the level of categories of signatures, and of the sentence and model functors. Here it is important to distinguish between the abstract and the concrete level, because in the latter, of course many-valued truth is present, usually on the semantics (models) side. However, many-valued truth is present in the satisfaction relation both in the case of abstract and concrete $\mathcal{L}$-institutions.

The category of MV-comorphisms. The proofs of the following couple of propositions consist of straightforward calculations very similar to those for the binary comorphisms, so we omit them here.

Proposition 13.40 (The category of MV-comorphisms)**.** *Given MV-comorphisms*

$$(\Phi, \alpha, \beta, \lambda) : I[\mathcal{L}] \to I'[\mathcal{L}'] \;\; \textit{and} \;\; (\Phi', \alpha', \beta', \lambda') : I'[\mathcal{L}'] \to I''[\mathcal{L}''],$$

the 4-tuple

$$(\Phi;\Phi',\ \alpha;\Phi\alpha',\ \Phi\beta';\beta,\ \lambda \circ \lambda')$$

is an MV-comorphism $I[\mathcal{L}] \to I''[\mathcal{L}'']$ which is called the composition *of the two MV-comorphisms. Moreover, the composition of MV-comorphisms is associative and has identities.*

The following examples emphasise various different aspects of MV-comorphisms. The first one emphasises the natural possibility to have a proper (non-identity) λ. The second example presents a 'theoroidal' MV-comorphism, i.e. when signatures get mapped to fuzzy theories. The third example generalises to many-valued truth the binary comorphism that constitutes the foundations for regarding the ordinary logic programming paradigm as a particular form of *equational* logic programming (this can be found under the title 'Encoding relations as operations in $\mathcal{FOL}$' in Sec. 3.3).

Changing truth values. This example has the potential to be replicated successfully to many concrete $\mathcal{L}$-institutions of interest, such as $\mathcal{FEL}$. Here we develop it for the emblematic case of $\mathcal{MVL}$, this being also the simplest non-trivial example where changing truth values can be illustrated properly. As we are already aware, $\mathcal{MVL}$ represents in fact a class of $\mathcal{L}$-institutions as $\mathcal{L}$ is a parameter of $\mathcal{MVL}$. Then each homomorphism of residuated lattices $\lambda : \mathcal{L}' \to \mathcal{L}$ determines a comorphism $(\Phi, \alpha, \beta) : \mathcal{MVL}[\mathcal{L}] \to \mathcal{MVL}[\mathcal{L}']$ where

- Φ and α are identities;
- for each (S, C, P)-model M', $\beta_{(S,C,P)}M'$ only re-assigns the truth values in M' according to λ, i.e., for each π in P_w and $m \in M'_w$, $(\beta_{(S,C,P)}M')_\pi m = \lambda(M'_\pi m)$.
- Then the Satisfaction Condition of this comorphism can be proved by induction on the structure of the sentences. The base case is given by the mere definition of β, while the step case relies on the homomorphism properties of λ.

Encoding $\mathcal{FEL}$ into many-valued first order logic. Fuzzy equational logic $\mathcal{FEL}$ can be "encoded" into many-valued first-order logic under the following conditions:

- We consider also non-constant operation symbols, so now a signature is just a $\mathcal{FOL}$ signature (S, F, P). The models interpret the operation symbols ina crisp way, like $\mathcal{FOL}$ models do.
- We add crisp equality to $\mathcal{MVL}$. We may denote this extended $\mathcal{L}$-institution by $\mathcal{MVL}^=$.
- We assume that $\mathcal{L}$ is a residuated lattice such that there exists a natural number n such that for each $x \in L \setminus \{1\}$, $x^n = 0$ (where $x^0 = 1$, $x^{n+1} = x^n * x$).

Both conditions above are required by our encoding of the fuzzy equality. Prominent examples of $\mathcal{L}$ that satisfy the second condition above are the discrete Łukasiewicz residuated lattices. On the other hand, Goguen and Gödel residuated lattices do not satisfy it.

The encoding under discussion can be presented as an $\mathcal{L}$-comorphism $(\Phi, \alpha, \beta) : \mathcal{FEL} \to (\mathcal{MVL}^=)^\bullet$ where $(\_)^\bullet$ may be any closure system on $\mathcal{MVL}^=$ that is lower than the Galois connection closure system. This goes as follows.

- Let (S, F) be any $\mathcal{FEL}$-signature. Then $\Phi(S, F) = ((S, F, \approx), T(F))$ where $\approx\, = (\approx_s)_{s \in S}$ is a family of relation / predicate symbols such that the arity of $\approx_s$ is ss, and $T(F)$ is the $\mathcal{MVL}^=$ theory defined by

$$T(F)(\forall x)x \approx x \;=\; 1.$$
$$T(F)(\forall x, y)x \approx y \;\Rightarrow\; y \approx x \;=\; 1.$$
$$T(F)(\forall x, y, z)(x \approx y) * (y \approx z) \;\Rightarrow\; (x \approx z) \;=\; 1.$$
$$T(F)(\forall x, y)\underbrace{(x \approx y) * \ldots * (x \approx y)}_{\times n} \;\Rightarrow\; (x = y) \;=\; 1.$$
$$T(F)\rho \;=\; 0 \qquad \text{when } \rho \text{ is a sentence different from any of the four sentences above.}$$

The first three axioms above are obvious; they represent the encoding of the fact that $\approx$ is a fuzzy equivalence. The fourth axiom is an encoding of the reverse of the fuzzy reflexivity implication, ie., of $(x \approx y) = 1$ implies $x = y$. Since this cannot be encoded as such in $\mathcal{MVL}_1^{=}$, the fourth axiom represents a pragmatic solution to this because $\underbrace{(x \approx y) * \ldots * (x \approx y)}_{\times n}$ is 1 if and only if $(x \approx y) = 1$.

- On the atomic sentences the sentence translations are defined by $\alpha_F(t \approx t') = (t \approx t')$. Note that in this equation the symbol $\approx$ is overloaded, in its left-hand side it stands for the meta-symbol of fuzzy equality while in its right-hand side it represents the binary relation / predicate symbol in the encoded signature. Then α extends in the obvious way to quantified fuzzy equations.
- For each $(S,F,\approx)$-model M such that $M^* \geq T(F)$, its translation $\beta_{(S,F)}M$ does not achieve anything different from M. However here it is essential to note that $\approx_M$ is a fuzzy equality indeed as a consequence of $M^* \geq T(F)$.

Encoding many-valued Horn clause logic into conditional fuzzy equational logic. Let $\mathcal{HMVL}$ be the Horn clause fragment of $\mathcal{MVL}$ and let $\mathcal{CFEL}$ be the conditional extension of $\mathcal{FEL}$. This means that in both $\mathcal{HMVL}$ and $\mathcal{CFEL}$ the sentences are of the form $(\forall X)H \Rightarrow C$ where C is an atomic sentence and H is a finite conjunction of atomic sentences. We define an $\mathcal{L}$-comorphism $(\Phi,\alpha,\beta) : \ \mathcal{HMVL} \to \mathcal{CFEL}$ as follows:

- Each $\mathcal{MVL}$ signature (S,C,P) gets mapped to $\Phi(S,C,P) = (S + \{b\}, \Omega)$ where $\Omega = C + \overline{P} + \mathsf{true}$ such that $\overline{P}_{w \to s} = P_w$ is $s = b$ and $\overline{P}_{w \to s} = \emptyset$ otherwise. Thus Φ is just like in the corresponding example from Sec. 3.3.
- $\alpha_{(S,C,P)}(\pi \underline{x}) = (\pi \underline{x} \approx \mathsf{true})$ for each appropriate finite sequence $\underline{x}$ of constants. Then $\alpha_{(S,C,P)}$ extends canonically to non-atomic sentences.
- For each $\Phi(S,C,P)$-model M,
  - First $\beta_{(S,C,P)}M$ erases the interpretation of the sort b,
  - then sets the underlying set of $\beta_{(S,C,P)}M$ to M_s,
  - for each σ in C, it defines $(\beta_{(S,C,P)}M)_\sigma = M_\sigma$, and
  - for each π in P, it defines $(\beta_{(S,C,P)}M)_\pi \underline{m} = (M_\pi \underline{m} \approx_M M_{\mathsf{true}})$.

Preservation of graded semantic consequence. In the many-valued case the preservation of semantic consequence means the preservation of the degree of consequence. Like in the binary case, this is a basic expected property of MV-comorphisms for which the translation of truth values (λ) preserve the residuated lattice operations on the truth values. The following result does this, but we leave its proof as an exercise.

Proposition 13.41 (Preservation of the graded semantic consequence)**.** *Let* $(\Phi,\alpha,\beta,\lambda): \ I[\mathcal{L}] \to I'[\mathcal{L}']$ *be an MV-comorphism such that* λ *is a homomorphism of complete residuated lattices. Then for any* I*-signature* Σ*, any* $\Gamma \subseteq Sen\Sigma, \gamma \in Sen\Sigma$

$$\Gamma \models_\Sigma \gamma \ \leq \ \lambda(\alpha_\Sigma\Gamma \models'_{\Phi\Sigma} \alpha_\Sigma\gamma).$$

Conservativity. Like with binary comorphisms, in the case of MV-comorphisms, conservativity is the property that guarantees that by translating we cannot deduce more. This is crucial when doing logic-by-translation, once we establish a consequence degree we can return it to the translated (sets of) sentences. For MV-comorphisms, conservativity is expressed as an equality of truth degrees. An MV-comorphism $(\Phi,\alpha,\beta,\lambda): \ I[\mathcal{L}] \to I'[\mathcal{L}']$ is *conservative* when $\mathcal{L}$ and $\mathcal{L}'$ are complete residuated lattices and for each $\Sigma \in |Sig^I|$, $\Gamma \subseteq Sen^I\Sigma$, $\gamma \in Sen^I\Sigma$.

$$\Gamma \models_\Sigma \gamma \ = \ \lambda(\alpha_\Sigma\Gamma \models'_{\Phi\Sigma} \alpha_\Sigma\gamma).$$

Also, like in the binary situation, the most convenient way to obtain conservativity of MV-comorphisms is via the model expansion property. Itn is convenient technically at the general level, but, more importantly, it is also convenient in the sense that it commonly holds in the concrete applications. An MV-comorphism $(\Phi,\alpha,\beta,\lambda)$ has the *model expansion* property when for each Σ-model M in I there exists a $\Phi\Sigma$-model M' in I' such that $\beta_\Sigma M' = M$. The proof of the following result is straightforward and is left as exercise.

Proposition 13.42. *An MV-comorphism* $(\Phi,\alpha,\beta,\lambda): \ I[\mathcal{L}] \to I'[\mathcal{L}']$ *such that* λ *is a homomorphism of complete residuated lattices is conservative when it has the model expansion property.*

All three MV-comorphisms given as examples above can be subjects of Prop. 13.42, hence they are conservative, with a special mention for the first example where this property holds conditionally.

- Within the framework of the first example we note that the comorphism $\mathcal{MVL}[\mathcal{L}] \to \mathcal{MVL}[\mathcal{L}']$ has the model expansion property if and only if $\lambda: \ L' \to L$ is surjective.
- The MV-comorphism of the second example is conservative since the components of β are isomorphisms.
- Let (S,C,P) be any $\mathcal{MVL}$-signature and let M be any (S,C,P)-model. We define an $\Phi(S,C,P)$-model M' such that $M = \beta_{(S,C,P)}M'$ as follows.
  - For each $s \in S$, $M'_s = M_s$ and $M'_b = L$.
  - For each $s \in S$, $(\approx_{M'})_s$ is the diagonal of M'_s and on M'_b, $x \approx_{M'} y$ is 1 when $x = y$ and is $x * y$ otherwise.
  - $M'_{\mathsf{true}} = 1$.
  - For each σ in C, $M'_\sigma = M_\sigma$.
  - For each π in P, $M'_\pi \underline{m} = M_\pi \underline{m}$.

  Then it is straightforward to check that M' is well defined (for instance that $\approx$ is a many-sorted fuzzy equality) and that it serves the purpose.

Exercises

13.37. [91] **Binary theory morphisms as fuzzy theory morphisms**
If $(\_)^\bullet$ is a closure system that is higher than the Galois connection closure system $(\_)^{**}$ then there exists a functor $\Phi^\sharp : \ Th^\sharp \to Th^\bullet$ such that $\Phi^\sharp(\Sigma, E) = (\Sigma, \overline{E})$, where for each Σ-sentence ρ

$$\overline{E}\rho = \bigvee\{\kappa \mid (\rho,\kappa) \in E\}.$$

Moreover, prove that $\Phi^\sharp : \ Th^\sharp \to Th^{**}$ is a right-adjoint right-inverse to $\Phi^{**} : \ Th^{**} \to Th^\sharp$.

13.38. [91] **Limits of fuzzy theories**
The forgetful functor $\Pi^\bullet : \ Th^\bullet \to Sig$ lifts limits.

13.39. [91] **Conservative fuzzy theory morphisms**
In any $\mathcal{L}$-institution a signature morphism $\varphi : \ \Sigma \to \Sigma'$ is

- is *conservative* when $(\Gamma \models \gamma) \ = \ (\varphi\Gamma \models \varphi\gamma)$ for each $\Gamma \subseteq Sen\Sigma$, $\gamma \in Sen\Sigma$, and
- has the *model expansion* property when each Σ-model has a φ-expansion.

Show that a fuzzy theory morphism is conservative when $\varphi : \ \Sigma \to \Sigma'$ has the model expansion property and $T'^\bullet = (\varphi T)^\bullet$.

13.40. [91] **Binary flattening of MV-comorphisms**
For any MV-comorphism $(\Phi,\alpha,\beta,\lambda) : \ I[\mathcal{L}] \to I'[\mathcal{L}']$ such that the pair (λ',λ) is a Galois connection between $\mathcal{L}$ and $\mathcal{L}'$ we have that $(\Phi,\alpha^\sharp,\beta)$ is a (binary) comorphism $I^\sharp \to I'^\sharp$ where $\alpha^\sharp$ is defined by $\alpha^\sharp_\Sigma(\rho,\kappa) = (\alpha_\Sigma\rho, \lambda'\kappa)$.

13.41. [91] **Embedding $\mathcal{L}$-institutions into $\mathcal{L}$-institutions of fuzzy theories**
Consider a closure system $(\_)^\bullet$ in an $\mathcal{L}$-institution I such that it is lower than the Galois connection closure system. Then there exists a canonical $\mathcal{L}$-comorphism $(\Phi,\alpha,\beta) : \ I \to I^\bullet$. (*Hint:* The functor $\Phi : \ Sig \to Th^\bullet$ maps each signature Σ to the lowest fuzzy theory, i.e., $\Phi\Sigma = (\Sigma, \bot_\Sigma)$ where $\bot_\Sigma\rho = 0$ for each Σ-sentence ρ.)

13.42. Develop the proof of Prop. 13.41.

13.43. Develop the proof of Prop. 13.42.

Notes. The step from binary institutions to many-valued institutions is hardly new; this idea had appeared already in the early age of institution theory in the form of the so-called 'galleries' of [172]. The recently introduced 'generalized institutions' of [105] are very similar to $\mathcal{L}$-institutions, however they introduce an additional monadic structure on the sentence functor meant to model substitution systems. A fully abstract treatment of many-valued semantics appears very early in [200], however it differs form the approach of $\mathcal{L}$-institutions in two quite important aspects. One is its single-signature feature. The other is the collapse of model theory modulo elementary equivalence, which makes it unusable for the development of a proper fully abstract many-valued model theory. In other words, Pavelka's approach in [200] would correspond to an $\mathcal{L}$-institution that has only one signature Σ and also such that $|Mod\Sigma| \subseteq L^{Sen\Sigma}$. The terminology '$\mathcal{L}$-institution' is reminiscent of the terminology '$\mathcal{L}$-sets' of Goguen in [116]; in fact they have quite similar motivations.

Residuated lattices have been introduced in [243]; a survey on residuated lattices is [111]. Following Goguen's seminal work [118] they have been widely adopted in many-valued truth logic as the most prominent abstract algebraic structure for truth values. As the reader has probably already noticed we have been using 'fuzzy' and 'many-valued' in an interchangeable manner as we consider 'fuzzy' in the wide sense like in the pioneering works [118, 116, 200] or in the more recent works such as [38], etc., where the truth values are considered in a general (residuated) lattice.

It would be possible to make the definition of many-valued truth institution more general by letting the space L of truth values float by making L a component of the concept of signature (somehow in the spirit of [172]). However because of reasons of simplicity of presentation we refrain here from that kind of generalisation, which anyway may be canonically achieved from our definitions by a Grothendieck flattening in the style of [62]. The single sorted variant of $\mathcal{MVL}$ has been studied in [44]. $\mathcal{FMA}$ and $\mathcal{TL}$ have been defined in [78] and [79], respectively. $\mathcal{FEL}$ is the topic of the monograph [38].

The main idea of the flattening of $\mathcal{L}$-institutions to binary institutions, which is to consider the pairs (ρ,κ), has been present in several places in the fuzzy logic literature. In [112] the pairs (ρ,κ) are called 'signed formulas' and given the same interpretation as here.

The entailment systems of [175] or the π-institutions of [108] are $\mathcal{L}$-entailment systems when $\mathcal{L}$ is the binary Boolean algebra. Very related and important early work on abstract entailments is also due to Dana Scott [222]. The restriction of the concept of $\mathcal{L}$-entailment system to a single signature (the main implication being the absence of *translation*) is essentially the same as the 'graded consequences' of Chakraborty [40, 41]. Our $\mathcal{L}$-entailment systems come very close to the so-called 'generalized entailment systems' of [105]; however here we do not assume a monad structure at the level of the abstract syntax. But the major difference with respect to the corresponding concept from [105] occurs in the rule *transitivity* which in our definition relies upon an abstract binary operation $*$ (which in the case of a residuated lattice is its residual conjunction) rather than meet operation $\wedge$, and essential aspect that our definition shares with the graded consequences of [40, 41]. Of course, this is irrelevant when $*$ and $\wedge$ coincide, such as in the case of Heyting algebras, but it makes an important difference in the other situations. Our graded concept of semantic consequence, which subsumes the semantic consequence in binary institutions [124], appears in a disguised form in [200] within the context of Pavelka's theory of fuzzy consequence operators and in a form that is more explicitly similar to ours in [41] within the framework of 'graded consequence relations'. However the semantic frameworks of [200] and [41] are very similar but less general than ours, in both of them models being in fact fuzzy theories. The result of Thm. 13.1 has been proved in [41] but within a single signature context.

In the fuzzy sets literature terminology Σ-theories are called *L-sets*, our definition generalizing concrete concepts of fuzzy theories (e.g. [44]).

The theory of closure or consequence has been introduced by Tarski [233]. In [200] Pavleka had used Tarski's closure operators on $\mathcal{L}$-sets in order to provide a suitable concept of consequence operator for the many-valued framework. $\mathcal{L}$-closure operators extend the concept introduced in [200] to the multi-signature framework by adding the *C-translation* axiom. The Galois connection closures and the Goguen closures have been introduced in [79]. The name of the latter is motivated by the fact that it owes inspiration to Goguen's many-valued interpretation of Modus Ponens [118]. For any fuzzy theory X, the Galois connection closure X^{**} is essentially the same with the semantic consequence of X in [200]. Concepts of compactness extending those from binary institution theory to many-valued truth have been introduced in [79].

The graded consequence-theoretic connectives and the model theoretic connectives have been introduced and studied in [79] and further used in [90]. Basic sentences in $\mathcal{L}$-institutions have been introduced in [90], where the preservation results by filtered products / factors have also been developed.

Graded interpolation, graded Robinson consistency, and graded definability have been introduced and studied in [89]. It is there where the causality relations between these as known from the binary truth situation have been recovered at the many-valued truth level.

Fuzzy theory morphisms and MV-comorphisms have been introduced and studied in [91]. Colimits of fuzzy theory morphisms and model amalgamation properties are important for the modularisation systems of fuzzy logic based specification and programming languages. In [78] it was shown how $\mathcal{MVL}$ and $\mathcal{FMA}$ can be embedded conservatively in a generic abstract $\mathcal{L}$-institution, in the terminology of this chapter both embeddings being conservative $\mathcal{L}$-comorphisms.

Part IV

Applications to computing

Chapter 14

Grothendieck Institutions

Suppose we have a network of institutions that are connected through some kind of mappings, such as institution comorphisms, for instance. This may represent a *heterogeneous* logical environment that is composed from various logical systems, captured as institutions, in which some of them are related via comorphisms. In modern logic-based computing environments this situation is not uncommon because with each 'local' logical system we target optimally a particular application domain, and, moreover, the translations involved enable the communication between applications with the possibility to transfer solutions across the environment. Such heterogeneous environments offer the kind of flexibility required by the current logical distributed complexity of computing applications. These environments are technically called 'indexed institutions'.

Often it is important to be able to relate to such heterogeneous environments in a homogeneous way. For instance, when we have a single computing language that is based on such as heterogeneous environment, we still need a homogeneous module system that can aggregate software modules that may be based on different logical components of the environment. Technically, we need a *single* institution that can represent the whole logical environment (indexed institution) without collapsing data, such that the identity of each component institution is fully maintained within the 'global' institution. Furthermore, it is also important that model-theoretic properties that hold at the 'local' level can be lifted to the 'global' level. In this chapter we see how we can do this as follows.

1. The developments in this chapter evolve around one construction, called 'Grothendieck institution', which is an institution-theoretic generalisation of a category-theoretic concept that comes from algebraic geometry and is called the 'Grothendieck construction'. In the first section we introduce the Grothendieck institutions from different angles. On the one hand, everything depends on what kind of category of institutions we chose. We discuss two situations, when we work with morphisms of institutions, or else with comorphisms. On the other hand, the Grothendieck institution construction is a bottom-up side of a coin, the top-down side of the same coin being the so-called 'fibred institutions'. This is perfectly similar to the duality between fibra-

tions and Grothendieck constructions in category theory. Often, in category theory, due to technical convenience, the fibration perspective is favoured with respect to the Grothendieck construction perspective. In institution theory this is reversed because of the applications.

2. A section is dedicated to the lifting of theory co-limits and of model amalgamation from the 'local' level to the 'global' level of the Grothendieck institutions. These two themes, considered together, are recurrent in this book because of their importance in computing applications of institution theory.

3. The final section of this chapter is dedicated to the lifting of interpolation properties from 'local' to 'global' . Apart of its importance to computing, interpolation has an intrinsic importance in logic and model theory as such. We present a stunning and unlikely application of a general Grothendieck institution interpolation result, namely a Craig-Robinson interpolation result based on Craig interpolation in institutions *without* implication. We know that the standard and technically most convenient route to Craig-Robinson interpolation is through Craig interpolation plus implications (see Sec. 9.5) but this is limiting in the computing applications. For instance, due to an unique combination of good model-theoretic and computational properties, $\mathcal{HCL}$ is perhaps one of the most important computing logics. It enjoys some helpful Craig interpolation properties (see Chap. 9) but it does not have semantic implications. On the other hand, in general, Craig-Robinson interpolation is what is really needed as logical support for advanced modularisation systems.

It may be difficult to argue better in favour of the concept of institution and its abstract nature than with Grothendieck institutions. Because, as institutions, through axiomatisation and abstraction, they internalise a lot of concepts from logic and model theory as such and therefore we can have all these available and ready-to-use in spite of the deeply non-logical nature of the Grothendieck institutions. Albeit in the applications being constructed on the basis of logical institutions, Grothendieck institutions do not correspond to logical systems in the common acceptation. Furthermore, we can go in the other direction and use the model theory of Grothendieck institutions to derive interesting results in concrete logics, such as the above-mentioned Craig-Robinson interpolation property in $\mathcal{HCL}$.

The material of this chapter requires some degree of fluency with indexed categories and with fibrations. A concise introduction to these topics can be found in Sec. 2.5. Perhaps this is the chapter of this book that engages category theory in the hardest technical way. Otherwise, Sec. 14.1 requires only familiarity with material of Chap. 3, Sec. 14.2 with material from Sections 4.1, 4.2, 4.3, and Sec. 14.3 with some basic concepts from Chap. 9.

14.1 Fibred and Grothendieck institutions

This section is dedicated to the construction of the Grothendieck institutions.

1. First, we introduce the concept of 'fibred institution' through the example of many-sortedness in $\mathcal{FOL}$.
2. Then, we turn around the tables and rather than considering the fibred institutions just as institutions for which the categories of the signatures come as fibrations, we *construct* fibred institutions as 'Grothendieck institutions'. We show that these are equivalent concepts, the main difference between them being that of the perspective.
3. Fibred institutions are easier to understand when using institution morphisms. However, in terms heterogeneous logical environments, the comorphism-based Grothendieck institutions are a better alternative to the morphism-based ones. We provide the construction of comorphism-based Grothendieck institutions.
4. Finally, we show that both alternative Grothendieck constructions lead to the same result when the morphisms and the comorphisms involved come in 'adjoint pairs' satisfying a natural coherence property.

Fibred institutions

$\mathcal{FOL}$ as fibred institution. Fibred institutions arise in a variety of contexts, one of the most familiar one to us being the many-sorted logical systems. For any set S, let the institution of *S-sorted first-order logic* $\mathcal{FOL}^S = (Sig^S, Sen^S, Mod^S, \models)$ be the sub-institution of $\mathcal{FOL}$ determined by fixing the set of sort symbols to S. Thus the category of signatures Sig^S consists of all pairs (F,P) where (S,F,P) is a $\mathcal{FOL}$ signature, the morphisms of signatures in Sig^S being just the morphism of signatures φ in $\mathcal{FOL}$ which are identities on the sets S of sort symbols, i.e., $\varphi^{\mathrm{st}} = 1_S$. Then, in $\mathcal{FOL}^S$, the (F,P)-sentences, respectively models, are the $\mathcal{FOL}$ (S,F,P)-sentences, respectively models, in $\mathcal{FOL}$. The satisfaction relation between models and sentences is of course inherited from $\mathcal{FOL}$.

Fact 14.1. *Any function $u\colon\ S \to S'$ determines an institution morphism $(\Phi^u, \alpha^u, \beta^u)$: $\mathcal{FOL}^{S'} \to \mathcal{FOL}^S$ such that for each $\mathcal{FOL}^{S'}$ signature (F',P')*

- *$\Phi^u(F',P') = (F,P)$ with $F_{w\to s} = F'_{u(w)\to u(s)}$ and $P_w = P'_{u(w)}$ for each string of sort symbols $w \in S^*$ and each sort symbol $s \in S$.*
- *The canonical $\mathcal{FOL}$ signature morphism $(S,F,P) \to (S',F',P')$ thus determined by Φ^u is denoted by $\varphi^u_{(F',P')}$. Then $(\varphi^u_{(F',P')})^{\mathrm{st}} = u$ and its other components consist of identities.*
- *$\alpha^u_{(F',P')}\colon\ Sen^S(F,P) \to Sen^{S'}(F',P')$ is defined as $Sen^{\mathcal{FOL}}\varphi^u_{(F',P')}$, informally, it maps each (F,P)-sentence to itself but regarded as an (F',P')-sentence, and*
- *$\beta^u_{(F',P')}\colon\ Mod^{S'}(F',P') \to Mod^S(F,P)$ is defined as $Mod^{\mathcal{FOL}}\varphi^u_{(F',P')}$.*

The functors Φ^u have the flavour of 'reducts', somehow reminding us of the model reduct functors. The situation described by Fact 14.1 is common to all 'many-sorted' logics formalized as institutions, follows from the fact that $Sig^{\mathcal{FOL}}$ is fibred over $\mathbb{S}et$ by the projection Π of each signature to its set of sorts (defined by $\Pi(S,F,P) = S$ on signatures and $\Pi(\phi) = \phi^{\mathrm{st}}$ on signature morphisms). At this point, it is important to recall from Sec. 2.5 concepts from fibred category theory.

Fact 14.2. *The fibration $\Pi : Sig^{\mathcal{FOL}} \to \mathbb{S}et$ is split. Moreover, a $\mathcal{FOL}$ signature morphism ϕ is cartesian when ϕ^{op} and ϕ^{rl} are bijections, and $\varphi^u_{(F',P')}$ is the distinguished cartesian lifting of u for each function $u : S \to S'$ and each $\mathcal{FOL}$-signature (S',F',P').*

Fibred institutions in general. By abstracting the forgetful functor $\Pi : Sig^{\mathcal{FOL}} \to \mathbb{S}et$ above to any fibration, we can formulate the general concept of 'fibred institution' as follows. Given a category I, a *fibred institution over the base I* is a tuple $(\Pi : Sig \to I, Mod, Sen, \models)$ such that

- $\Pi : Sig \to I$ is a fibred category, and
- $(Sig, Mod, Sen, \models)$ is an institution.

Standard concepts from fibred category theory lift immediately to institutions. The fibred institution is *split* when the fibration Π is split. A *cartesian institution morphism* is an institution morphism between fibred institutions for which the signature mapping functor is a cartesian functor between the corresponding fibred categories of signatures.

Given a fibred institution $\mathcal{J} = (\Pi : Sig \to I, Mod, Sen, \models)$, for each object $i \in |I|$, the *fibre of $\mathcal{J}$ at i* is the institution $\mathcal{J}^i = (Sig^i, Mod^i, Sen^i, \models^i)$ where

- Sig^i is the fibre of Π at i, and
- Mod^i, Sen^i, and $\models^i$ are the restrictions of Mod, Sen, and respectively $\models$ to Sig^i.

By applying this terminology to the $\mathcal{FOL}$ case, we can therefore say that $\mathcal{FOL}$ is fibred over $\mathbb{S}et$ with its fibre at a set S being the institution $\mathcal{FOL}^S$ of S-sorted first order logic. The following generalizes Fact 14.1 to any fibred institution.

Proposition 14.3. *Given a fibred institution $\mathcal{J} = (\Pi : Sig \to I, Mod, Sen, \models)$, for each arrow $u \in I(i,j)$, any inverse image functor $\Phi^u : Sig^j \to Sig^i$ (with distinguished cartesian morphisms $\varphi^u_{\Sigma'} : \Phi^u\Sigma' \to \Sigma'$, $\Sigma' \in |Sig^j|$) determines a canonical institution morphism $(\Phi^u, \alpha^u, \beta^u) : \mathcal{J}^j \to \mathcal{J}^i$ between the fibres of $\mathcal{J}$, where for each signature Σ' in the fibre Sig^j at j, $\alpha^u_{\Sigma'} = Sen\varphi^u_{\Sigma'}$ and $\beta^u_{\Sigma'} = Mod\varphi^u_{\Sigma'}$.*

Proof.
- The naturalities of α^u and β^u follow directly from the way the family of distinguished cartesian morphisms $(\varphi^u_{\Sigma'})_{\Sigma' \in |Sig^j|}$ determine the functor Φ^u, and by applying the sentence functor and the model functor, respectively, to the left-hand side commutative diagram below.

$$
\begin{array}{ccc}
\Phi^u\Sigma' & \xrightarrow{\varphi^u_{\Sigma'}} & \Sigma' \\
{\scriptstyle \Phi^u\theta}\big\downarrow & & \big\downarrow{\scriptstyle \theta} \\
\Phi^u\Sigma'_1 & \xrightarrow[\varphi^u_{\Sigma'_1}]{} & \Sigma'_1
\end{array}
\qquad
\begin{array}{ccc}
Sen^i(\Phi^u\Sigma') & \xrightarrow{\alpha^u_{\Sigma'}=Sen\varphi^u_{\Sigma'}} & Sen^j\Sigma' \\
{\scriptstyle Sen^i(\Phi^u\theta)}\big\downarrow & & \big\downarrow{\scriptstyle Sen^j\theta} \\
Sen^i(\Phi^u\Sigma'_1) & \xrightarrow[\alpha^u_{\Sigma'_1}=Sen\varphi^u_{\Sigma'}]{} & Sen^j\Sigma'_1
\end{array}
$$

$$
\begin{array}{ccc}
Mod^i(\Phi^u\Sigma') & \xleftarrow{\beta^u_{\Sigma'}=Mod\varphi^u_{\Sigma'}} & Mod^j\Sigma' \\
{\scriptstyle Mod^i(\Phi^u\theta)}\big\uparrow & & \big\uparrow{\scriptstyle Mod^j\theta} \\
Mod^i(\Phi^u\Sigma'_1) & \xleftarrow[\beta^u_{\Sigma'_1}=Mod\varphi^u_{\Sigma'_1}]{} & Mod^j\Sigma'_1
\end{array}
$$

- The Satisfaction Condition for the institution morphism $(\Phi^u, \alpha^u, \beta^u)$ follows from the Satisfaction Condition of the fibred institution $\mathcal{J}$ applied for the distinguished cartesian morphisms. Consider a Σ'-model M' and a $\Phi^u\Sigma'$-sentence ρ. Then

$$
\begin{array}{rcll}
M' \models^j_{\Sigma'} \alpha^u_{\Sigma'}\rho & = & M' \models_{\Sigma'} (Sen\varphi^u_{\Sigma'})\rho & \text{definition of } \alpha^u_{\Sigma'} \\
& = & (Mod\varphi^u_{\Sigma'})M' \models_{\Phi^u\Sigma'} \rho & \text{Satisfaction Condition in } \mathcal{J} \\
& = & \beta^u_{\Sigma'}M' \models^i_{\Phi^u\Sigma'} \rho & \text{definition of } \beta^u_{\Sigma'}.
\end{array}
$$

□

The case of the Satisfaction Condition of $(\Phi^u, \alpha^u, \beta^u)$ in Prop. 14.3 shows how fibred institutions tell us that the Satisfaction Condition for institution morphisms arises as a Satisfaction Condition of an institution. Grothendieck institutions will show us the reverse relationship between the two kinds of Satisfaction Condition.

Indexed and Grothendieck institutions

'Indexed institutions' lift the concept of indexed category to institutions. 'Grothendieck institutions' lift the Grothendieck construction on categories to a construction on institutions. Here, the idea of 'lifting' can be taken quite literally as the basis of Grothendieck institutions is a Grothendieck construction on the respective indexed category of the signatures. Then the equivalence between fibred categories, on the one hand, and Grothendieck constructions, on the other hand, lifts from categories to institutions.

Indexed institutions. Given a category I of indices, an *indexed institution* $\mathcal{J}$ is a functor $\mathcal{J} : I^{\mathrm{op}} \to \mathbb{I}ns$ (the category of institution morphisms). For each index $i \in |I|$ we denote the institution $\mathcal{J}^i$ by $(Sig^i, Mod^i, Sen^i, \models^i)$ and for each index morphism $u \in I$ we denote the institution morphism $\mathcal{J}^u$ by $(\Phi^u, \alpha^u, \beta^u)$. $\mathcal{FOL}$ provides an expected example of an indexed institution (with $\mathbb{S}et$ in the role of the category of indices), denoted *fol*. Hence, $fol : \mathbb{S}et^{\mathrm{op}} \to \mathbb{I}ns$, where for each set S, $fol(S) = \mathcal{FOL}^S$ and for each $u : S \to S'$ function

$fol(u) = (\Phi^u, \alpha^u, \beta^u)$ as defined by Fact 14.1. Thye 'reduct' feeling of $(\Phi^u, \alpha^u, \beta^u)$ is related to the contravariance of *fol*. A warning about terminology: 'indexed institution' may be slightly confusing because indexed institutions are not institutions, but rather 'diagrams' of institutions.

Grothendieck institutions. Given an indexed institution $\mathcal{J} : I^{\mathrm{op}} \to \mathbb{I}ns$, its *Grothendieck institution* $\mathcal{J}^\sharp = (Sig^\sharp, Sen^\sharp, Mod^\sharp, \models^\sharp)$ is defined as follows:

1. Let $Sig : I^{\mathrm{op}} \to \mathbb{C}at$ be the indexed category mapping each index i to Sig^i and each index morphism u to Φ^u; then the category $Sig^\sharp$ of the signatures of $\mathcal{J}^\sharp$ is the Grothendieck category $Sig^\sharp$. Thus the signatures of $\mathcal{J}^\sharp$ consist of pairs $\langle i, \Sigma\rangle$ with i index and $\Sigma \in |Sig^i|$ and the signature morphisms $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ consist of index morphisms $u : i \to i'$ and signature morphisms $\varphi : \Sigma \to \Phi^u\Sigma'$.
2. The sentence functor $Sen^\sharp : Sig^\sharp \to \mathbb{S}et$ is given by
   - $Sen^\sharp\langle i, \Sigma\rangle = Sen^i\Sigma$ for each index $i \in |I|$ and signature $\Sigma \in |Sig^i|$, and
   - $Sen^\sharp\langle u, \varphi\rangle = Sen^i\varphi \,;\, \alpha^u_{\Sigma'}$ for each $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$.
3. The model functor $Mod^\sharp : (Sig^\sharp)^{\mathrm{op}} \to \mathbb{C}at$ is given by
   - $Mod^\sharp\langle i, \Sigma\rangle = Mod^i\Sigma$ for each index $i \in |I|$ and signature $\Sigma \in |Sig^i|$, and
   - $Mod^\sharp\langle u, \varphi\rangle = \beta^u_{\Sigma'} \,;\, Mod^i\varphi$ for each $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$.
4. The satisfaction relation is given, for each $i \in |I|$, $\Sigma \in |Sig^i|$, $M \in |Mod^\sharp\langle i, \Sigma\rangle|$, and $e \in Sen^\sharp\langle i, \Sigma\rangle$ by

$$M \models^\sharp_{\langle i,\Sigma\rangle} e \quad \text{if and only if} \quad M \models^i_\Sigma e.$$

The following shows that the above construction gives an institution indeed.

Proposition 14.4. *$\mathcal{J}^\sharp$ is an institution. Moreover, for each index $i \in |I|$ there exists a canonical institution morphism $(\Phi^i, \alpha^i, \beta^i) : \mathcal{J}^i \to \mathcal{J}^\sharp$ mapping any signature $\Sigma \in |Sig^i|$ to $\langle i, \Sigma\rangle \in |Sig^\sharp|$ and such that the components of α^i and β^i are identities.*

Proof. We have to prove the Satisfaction Condition of $\mathcal{J}^\sharp$. Consider a signature morphism $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$, a $\langle i', \Sigma'\rangle$-model M' and a $\langle i, \Sigma\rangle$-sentence e. Then

$$\begin{array}{rcll} M' \models^\sharp_{\langle i',\Sigma'\rangle} (Sen^\sharp\langle u, \varphi\rangle)e & = & M' \models^{i'}_{\Sigma'} \alpha^u_{\Sigma'}((Sen^i\varphi)e) & \text{definitions of } \models^\sharp \text{ and of } Sen^\sharp \\ & = & \beta^u_{\Sigma'}M' \models^i_\Sigma (Sen^i\varphi)e & \text{Satisfaction Condition for } (\Phi^u, \alpha^u, \beta^u) \\ & = & (Mod^i\varphi)(\beta^u_{\Sigma'}M') \models^i_{\Phi^u\Sigma'} e & \text{Satisfaction Condition for } \varphi \\ & = & (Mod^\sharp\langle u, \varphi\rangle)M' \models^\sharp_{\langle i',\Sigma'\rangle} e & \text{definitions of } \models^\sharp \text{ and of } Mod^\sharp. \end{array}$$

□

That Grothendieck constructions are split fibrations (first part of Prop. 2.9) extends immediately to Grothendieck institutions.

Fact 14.5. *The Grothendieck institution $\mathcal{J}^\sharp$ of an indexed institution $\mathcal{J}: I^{\mathrm{op}} \to \mathbb{I}ns$ is a split fibred institution $(\Pi: Sig^\sharp \to I, Mod^\sharp, Sen^\sharp, \models^\sharp)$, where $\Pi: Sig^\sharp \to I$ is the fibration projection from the Grothendieck category $Sig^\sharp$ to its index category.*

The other way around (corresponding to the second conclusion of Prop. 2.9) works also as expected. Cf. Prop. 14.3, each split fibred institution determines an indexed institution and consequently a Grothendieck institution. It is easy to see that the mappings from Grothendieck institutions to split fibred institutions and opposite are inverse to each other. For instance, $\mathcal{FOL}$ can be recovered as the Grothendieck institution $fol^\sharp$.

Fact 14.6. *For any category I, there exists a natural isomorphism between the category of split fibred institutions over I (with cartesian institution morphisms as arrows) and the category of I-indexed institutions (with natural transformation between the indexing functors as arrows).*

Recall from Sec. 3.3 that an institution morphism (Φ, α, β) is an *equivalence of institutions* when

- Φ is an equivalences of categories,
- α_Σ has an inverse up to semantic equivalence α'_Σ, which is natural in Σ, and
- β_Σ is an equivalence of categories, such that its inverse up to isomorphism and the corresponding isomorphism natural transformations are natural in Σ.

Because each fibred institution is equivalent to a split fibred institution, we have the following corollary (its correspondent in the realm of categories is to the last conclusion of Prop. 2.9).

Corollary 14.7. *Each fibred institution is equivalent to a Grothendieck institution.*

Comorphism-based Grothendieck institutions

Grothendieck institutions can be constructed using comorphisms instead of morphisms. Comorphism-based Grothendieck institutions may be more friendly towards some model theoretic properties than the morphism-based ones. This construction mimics the morphism-based variant, modulo reversing some directions.

Given a category I of indices, an *indexed comorphism-based institution* $\mathcal{J}$, in short called *indexed co-institution*, is a functor $\mathcal{J}: I^{\mathrm{op}} \to co\mathbb{I}ns$. (Recall that $co\mathbb{I}ns$ is the category having institutions as objects and institution comorphisms as arrows). Its Grothendieck institution $\mathcal{J}^\sharp = (Sig^\sharp, Sen^\sharp, Mod^\sharp, \models^\sharp)$ is defined as follows (where $\mathcal{J}^i = (Sig^i, Mod^i, Sen^i, \models^i)$ for each index $i \in |I|$ and $\mathcal{J}^u = (\Phi^u, \alpha^u, \beta^u)$ for $u \in I$ index morphism):

1. Its category of signatures $Sig^\sharp$ is $((Sig; (\_)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$ where $Sig: I^{\mathrm{op}} \to \mathbb{C}at$ is the *indexed* category of signatures of the indexed co-institution $\mathcal{J}$, $(\_)^{\mathrm{op}}: \mathbb{C}at \to \mathbb{C}at$ is the 'opposite' functor that reverses the directions of the arrows in a category, and $(Sig; (\_)^{\mathrm{op}})^\sharp$ is its Grothendieck category; this means that
   - signatures are pairs $\langle i, \Sigma \rangle$ for $i \in |I|$ index and $\Sigma \in |Sig^i|$, and

- signature morphisms are pairs $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ where $u \in I(i', i)$ and $\varphi \in Sig^{i'}(\Phi^u\Sigma, \Sigma')$. We get to this by the following sequence of equivalent facts:
  - $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle \in ((Sig; (\_)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$
  - $\langle u, \varphi\rangle : \langle i', \Sigma'\rangle \to \langle i, \Sigma\rangle \in (Sig; (\_)^{\mathrm{op}})^\sharp$,
  - $u \in I(i', i)$, $\varphi \in (Sig^{i'})^{\mathrm{op}}(\Sigma', \Phi^u\Sigma)$,
  - $u \in I(i', i)$, $\varphi \in Sig^{i'}(\Phi^u\Sigma, \Sigma')$.

2. Its sentence functor $Sen^\sharp : ((Sig; (\_)^{\mathrm{op}})^\sharp)^{\mathrm{op}} \to \mathbb{S}et$ is given by
   - $Sen^\sharp\langle i, \Sigma\rangle = Sen^i\Sigma$ for each index $i \in |I|$ and signature $\Sigma \in |Sig^i|$, and
   - $Sen^\sharp\langle u, \varphi\rangle = \alpha^u_\Sigma \,;\, Sen^{i'}\varphi$ for each $\langle u, \varphi\rangle : \langle i', \Sigma'\rangle \to \langle i, \Sigma\rangle$.
3. Its model functor $Mod^\sharp : (Sig; (\_)^{\mathrm{op}})^\sharp \to \mathbb{C}at$ is given by
   - $Mod^\sharp\langle i, \Sigma\rangle = Mod^i\Sigma$ for each index $i \in |I|$ and signature $\Sigma \in |Sig^i|$, and
   - $Mod^\sharp\langle u, \varphi\rangle = Mod^{i'}\varphi \,;\, \beta^u_\Sigma$ for each $\langle u, \varphi\rangle : \langle i', \Sigma'\rangle \to \langle i, \Sigma\rangle$.
4. $M \models^\sharp_{\langle i,\Sigma\rangle} e$ if and only if $M \models^i_\Sigma e$ for each $i \in |I|$, $\Sigma \in |Sig^i|$, $M \in |Mod^\sharp\langle i, \Sigma\rangle|$, and $e \in Sen^\sharp\langle i, \Sigma\rangle$.

Routine calculations similar to those of Prop. 14.4 show that:

Proposition 14.8. *The comorphism-based Grothendieck institution $\mathcal{I}^\sharp$ is indeed an institution, i.e., the Satisfaction Condition holds.*

Adjoint-indexed institutions. What si the relationship between morphism- and comorphism-based Grothendieck institutions? When do both constructions yield the same result? In what follows we will show that when the respective indexed institutions and co-institutions are in an 'adjoint duality' situation, then the respective Grothendieck institutions are isomorphic.

An indexed institution $\mathcal{I} : I^{\mathrm{op}} \to \mathbb{I}ns$ is *adjoint-indexed* when, for all $u \in I$, the institution morphisms $\mathcal{I}^u$ are adjoint morphisms (in the sense defined in Sec. 3.3). An adjoint-indexed institution $\mathcal{I} : I^{\mathrm{op}} \to \mathbb{I}ns$ is *coherent* when the adjunctions between the categories of signatures are designated, and for each composable pair of index morphisms $u : i \to i'$ and $u' : i' \to i''$, the adjunction from Sig^i to $Sig^{i''}$ corresponding to $u;u'$ is the composition of the adjunctions corresponding to u and u'.

For example, the $\mathbb{S}et$-indexed institution *fol* determined by the fibred institution $\mathcal{FOL}$ is adjoint-indexed. For each function $u : S \to S'$, let $\overline{\Phi^u} : Sig^S \to Sig^{S'}$ map each $\mathcal{FOL}^S$ signature (F, P) to the $\mathcal{FOL}^{S'}$ signature (F^u, P^u) defined by $F^u_{w'\to s'} = \uplus_{u(ws)=w's'} F_{w\to s}$ and $P^u_{w'} = \uplus_{u(w)=w'} P_w$ for each string of sort symbols $w \in S^*$ and sort symbol $s \in S$.

Fact 14.9. *$\overline{\Phi^u}$ is a left adjoint to the 'forgetful' functor $\Phi^u : Sig^{S'} \to Sig^S$.*

However, *fol* is *not* coherent because the composition of left adjoints corresponding to u and u' is only isomorphic to the left adjoint corresponding to $u;u'$. In order to achieve coherence in this example we have two solutions.

1. We either restrict the signature morphisms $\varphi : (S,F,P) \to (S',F',P')$ to those that are either injective on the sorts,
2. or else to those that preserve ad-hoc overloading, i.e. $\varphi_{w1\to s1}\sigma = \varphi_{w2\to s2}\sigma$ for $\sigma \in F_{w1\to s1} \cap F_{w2\to s2}$ and $\varphi_{w1}\pi = \varphi_{w2}\pi$ for $\pi \in P_{w1} \cap P_{w2}$ whenever the length of $w1$ and of $w2$ coincide.

In both cases we can define (F^u, P^u) by $F^u_{w'\to s'} = \cup_{u(ws)=w's'} F_{w\to s}$ and $P^u_{w'} = \cup_{u(w)=w'} P_w$, which yields the coherence. This coherence holds because of the uniqueness of ordinary union, a property that is not enjoyed by the disjoint unions. Let us denote by *foli* and *folp* the coherent adjoint-indexed institutions obtained by restricting *fol* to the injective u's and to the signature morphism preserving ad-hoc overloading, respectively.

Adjoint-indexed co-institutions are defined similarly to adjoint-indexed institutions. Notice that each adjoint-indexed institution $\mathcal{J} : I^{\mathrm{op}} \to \mathbb{I}ns$ determines an adjoint-indexed co-institution $\overline{\mathcal{J}} : (I^{\mathrm{op}})^{\mathrm{op}} \to co\mathbb{I}ns$ such that

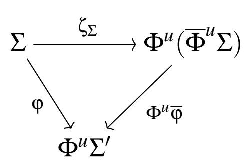
- for each index $i \in I$, $\overline{\mathcal{J}}^i = \mathcal{J}^i$, and
- for each index morphism u, $\overline{\mathcal{J}}^u$ is the comorphism adjoint to the morphism $\mathcal{J}^u$ (as given by Thm. 3.9).

Therefore the duality relation between institution morphisms and comorphisms determines a similar duality relation between adjoint-indexed institutions and adjoint-indexed coinstitutions.

The Grothendieck institution construction is invariant with respect to the duality between the concepts of institution morphism and institution comorphism:

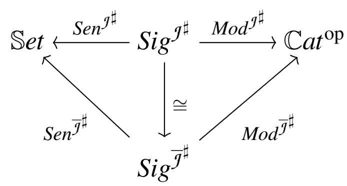
Proposition 14.10. *For each dual pair of an adjoint-indexed institution $\mathcal{J}$ and an adjoint-indexed co-institution $\overline{\mathcal{J}}$ their Grothendieck institutions $\mathcal{J}^\sharp$ and $\overline{\mathcal{J}}^\sharp$ are isomorphic.*

Proof. The isomorphism $Sig^{\mathcal{J}^\sharp} \cong Sig^{\overline{\mathcal{J}}^\sharp}$ maps each $Sig^{\mathcal{J}^\sharp}$-signature morphism $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ to the $Sig^{\overline{\mathcal{J}}^\sharp}$-signature morphism $\langle u, \overline{\varphi}\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ where $\varphi : \Sigma \to \Phi^u\Sigma'$ and $\overline{\varphi} : \overline{\Phi}^u\Sigma \to \Sigma'$ are such that $\varphi = \zeta_\Sigma \,;\, \Phi^u\overline{\varphi}$



with ζ being the unit of the adjunction between Sig^i and $Sig^{i'}$.

The conclusion of the proposition follows by the commutativity of the diagram



which is obtained by routine calculations. □

Instances of Prop. 14.10 show that the sub-institutions of $\mathcal{FOL}$ corresponding to restricting the signatures morphism to those that are injective on the sorts ($\mathcal{FOL}i$) or to those that preserve ad-hoc overloading ($\mathcal{FOL}p$) can be obtained both as morphism-based Grothendieck institutions ($foli^\sharp$ and $folp^\sharp$, respectively) or as comorphism-based ones ($\overline{foli}^\sharp$ and $\overline{folp}^\sharp$, respectively). In both cases, the morphism-based Grothendieck construction seems rather simpler and more natural than the comorphism-based one. Moreover, $\mathcal{FOL}$ can be obtained only as a morphism-based Grothendieck institution, namely $fol^\sharp$, as *fol* lacks coherence and hence does not admit a corresponding indexed co-institution.

Exercises

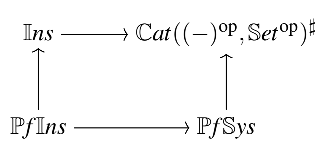
14.1. (a) The Satisfaction Condition of institution morphisms is a special case of the Satisfaction Condition of institutions. (*Hint:* For any institution morphism (Φ,α,β) consider the Grothendieck institution determined by the indexed institution $(\bullet \xrightarrow{u} \bullet) \to \mathbb{I}ns$ which maps u to (Φ,α,β).)
(b) The opposite also holds, the Satisfaction Condition of institutions is a special case of the Satisfaction Condition of institution morphisms. (*Hint:* Each institution is a trivially split fibred institution over its own category of signatures.)

14.2. (a) Let K be any 2-category and I_K be the Grothendieck 2-category for the 2-functor $\mathbb{C}at((-)^{\mathrm{op}},K)\colon\ \mathbb{C}at^* \to \mathbb{C}at$ mapping each category S to $\mathbb{C}at(S^{\mathrm{op}},K)$, and each functor Φ to $(\Phi^{\mathrm{op}};-)$ (which maps each $I\colon\ S^{\mathrm{op}} \to K$ to $\Phi^{\mathrm{op}};I$). Then the fibration $\Pi_K\colon\ I_K \to \mathbb{C}at$ creates Grothendieck constructions for each functor $\mathcal{J}\colon\ I^{\mathrm{op}} \to I_K$.
(b) Conclude that the 2-category of institutions $\mathbb{I}ns$ admits Grothendieck constructions with the Grothendieck institutions as the Grothendieck objects of $\mathbb{I}ns$. (*Hint:* $\mathbb{I}ns = I_{\mathbb{R}oom}$.)

14.3. The comorphism-based Grothendieck institutions are Grothendieck objects in the 2-category $co\mathbb{I}ns$ of institution comorphisms.

14.4. Define $\mathbb{P}f\mathbb{S}ys$, $\mathbb{R}l\mathbb{S}ys$, $co\mathbb{P}f\mathbb{S}ys$ and $co\mathbb{R}l\mathbb{S}ys$ as Grothendieck categories in the style of Fact 3.11. Consequently, establish the completeness properties for $co\mathbb{P}f\mathbb{S}ys$ and $co\mathbb{R}l\mathbb{S}ys$.

14.5. The category $\mathbb{P}f\mathbb{I}ns$ of institutions with proofs is the pullback of the category $\mathbb{I}ns$ of institutions and the category $\mathbb{P}f\mathbb{S}ys$ of proof systems over the Grothendieck category $\mathbb{C}at((-)^{\mathrm{op}},\mathbb{S}et^{\mathrm{op}})^\sharp$ of the functor $\mathbb{C}at((-)^{\mathrm{op}},\mathbb{S}et^{\mathrm{op}})\colon\ \mathbb{C}at^{\mathrm{op}} \to \mathbb{C}at$.



14.6. The category $\mathbb{P}f\mathbb{I}ns$ of institutions with proofs admits Grothendieck objects. This gives the construction for Grothendieck institutions with proofs; describe them directly.

14.2 Theory co-limits and model amalgamation

For the practical applications of Grothendieck institutions in computing, the issue of establishing model-theoretic properties that are crucial for the foundations of specification and programming is a very important one. In this section we provide sufficient pragmatic conditions for the existence of theory co-limits and of model amalgamation for Grothendieck institutions. The main idea is to lift them from the 'local' level of the indexed institutions to the 'global' level of the corresponding Grothendieck institution. For these problems we will rely on the comorphism-based variant of the Grothendieck institutions, as in this case this is technically more convenient than the morphism-based variant.

Theory co-limits

We know how in any institution co-limits get lifted from the categories of the signatures to the category of the theories (Prop. 4.2). Since Grothendieck institutions are institutions, this of course applies to Grothendieck institutions too. Hence, we have only to address the problem of co-limits of signatures in Grothendieck institutions. This is a purely categorical problem, which is already solved and the solution is well known in the context of the Grothendieck construction for categories (see Prop. 2.10). Here we adapt this solution to the context of comorphism-based Grothendieck institutions by expressing the conditions and by rephrasing the construction in this context.

Supporting co-limits. For any category J we say that an indexed co-institution $\mathcal{I}\colon\ I^{\mathrm{op}} \to co\mathbb{I}ns$ *supports* J*-co-limits* when

- the index category I is J-complete, i.e., has J-limits,
- the indexed category of signatures $Sig\colon\ I^{\mathrm{op}} \to \mathbb{C}at$ of $\mathcal{I}$ is *locally* J*-co-complete*, i.e., Sig^i has all J-co-limits for each index $i \in |J|$, and
- for each index morphism u, the comorphism $\mathcal{I}^u$ preserves J-co-limits of signatures (meaning that the corresponding sentence translation functors Φ^u preserve J-co-limits).

Theorem 14.11. *The category of theories* $Th^\sharp$ *of a comorphism-based Grothendieck institution* $\mathcal{I}^\sharp$ *has* J*-co-limits if the indexed co-institution* $\mathcal{I}$ *supports* J*-co-limits.*

Proof. By the fundamental result that in any institution the forgetful functor from theories to signatures lifts co-limits (Prop. 4.2), we have only to show that the category of the signatures of the Grothendieck institution $\mathcal{I}^\sharp$ has J-co-limits. But the category $Sig^\sharp$ of the signatures of $\mathcal{I}^\sharp$ is the opposite of the Grothendieck category $(Sig;(-)^{\mathrm{op}})^\sharp$. The conclusion of the theorem now follows immediately from the general result on the existence of limits in Grothendieck categories (Thm. 2.10, the limit part), also by noting that co-limits in Sig^i means limits in $(Sig^i)^{\mathrm{op}}$. In the following we review that construction in order to understand the nature of aggregation of 'local' signatures across indexed co-institutions.

Let J be a small category and $F\colon\ J \to Sig^\sharp$ any functor. Let $K = F;\Pi$ where $\Pi\colon\ Sig^\sharp = ((Sig;(-)^{\mathrm{op}})^\sharp)^{\mathrm{op}} \to I^{\mathrm{op}}$ is the projection that maps each $\langle i, \Sigma\rangle$ to i. Let us write $Fj = \langle K_j, \Sigma_j\rangle$ for each $j \in |J|$ and $Fu = \langle K_u, \varphi_u\rangle$ for each morphism $u \in J$.

- Any co-cone $\nu\colon\ K \Rightarrow i$ in I^{op} determines a functor $F^\nu\colon\ J \to Sig^i$ defined by $F^\nu j = \Phi^{\nu_j}\Sigma_j$ for each $j \in |J|$ and by $F^\nu u = \Phi^{\nu_{j'}}\varphi_u$ for each $u \in J(j, j')$.

$$\begin{array}{ccc} K_j & \xrightarrow{K_u} & K_{j'} \\ & {\scriptstyle \nu_j}\searrow \quad \swarrow{\scriptstyle \nu_{j'}} & \\ & i & \end{array} \qquad \begin{array}{ccc} Sig^{K_j} & \xrightarrow{\Phi^{K_u}} & Sig^{K_{j'}} \\ & {\scriptstyle \Phi^{\nu_j}}\searrow \quad \swarrow{\scriptstyle \Phi^{\nu_{j'}}} & \\ & Sig^i & \end{array} \tag{14.1}$$

We prove that F^ν is a functor indeed. For each $u \in J(j, j')$ and $u' \in J(j', j'')$,

$$\begin{aligned} F^\nu u \,;\, F^\nu u' &= \Phi^{\nu_{j'}}\varphi_u \,;\, \Phi^{\nu_{j''}}\varphi_{u'} && \text{definition of } F^\nu \\ &= \Phi^{\nu_{j''}}(\Phi^{K_{u'}}\varphi_u) \,;\, \Phi^{\nu_{j''}}\varphi_{u'} && \text{commutativity of (14.1)} \\ &= \Phi^{\nu_{j''}}(\Phi^{K_{u'}}\varphi_u \,;\, \varphi_{u'}) && \Phi^{\nu_{j''}} \text{ functor} \\ &= \Phi^{\nu_{j''}}\varphi_{u;u'} && Fu \,;\, Fu' = F(u;u') \ (F \text{ functor}) \\ &= F^\nu(u;u') && \text{definition of } F^\nu. \end{aligned}$$

- The co-limit $\mu\colon\ F \Rightarrow \langle i, \Sigma\rangle$ is defined by
  - $\mu_j = \langle \nu_j, \theta_j\rangle\colon\ Fj = \langle K_j, \Sigma_j\rangle \to \langle i, \Sigma\rangle$ where $\nu\colon\ F;\Pi = K \Rightarrow i$ is the co-limiting co-cone of $F;\Pi$, and
  - $\theta\colon\ F^\nu \Rightarrow \Sigma$ is the co-limiting co-cone of F^ν.

$$\begin{array}{ccc} K_j & & \\ {\scriptstyle K_u}\downarrow & \searrow{\scriptstyle \nu_j} & \\ K_{j'} & \xrightarrow[\nu_{j'}]{} & i \end{array} \qquad \begin{array}{ccc} F^\nu_j = \Phi^{\nu_j}\Sigma_j & & \\ {\scriptstyle F^\nu_u = \Phi^{\nu_{j'}}\varphi_u}\downarrow & \searrow{\scriptstyle \theta_j} & \\ F^\nu_{j'} = \Phi^{\nu_{j'}}\Sigma_{j'} & \xrightarrow[\theta_{j'}]{} & \Sigma \end{array} \tag{14.2}$$

- First we show that μ is a co-cone, which, under the usual notations, amounts to showing the commutativity of the following triangle for $u \in J(j, j')$.

$$\begin{array}{ccc} Fj = \langle K_j, \Sigma_j\rangle & \xrightarrow{Fu = \langle K_u, \varphi_u\rangle} & Fj' = \langle K_{j'}, \Sigma_{j'}\rangle \\ & {\scriptstyle \mu_j = \langle \nu_j, \theta_j\rangle}\searrow \quad \swarrow{\scriptstyle \mu_{j'} = \langle \nu_{j'}, \theta_{j'}\rangle} & \\ & \langle i, \Sigma\rangle & \end{array}$$

This goes as follows:

$$\begin{aligned}
\langle K_u, \varphi_u\rangle \,;\, \langle \nu_{j'}, \theta_{j'}\rangle &= \langle K_u;\nu_{j'}, \Phi^{\nu_{j'}}\varphi_u;\theta_{j'}\rangle && \text{composition in } Sig^{\mathcal{J}^\sharp} \\
&= \langle \nu_j, \theta_j\rangle && \text{commutativity of (14.1).}
\end{aligned}$$

- For any co-cone $\mu' : F \Rightarrow \langle i', \Sigma'\rangle$, let $\mu'_j = \langle \nu'_j, \theta'_j\rangle$ for each $j \in |J|$.
  - Then $\nu' : K \Rightarrow i'$ is a co-cone in I^{op}. Let $w : i \to i'$ be the unique arrow such that $\nu' = \nu; w$.
  - Because Φ^w preserves J-co-limits, $\theta\Phi^w$ is a co-limit for $F^\nu;\Phi^w$.
  - Note that $F^{\nu'} = F^\nu;\Phi^w$. Since θ' is a co-cone $F^{\nu'} \Rightarrow \Sigma'$, let $\varphi : \Phi^w\Sigma \to \Sigma'$ be the unique arrow such that $\theta\Phi^w;\varphi = \theta'$.
  - Then $\langle w, \varphi\rangle$ is the unique morphism of Grothendieck signatures $\langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ such that $\mu \,;\, \langle w, \varphi\rangle = \mu'$.

□

In the applications, the co-limits of theories in Grothendieck institutions are much more relevant than the limits. In the comorphism-based variant, co-limits in the category $Sig^\sharp$ of the signatures amount to limits in the Grothendieck construction $(Sig;(-)^{\mathrm{op}})^\sharp$, which require less stringent conditions and are simpler than the co-limits (compare the two conclusions of Prop. 2.10). This is one of the benefits of comorphism-based over the morphism-based Grothendieck institutions.

Model amalgamation

Model amalgamation in a Grothendieck institution can be treated in a manner similar to theory co-limits by reducing the problem to model amalgamation properties at the 'local' level of the component institutions and at the level of the indexed co-institution. We treat here only the semi-exactness property as it is simpler to understand than the more general forms and moreover, it is the kind of model amalgamation which is most used in the applications. The versions of model amalgamation that are weaker than semi-exactness can be handled in a similar way.

The underlying framework of model amalgamation in Grothendieck institutions is that which supports co-limits of theories. In the case of semi-exactness this means supporting pushouts. This is expected because we know from Sec. 4.3 that in general, model amalgamation is intimately related to co-limits of signatures / theories. So, on the basis of the indexed co-institution supporting pushouts, we formulate a set of three conditions that are both necessary and sufficient for semi-exactness. These three conditions parallel the three conditions that define the concept of 'supporting pushouts' by an indexed co-institution.

Semi-exactness of indexed co-institutions. An indexed co-institution $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ is *semi-exact* if and only if for each pullback

$$\begin{array}{ccc} i & \xleftarrow{u1} & j1 \\ {\scriptstyle u2}\uparrow & & \uparrow{\scriptstyle v1} \\ j2 & \xleftarrow[v2]{} & k \end{array}$$

in I and each signature Σ in I^i, the commutative square

$$\begin{array}{ccc} Mod^i\Sigma & \xleftarrow{\beta^{u1}_{\Sigma}} & Mod^{j1}(\Phi^{u1}\Sigma) \\ {\scriptstyle \beta^{u2}_{\Sigma}}\uparrow & & \uparrow{\scriptstyle \beta^{v1}_{\Phi^{u1}\Sigma}} \\ Mod^{j2}(\Phi^{u2}\Sigma) & \xleftarrow[\beta^{v2}_{\Phi^{u2}\Sigma}]{} & Mod^k(\Phi^{vi}(\Phi^{ui}\Sigma)) \end{array}$$

is a pullback.

Proposition 14.12. *If the Grothendieck institution $\mathcal{J}^\sharp$ of an indexed co-institution $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ which supports pushouts is semi-exact, then $\mathcal{J}$ is also semi-exact.*

Proof. Consider $(v1, v2)$ a pullback of $(u1, u2)$ in the index category I. By the construction of Thm. 14.11 the square

$$\begin{array}{ccc} \langle i, \Sigma\rangle & \xrightarrow{\langle u1, 1_{\Phi^{u1}\Sigma}\rangle} & \langle j1, \Phi^{u1}\Sigma\rangle \\ {\scriptstyle \langle u2, 1_{\Phi^{u2}\Sigma}\rangle}\downarrow & & \downarrow{\scriptstyle \langle v1, 1_{\Phi^{v1}(\Phi^{u1}\Sigma)}\rangle} \\ \langle j2, \Phi^{u2}\Sigma\rangle & \xrightarrow[\langle v2, 1_{\Phi^{v2}(\Phi^{u2}\Sigma)}\rangle]{} & \langle k, \Phi^{vi}(\Phi^{ui}\Sigma)\rangle \end{array}$$

is a pushout in the category of signatures $((Sig; (-)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$ of the Grothendieck institution. Because the Grothendieck institution is semi-exact, the Grothendieck model functor $Mod^\sharp$ maps this pushout square to a pullback square, which is precisely the square giving the semi-exactness of the indexed co-institution $\mathcal{J}$. □

Local semi-exactness. An indexed co-institution $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ is *locally (semi-)exact* if and only if the institution $\mathcal{J}^i$ is (semi-)exact for each index $i \in I$. The following shows that this is a necessary condition for the (semi-)exactness of the Grothendieck institution.

Proposition 14.13. *Let $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ be an indexed co-institution which supports pushouts. Then the semi-exactness of the Grothendieck institution $\mathcal{J}^\sharp$ implies the local semi-exactness of $\mathcal{J}$.*

Proof. For each index i, the model functor Mod^i is the restriction $Mod^\sharp\langle i, -\rangle$ of the model functor of the Grothendieck institution to Sig^i regarded as a sub-category of $((Sig;(-)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$, the category of signatures of the Grothendieck institution.

$$\begin{array}{ccc} (Sig^i)^{\mathrm{op}} & \longrightarrow & (Sig;(-)^{\mathrm{op}})^\sharp \\ & \searrow^{Mod^i} & \downarrow{\scriptstyle Mod^\sharp} \\ & & \mathbb{C}at \end{array}$$

Because the comorphisms $\mathcal{J}^u$ preserve the pushouts of signatures, by a simple calculation we can establish that the canonical injection $Sig^i \to ((Sig;(-)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$ preserves pushouts too. Therefore we have that Mod^i preserves pullbacks as a composition of two preserving pullback functors. □

Exactness of the institution comorphisms. Recall from Sec. 4.3 that an institution comorphism $(\Phi, \alpha, \beta)\colon\ I \to I'$ is *exact* if for each I-signature morphism $\varphi\colon\ \Sigma_1 \to \Sigma_2$ the naturality square

$$\begin{array}{ccc} Mod\Sigma_1 & \xleftarrow{\ \beta_{\Sigma_1}\ } & Mod'(\Phi\Sigma_1) \\ {\scriptstyle Mod\varphi}\uparrow & & \uparrow{\scriptstyle Mod'(\Phi\varphi)} \\ Mod\Sigma_2 & \xleftarrow[\ \beta_{\Sigma_2}\ ]{} & Mod'(\Phi\Sigma_2) \end{array}$$

is a pullback.

Proposition 14.14. *If the Grothendieck institution of an indexed co-institution $\mathcal{J}$ which supports pushouts is semi-exact, then each institution comorphism $\mathcal{J}^u = (\Phi^u, \alpha^u, \beta^u)$ is exact.*

Proof. Consider an index morphism $u\colon\ i' \to i$ and an arbitrary signature morphism $\varphi\colon\ \Sigma_1 \to \Sigma_2$ in $\mathcal{J}^i$. Then, by following the construction in Thm. 14.11, the commutative square

$$\begin{array}{ccc} \langle i, \Sigma_1\rangle & \xrightarrow{\ \langle 1_i, \varphi\rangle\ } & \langle i, \Sigma_2\rangle \\ {\scriptstyle \langle u, 1_{\Phi^u\Sigma_1}\rangle}\downarrow & & \downarrow{\scriptstyle \langle u, 1_{\Phi^u\Sigma_2}\rangle} \\ \langle i', \Phi^u\Sigma_1\rangle & \xrightarrow[\ \langle 1_{i'}, \Phi^u\varphi\rangle\ ]{} & \langle i', \Phi^u\Sigma_2\rangle \end{array}$$

is a pushout in the category of signatures of the Grothendieck institution. Because the Grothendieck institution is semi-exact, this pushout is mapped by the (Grothendieck) model functor to a pullback square, thus giving the exactness of the institution comorphism $\mathcal{J}^u$. □

The sufficient theorem. We have seen that the semi-exactness, the local semi-exactness of the indexed coinstitution, and the exactness of all its comorphisms are necessary conditions for the semi-exactness of the corresponding Grothendieck institution. The following establishes that these conditions are also sufficient.

Theorem 14.15. *Let $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ be an indexed co-institution which supports pushouts. Then the Grothendieck institution $\mathcal{J}^\sharp$ is semi-exact if and only if*

1. *the indexed co-institution $\mathcal{J}$ is locally semi-exact,*
2. *the indexed co-institution $\mathcal{J}$ is semi-exact, and*
3. *all institution comorphisms of $\mathcal{J}$ are exact.*

Proof. The 'necessary' part of this theorem holds by Propositions 14.13, 14.12, and 14.14. For the 'sufficient' part, we consider an arbitrary pushout of signatures in the Grothendieck institution

$$\begin{array}{ccc} \langle i_0, \Sigma_0\rangle & \xrightarrow{\langle u1, \varphi_1\rangle} & \langle i_1, \Sigma_1\rangle \\ {\scriptstyle\langle u2, \varphi_2\rangle}\downarrow & & \downarrow{\scriptstyle\langle v1, \theta_1\rangle} \\ \langle i_2, \Sigma_2\rangle & \xrightarrow[\langle v2, \theta_2\rangle]{} & \langle i, \Sigma\rangle \end{array} \tag{14.3}$$

The main idea behind this proof is that the pushout square (14.3) can be expressed as the following composition of four pushout squares:

$$\begin{array}{ccccc} \langle i_0, \Sigma_0\rangle & \xrightarrow{\langle u1, 1_{\Phi^{u1}\Sigma_0}\rangle} & \langle i_1, \Phi^{u1}\Sigma_0\rangle & \xrightarrow{\langle 1_{i_1}, \varphi_1\rangle} & \langle i_1, \Sigma_1\rangle \\ {\scriptstyle\langle u2, 1_{\Phi^{u2}\Sigma_0}\rangle}\downarrow & & \downarrow{\scriptstyle\langle v1, 1_{\Phi^{v1}(\Phi^{u1}\Sigma_0)}\rangle} & & \downarrow{\scriptstyle\langle v1, 1_{\Phi^{v1}\Sigma_1}\rangle} \\ \langle i_2, \Phi^{u2}\Sigma_0\rangle & \xrightarrow[\langle v2, 1_{\Phi^{v2}(\Phi^{u2}\Sigma_0)}\rangle]{} & \langle i, \Phi^{vi}(\Phi^{ui}\Sigma_0)\rangle & \xrightarrow[\langle 1_i, \Phi^{v1}\varphi_1\rangle]{} & \langle i, \Phi^{v1}\Sigma_1\rangle \\ {\scriptstyle\langle 1_{i_2}, \varphi_2\rangle}\downarrow & & \downarrow{\scriptstyle\langle 1_i, \Phi^{v2}\varphi_2\rangle} & & \downarrow{\scriptstyle\langle 1_i, \theta_1\rangle} \\ \langle i_2, \Sigma_2\rangle & \xrightarrow[\langle v2, 1_{\Phi^{v2}\Sigma_2}\rangle]{} & \langle i, \Phi^{v2}\Sigma_2\rangle & \xrightarrow[\langle 1_i, \theta_2\rangle]{} & \langle i, \Sigma\rangle \end{array} \tag{14.4}$$

Then the Grothendieck model functor

- maps the up-left pushout square to a pullback square because the indexed co-institution is semi-exact and because by the construction of co-limits of signatures in comorphism-based Grothendieck institutions, given by Thm. 14.11, we have that

$$\begin{array}{ccc} i_0 & \xleftarrow{u1} & i_1 \\ {\scriptstyle u2}\uparrow & & \uparrow{\scriptstyle v1} \\ i_2 & \xleftarrow[v2]{} & i \end{array}$$

is a pullback in the index category I.

- maps the down-right pushout square to a pullback square because the indexed co-institution is locally semi-exact, and
- maps the up-right and down-left pushout squares to pullback squares because the institution comorphisms $(\Phi^{v1}, \alpha^{v1}, \beta^{v1})$ and $(\Phi^{v2}, \alpha^{v2}, \beta^{v2})$ are exact.

Therefore, the Grothendieck model functor maps the original pushout square of signatures in the Grothendieck institution to a pullback square obtained as the composition of the four pullback squares resulting from mapping the four component pushout squares. □

Exercises

14.7. Establish the existence of pushouts of signatures in the subinstitutions of $\mathcal{FOL}$ corresponding to *foli* and *folp* as an instance of Thm. 14.11. Apply the model amalgamation Thm. 14.15 on these examples.

14.8. The institution comorphism $\mathcal{FOL} \to \mathcal{FOEQL}$ encoding relations as operations (see Sec. 3.3) preserves pushouts of signatures although it is not an adjoint institution comorphism.

14.9. The Grothendieck institution determined by the forgeful institution morphism $\mathcal{POA} \to \mathcal{FOL}$ has small co-limits.

14.10. Give a counterexample showing that even if the index category I is J-co-complete, the comorphism-based Grothendieck institution has J-co-limits of theories, and the institution comorphisms $\mathcal{I}^u$ preserve J-co-limits, the indexed co-institution $\mathcal{I}$ is not necessarily locally J-co-complete.

14.11. [62] Let $\mathcal{I}:\ I^{\mathrm{op}} \to \mathbb{I}ns$ be an adjoint-indexed (morphism-based) institution such that I is J-co-complete for a small category J, and the indexed category of signatures Sig of $\mathcal{I}$ is locally J-co-complete. Then the category of theories $Th^{\mathcal{I}^\sharp}$ of the (morphism-based) Grothendieck institution $\mathcal{I}^\sharp$ has J-co-limits.

14.12. The Grothendieck institution determined by the forgetful institution morphism from $\mathcal{POA}$ to $\mathcal{FOL}$ is *not* semi-exact due only to the failure of the exactness of the embedding institution comorphism $\mathcal{FOL} \to \mathcal{POA}$.

14.13. [62, 60] **Liberality in Grothendieck institutions**
An indexed institution $\mathcal{I}:\ I^{\mathrm{op}} \to \mathbb{I}ns$ is *locally liberal* if and only if the institution $\mathcal{I}^i$ is liberal for each index $i \in I$. The Grothendieck institution $\mathcal{I}^\sharp$ of an indexed institution $\mathcal{I}:\ I^{\mathrm{op}} \to \mathbb{I}ns$ is liberal if and only if $\mathcal{I}$ is liberal and each institution morphism $\mathcal{I}^u$ is liberal for each index morphism $u \in I$.

14.14. [73] **Grothendieck inclusion systems**
The category $\mathbb{IS}$ of inclusion systems can be endowed with a 2-categorical structure in which the 2-cells are inclusion natural transformation (i.e., such that all their components are inclusions) between inclusive functors. An adjunction in $\mathbb{IS}$ is thus just an ordinary adjunction (in $\mathbb{C}at$) such that all the components of the unit and of the counit of the adjunction are inclusions. An *enriched indexed inclusion system* is a functor $B:\ \langle I, \mathcal{E}\rangle \to \mathbb{IS}^{\mathrm{op}}$ from the opposite of the underlying category of an inclusion system 'of indices' to the category of inclusions systems and inclusive functors. An enriched indexed inclusion system is *invertible* when each inclusion system morphism B^u has

a $\mathbb{IS}$-left-adjoint $[-]^u$. It is $\mathcal{E}$-*invertible* when the $\mathbb{IS}$-left-adjoint to B^u exists for $u \in \mathcal{E}$ (and not necessarily for all index morphisms u). Show that for any $\mathcal{E}$-invertible enriched indexed inclusion system $B : \langle I, \mathcal{E} \rangle \to \mathbb{IS}^{\mathrm{op}}$ the Grothendieck category $B^\sharp$ of $B^{\mathrm{op}}; (\mathbb{IS} \to \mathbb{C}at) : \langle I, \mathcal{E} \rangle^{\mathrm{op}} \to \mathbb{C}at$ can be endowed with an inclusion system $\langle I^\sharp, \mathcal{E}^\sharp \rangle$ such that $\langle u, \varphi \rangle : \langle j, \Sigma \rangle \to \langle j', \Sigma' \rangle$ is

- *abstract inclusion* iff both u and φ are abstract inclusions, and
- *abstract surjection* iff u is abstract surjection and $\Sigma' = [\varphi(\Sigma)]^u$.

Show that the strong inclusion systems of $\mathcal{FOL}$-models and of theories (see Sect. 4.5) are instances of this Grothendieck inclusion system construction. What about the strong inclusion system of the $\mathcal{FOL}$-signatures?

14.15. [73] Show that for any invertible enriched indexed inclusion system $B : \langle I, \mathcal{E} \rangle \to \mathbb{IS}^{\mathrm{op}}$, the Grothendieck inclusion system $\langle I^\sharp, \mathcal{E}^\sharp \rangle$ (of Ex. 14.14) has unions if

- the inclusion system of indices $\langle I, \mathcal{E} \rangle$ has unions, and
- for each index j the 'local' inclusion system $B^j = \langle I^j, \mathcal{E}^j \rangle$ has unions.

14.16. [73] In addition to the conditions of Ex. 14.14 if the inclusion system of the indices $\langle I, \mathcal{E} \rangle$ is epic, $B^j = \langle I^j, \mathcal{E}^j \rangle$ is epic for each index j, and B^u are faithful for $u \in \mathcal{E}$, then the inclusion system $\langle I^\sharp, \mathcal{E}^\sharp \rangle$ defined in Ex. 14.14 is epic too.

14.17. [73] For any pair of functors $F, G : \langle I, \mathcal{E} \rangle \to \mathbb{IS}^{\mathrm{op}}$ (from the underlying category of an inclusion system $\langle I, \mathcal{E} \rangle$), a $\mathbb{IS}$-*lax natural transformation* $\mu : F \Rightarrow G$ is a lax natural transformation such that

- for any object j of $\langle I, \mathcal{E} \rangle$, the functor $\mu^j : Fj \to Gj$ is inclusive, and
- for any $u \in I$, the natural transformation μ^u is abstract inclusion (for the inclusion system of the corresponding functor category; see Ex. 4.61).

$\mathbb{IS}$-*lax co-cone* and $\mathbb{IS}$-*lax colimits*, respectively, are just lax co-cone and lax colimits, respectively, which are $\mathbb{IS}$-lax as natural transformations. Show that for any $\mathcal{E}$-invertible $\mathbb{IS}$-enriched indexed inclusion system $B : \langle I, \mathcal{E} \rangle \to \mathbb{IS}^{\mathrm{op}}$, the Grothendieck inclusion system $\langle I^\sharp, \mathcal{E}^\sharp \rangle$ defined by Ex. 14.14 is the $\mathbb{IS}$-lax co-limit of B.

14.18. [73] **Closed inclusion systems on Grothendieck categories**
Show that for any indexed category $B : \langle I, \mathcal{E} \rangle \to \mathbb{C}at^{\mathrm{op}}$ (functor from the underlying category of an inclusion system $\langle I, \mathcal{E} \rangle$ to the opposite of $\mathbb{C}at$), its Grothendieck category $B^\sharp$ admits an inclusion system such that $\langle u, \varphi \rangle : \langle j, \Sigma \rangle \to \langle j', \Sigma' \rangle$

- is abstract inclusion if and only if $u \in I$ and φ is identity, and
- is abstract surjection if and only if $u \in \mathcal{E}$.

Show that the closed inclusion systems of $\mathcal{FOL}$-signatures, of $\mathcal{FOL}$-models, and of theories (see Sect. 4.5) are instances of this general construction.

14.19. [85] **Inclusion systems via decomposition**
Given a decomposition of a stratified institution $\mathcal{S}$ such that both $\mathcal{S}^0$ and $\mathcal{B}$ are endowed with inclusion systems for their categories of models, follow the steps below for obtaining inclusion systems for the categories of models of $\mathcal{S}$.

- Find appropriate conditions for extending the respective inclusion systems from $\mathcal{B}$ to $\widetilde{\mathcal{B}}$. (*Hint:* Use the construction of Grothendieck inclusion systems of Exercise 14.14.)
- Extend this construction to $\widetilde{B}^C$.
- Aggregate the inclusion systems at the level of $\mathcal{S}^0$ and of $\widetilde{\mathcal{B}}^C$.

14.3 Interpolation

The interpolation problem for Grothendieck institutions is treated similarly to the model amalgamation problem by isolating a set of three sufficient and necessary conditions. These conditions are similar in flavor to those underlying the solution to the model amalgamation problem in Grothendieck institutions. We need the following interpolation concept for indexed co-institutions.

Interpolation squares of institution comorphisms. A commuting square of institution comorphisms

$$\begin{array}{ccc} I & \xrightarrow{(\Phi_1,\alpha_1,\beta_1)} & I_1 \\ {\scriptstyle (\Phi_2,\alpha_2,\beta_2)}\downarrow & & \downarrow{\scriptstyle (\Phi'_1,\alpha'_1,\beta'_1)} \\ I_2 & \xrightarrow[(\Phi'_2,\alpha'_2,\beta'_2)]{} & I' \end{array}$$

is a *Craig Interpolation square* if for each I-signature Σ, for each set E_1 of $\Phi_1\Sigma$-sentences and for each set E_2 of $\Phi_2\Sigma$-sentences, if $(\alpha'_1)_{\Phi_1\Sigma}E_1 \models' (\alpha'_2)_{\Phi_2\Sigma}E_2$, then there exists a set E of Σ-sentences such that $E_1 \models^{I_1} (\alpha_1)_\Sigma E$ and $(\alpha_2)_\Sigma E \models^{I_2} E_2$.

$$\begin{array}{ccc} Sen\Sigma & \xrightarrow{(\alpha_1)_\Sigma} & Sen^1(\Phi_1\Sigma) \\ {\scriptstyle (\alpha_2)_\Sigma}\downarrow & & \downarrow{\scriptstyle (\alpha'_1)_{\Phi_1\Sigma}} \\ Sen^2(\Phi_2\Sigma) & \xrightarrow[(\alpha'_2)_{\Phi_2\Sigma}]{} & Sen'(\Phi'_k(\Phi_k\Sigma)) \end{array}$$

Interpolation in Grothendieck institutions

The theorem below, giving a set of necessary and sufficient conditions for interpolation in Grothendieck institutions, involves the concept of left / right interpolation for institution comorphisms introduced in Sec. 9.6. Its proof follows the same technique like in the proof of the Grothendieck model amalgamation Thm. 14.15. The differences between the two proofs are rather superficial, in the case of model amalgamation pushout squares of signature morphisms yield pullback squares of categories of models, while in the case of interpolation they yield interpolation squares.

Theorem 14.16. *Consider an indexed co-institution* $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ *which supports pushouts, that is enhanced with*

- *fixed classes of index morphisms* $\mathcal{L}, \mathcal{R} \subseteq I$ *containing all identities, and*
- *for each index* $i \in |I|$*, fixed classes of signature morphisms* $\mathcal{L}^i, \mathcal{R}^i \subseteq Sig^i$ *containing all identities,*

such that

– *$\mathcal{L}$ and $\mathcal{R}$ are stable under pullbacks,*

– *$\Phi^u \mathcal{R}^i \subseteq \mathcal{R}^j$ for each index morphism $u\colon\ j \to i$ in $\mathcal{L}$, and*

– *$\Phi^u \mathcal{L}^i \subseteq \mathcal{L}^j$ for each index morphism $u\colon\ j \to i$ in $\mathcal{R}$.*

Let $\mathcal{L}^\sharp$ and $\mathcal{R}^\sharp$, be the classes of signature morphisms $\langle u\colon\ j \to i, \varphi\rangle$ of the Grothendieck institution such that $u \in \mathcal{L}$, $\varphi \in \mathcal{L}^j$, and $u \in \mathcal{R}$, $\varphi \in \mathcal{R}^j$, respectively. Then the Grothendieck institution $\mathcal{I}^\sharp$ has the Craig $(\mathcal{L}^\sharp, \mathcal{R}^\sharp)$-interpolation property if and only if

1. *for each $i \in |I|$, the institution $\mathcal{I}^i$ has the $(\mathcal{L}^i, \mathcal{R}^i)$-interpolation property,*

2. *each pullback square in I like below*

$$\begin{array}{ccc} \cdot & \xleftarrow{\ \mathcal{L}\ } & \cdot \\ {\scriptstyle \mathcal{R}}\big\uparrow & & \big\uparrow \\ \cdot & \longleftarrow & \cdot \end{array}$$

   determines a Craig interpolation square of institution comorphisms,

3. *for each $u\colon\ j \to i$ in $\mathcal{L}$ the institution comorphism $\mathcal{I}^u = (\Phi^u, \alpha^u, \beta^u)$ has the Craig $\mathcal{R}^i$-right interpolation property, and*

4. *for each $u\colon\ j \to i$ in $\mathcal{R}$ the institution comorphism $\mathcal{I}^u$ has the Craig $\mathcal{L}^i$-left interpolation property.*

Proof. Since the proof of this result follows the pattern of the proof of Thm. 14.15, we will not do it in details, and instead we will discuss its main steps.

- We consider an arbitrary pushout square [S] of signature morphisms in the Grothendieck institution like shown in the diagram (14.3); let us see it here also:

$$\begin{array}{ccc} \langle i_0, \Sigma_0\rangle & \xrightarrow{\langle u1, \varphi_1\rangle} & \langle i_1, \Sigma_1\rangle \\ {\scriptstyle \langle u2, \varphi_2\rangle}\big\downarrow & & \big\downarrow{\scriptstyle \langle v1, \theta_1\rangle} \\ \langle i_2, \Sigma_2\rangle & \xrightarrow[\langle v2, \theta_2\rangle]{} & \langle i, \Sigma\rangle \end{array}$$

  such that $u1 \in \mathcal{L}$, $\varphi_1 \in \mathcal{L}^{i_1}$, and $u2 \in \mathcal{R}$, $\varphi_2 \in \mathcal{R}^{i_2}$.

- The we represent that pushout square as the composition of four pushout squares like shown in diagram (14.4); let us see it here also:

$$\begin{array}{ccccc}
\langle i_0, \Sigma_0\rangle & \xrightarrow{\langle u1, 1_{\Phi^{u1}\Sigma_0}\rangle} & \langle i_1, \Phi^{u1}\Sigma_0\rangle & \xrightarrow{\langle 1_{i_1}, \varphi_1\rangle} & \langle i_1, \Sigma_1\rangle \\
{\scriptstyle \langle u2, 1_{\Phi^{u2}\Sigma_0}\rangle}\downarrow & & \downarrow{\scriptstyle \langle v1, 1_{\Phi^{v1}(\Phi^{u1}\Sigma_0)}\rangle} & & \downarrow{\scriptstyle \langle v1, 1_{\Phi^{v1}\Sigma_1}\rangle} \\
\langle i_2, \Phi^{u2}\Sigma_0\rangle & \xrightarrow[\langle v2, 1_{\Phi^{v2}(\Phi^{u2}\Sigma_0)}\rangle]{} & \langle i, \Phi^{vi}(\Phi^{ui}\Sigma_0)\rangle & \xrightarrow[\langle 1_i, \Phi^{v1}\varphi_1\rangle]{} & \langle i, \Phi^{v1}\Sigma_1\rangle \\
{\scriptstyle \langle 1_{i_2}, \varphi_2\rangle}\downarrow & & \downarrow{\scriptstyle \langle 1_i, \Phi^{v2}\varphi_2\rangle} & & \downarrow{\scriptstyle \langle 1_i, \theta_1\rangle} \\
\langle i_2, \Sigma_2\rangle & \xrightarrow[\langle v2, 1_{\Phi^{v2}\Sigma_2}\rangle]{} & \langle i, \Phi^{v2}\Sigma_2\rangle & \xrightarrow[\langle 1_i, \theta_2\rangle]{} & \langle i, \Sigma\rangle
\end{array}$$

- By using the hypotheses of the theorem we get that all four pushout square components of (14.4) are Ci squares.
- Since Ci squares compose 'horizontally' and 'vertically' (see Ex. 9.2) we get that [S] is a Ci square too. This argument completes the 'sufficient' part of the result.
- For the 'necessary' part of the result, we check that each of the four hypotheses of the theorem correspond to Ci squares in $\mathcal{J}^\sharp$, such that the respective interpolation properties express the four hypotheses. This is very similar to how the results of Propositions 14.13, 14.12, and 14.14, which combined give the 'necessary' part of the model amalgamation Thm. 14.15, have been obtained. In the case interpolation, two interpolation properties of comorphisms $\mathcal{J}^u$ correspond to the exactness property of $\mathcal{J}^u$ in Thm. 14.15 because of the left / right interpolation symmetry for comorphisms.

□

Craig-Robinson interpolation by Grothendieck interpolation

According to the result of Prop. 9.24, Craig-Robinson interpolation can be obtained from Craig interpolation when the institution has implications and it is quasi-compact (i.e., it is compact or has infinite conjunctions). The requirement on implications can be rather hard as it does not allow lifting Craig interpolation to Craig-Robinson interpolation by Prop. 9.24 in institutions such as $\mathcal{EQL}$ or $\mathcal{HCL}$. Grothendieck interpolation provides a general solution to this problem, in the form of the following result.

Theorem 14.17. *Consider a conservative institution comorphism* $(\Phi, \alpha, \beta) : I \to I'$ *and classes* $\mathcal{L}^i, \mathcal{R}^i$ *of signature morphisms in* I *such that*

1. *I and I' have pushouts of signature morphisms and Φ preserves pushout squares,*
2. *(Φ, α, β) has Craig $\mathcal{L}^i$-left interpolation,*
3. *I' has implications and is quasi-compact, and*
4. *I' has Craig $(\Phi\mathcal{L}^i, \Phi\mathcal{R}^i)$-interpolation.*

Then the institution I has Craig-Robinson $(\mathcal{L}^i, \mathcal{R}^i)$-interpolation.

Proof. The key to the proof of this theorem is that the Grothendieck institution determined by the comorphism (Φ, α, β) has Craig interpolation for pushout squares of the form

$$\begin{array}{ccc} \langle I, \Sigma \rangle & \xrightarrow{\langle 1_I, \varphi_1 \rangle} & \langle I, \Sigma_1 \rangle \\ {\scriptstyle \langle u, \Phi\varphi_2 \rangle} \downarrow & & \downarrow {\scriptstyle \langle u, \Phi\theta_1 \rangle} \\ \langle I', \Phi\Sigma_2 \rangle & \xrightarrow[\langle 1_{I'}, \Phi\theta_2 \rangle]{} & \langle I', \Phi\Sigma' \rangle \end{array} \tag{14.5}$$

where

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2} \downarrow & & \downarrow {\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array} \tag{14.6}$$

is a pushout of signature morphisms in I with $\varphi_1 \in \mathcal{L}^i$ and $\varphi_2 \in \mathbb{R}^i$.

- For this we first note that according to the construction from the proof of Thm. 14.11 the considered square of Grothendieck signature morphisms is indeed a pushout square.
- Then we apply the Grothendieck interpolation Thm. 14.16 as follows:
  - We take the category of indices to consist of two objects i and i' and one non-identity arrow u, the class of 'left' index arrows (denoted by $\mathcal{L}$ in Thm. 14.16) as $\{1_i, 1_{i'}\}$ and the class of 'right' index arrows (denoted by $\mathcal{R}$ in Thm. 14.16) as $\{1_i, 1_{i'}, u\}$.
  - We take $\mathcal{L}^{i'} = \Phi\mathcal{L}^i$ and $\mathcal{R}^{i'} = \Phi\mathcal{R}^i$.
  - I' has Craig $(\mathcal{L}^{i'}, \mathcal{R}^{i'})$-interpolation by hypothesis and I has Craig $(\mathcal{L}^i, \mathcal{R}^i)$-interpolation by the borrowing Prop. 9.31 by using the hypothesis that (Φ, α, β) has Craig $\mathcal{L}^i$-left interpolation.
  - The conditions on interpolation squares of institution comorphisms and on the right interpolation property for the comorphism are trivially fulfilled, while the condition on the left interpolation property for the comorphism is directly fulfilled by the hypothesis that (Φ, α, β) has Craig $\mathcal{L}^i$-left interpolation.

  By the conclusion of the Grothendieck interpolation Thm. 14.16 we obtain that (14.5) is indeed a Ci square. Now we proceed with the proof of the Craig-Robinson interpolation property for I.
- Consider a pushout square of signature morphisms in I as in (14.6) and $E_1 \subseteq Sen\Sigma_1$ and $E_2, \Gamma_2 \in Sen(\Sigma_2)$ such that

  1 $\quad \theta_1 E_1 \cup \theta_2 \Gamma_2 \models \theta_2 E_2$.

We have to find an interpolant $E \subseteq Sen\Sigma$ such that $E_1 \models \varphi_1 E$ and $\varphi_2 E \cup \Gamma_2 \models E_2$. As in the proof of Prop. 9.24 we may assume without loss of generality that E_2 is a singleton, i.e., consists of only one sentence (otherwise we take E to be the union of all interpolants obtained for the individual sentences). Then the original problem 1 translates to

2 $\alpha_{\Sigma'}(\theta_1 E_1) \cup \alpha_{\Sigma'}(\theta_2 \Gamma_2) \models' \alpha_{\Sigma'}(\theta_2 E_2)$ — 1, α preserves semantic consequence (Prop. 3.8)

3 $(\Phi\theta_1)(\alpha_{\Sigma_1} E_1) \cup (\Phi\theta_2)(\alpha_{\Sigma_2}\Gamma_2) \models' (\Phi\theta_2)(\alpha_{\Sigma_2} E_2)$ — 2, naturality of α.

- Because of the assumption that $E_2 = \{e\}$ is singleton and by compactness or by the existence of infinite conjunctions we may also assume that E_1 and Γ_2 are finite. Because I' has implications, let $\alpha_{\Sigma_2}\Gamma_2 \Rightarrow \alpha_{\Sigma_2} E_2$ denote the $\Phi\Sigma_2$-sentence $\gamma_1 \Rightarrow (\ldots \Rightarrow (\gamma_n \Rightarrow \alpha_{\Sigma_2} e))$ where $\alpha_{\Sigma_2}\Gamma_2 = \{\gamma_1, \ldots, \gamma_n\}$. Then

4 $\Phi\theta_1(\alpha_{\Sigma_1} E_1) \models' (\Phi\theta_2)(\alpha_{\Sigma_2}\Gamma_2 \Rightarrow \alpha_{\Sigma_2} E_2)$ — 3

which is a Grothendieck interpolation problem for the square (14.5) as follows:

5 $\langle u, \Phi\theta_1\rangle E_1 \models^\sharp \langle 1_{i'}, \Phi\theta_2\rangle(\alpha_{\Sigma_2}\Gamma_2 \Rightarrow \alpha_{\Sigma_2} E_2)$ — 4.

- Let E be an interpolant for 5. This means we have that

6 $E_1 \models^\sharp \langle 1_i, \varphi_1\rangle E$ and $\langle u, \Phi\varphi_2\rangle E \models^\sharp \alpha_{\Sigma_2}\Gamma_2 \Rightarrow \alpha_{\Sigma_2} E_2$.

- We show that E is also an interpolant for the original Craig-Robinson interpolation problem 1. Note that $E_1 \models \varphi_1 E$ is just $E_1 \models^\sharp \langle 1_i, \varphi_1\rangle E$. Hence it remains to prove:

7 $\varphi_2 E \cup \Gamma_2 \models E_2$.

This goes as follows:

8 $(\Phi\varphi_2)(\alpha_\Sigma E) \models' \alpha_{\Sigma_2}\Gamma_2 \Rightarrow \alpha_{\Sigma_2} E_2$ — second relation in 6, definition of $\models^\sharp$

9 $(\Phi\varphi_2)(\alpha_\Sigma E) = \alpha_{\Sigma_2}(\varphi_2 E)$ — naturality of α

10 $\alpha_{\Sigma_2}(\varphi_2 E) \models' \alpha_{\Sigma_2}\Gamma_2 \Rightarrow \alpha_{\Sigma_2} E_2$ — 8, 9

11 $\alpha_{\Sigma_2}(\varphi_2 E) \cup \alpha_{\Sigma_2}\Gamma_2 \models' \alpha_{\Sigma_2} E_2$ — 10

12 $\varphi_2 E \cup \Gamma_2 \models E_2$ — 11, conservative comorphism.

□

Some concrete consequences of Thm. 14.17. A straightforward way to obtain concrete CRi results from Thm. 14.17 is to rely on the left interpolation result of Prop. 9.35 for comorphisms. For example, by considering comorphisms $I \to \mathcal{FOL}$, the table of Cor. 9.14 can be also read as a table of Craig-Robinson interpolation properties as follows.

Corollary 14.18. *The following institutions have Craig-Robinson $(\mathcal{L},\mathcal{R})$-interpolation.*

institution	$\mathcal{L}$	$\mathcal{R}$
$\mathcal{EQL}$	*ie*	$**$
universal $\mathcal{FOL}$*-atoms*	*iei*	$***$
$\mathcal{HCL}$	*ie*$*$	$***$
$\mathcal{UNIV}$	*ie*$*$	$***$
$\forall\vee$	*ie*$*$	$***$

The examples of the CRi properties in Cor. 14.18 tell us again that many-sorted logics are non-trivial generalisations of their single-sorted variants. For instance, in the single-sorted situation, the encapsulation property (designated by 'e' in the place of y in (xyz)-morphisms) means that in the single-sorted version of $\mathcal{EQL}$, all (ie)-morphisms should be isomorphisms. This means that in some single-sorted logics CRi may become vacuous precisely because of the lack of many-sortedness.

One immediate consequence of the new Craig-Robinson interpolation properties given by Cor. 14.18 is that some of the definability results obtained as instances of the definability by axiomatizability Thm. 10.8 and given in Cor. 10.9 can also be obtained as instances of the definability by interpolation Thm. 10.5. However, there are many important consequences of the general result of Thm. 14.17 and its concrete consequences (such as those of Cor. 14.18), that have to do with the role played by CRi in computing. We will come back to this in Chap. 15.

Exercises

14.20. Check the details of the proof of Thm. 14.16.

14.21. For each pushout square of sets (as in the left diagram below), if $u1$ and $u2$ are injective, then its corresponding square of institution comorphisms (the right diagram below)

$$\begin{array}{ccc} S & \xrightarrow{u1} & S1 \\ {\scriptstyle u2}\downarrow & & \downarrow{\scriptstyle v1} \\ S2 & \xrightarrow[v2]{} & S' \end{array} \qquad \begin{array}{ccc} \mathcal{FOL}i^{S} & \xrightarrow{\overline{foli}^{u1}} & \mathcal{FOL}i^{S1} \\ {\scriptstyle \overline{foli}^{u2}}\downarrow & & \downarrow{\scriptstyle \overline{foli}^{v1}} \\ \mathcal{FOL}i^{S2} & \xrightarrow[\overline{foli}^{v2}]{} & \mathcal{FOL}i^{S'} \end{array}$$

is a Ci square.

Notes. The theory of (morphism-based) Grothendieck institutions developed by [62] was preceded by 'extra' theory morphisms across institution morphisms of [60] with the motivation to provide semantics for heterogeneous multi-logic specification with CafeOBJ [95]. Comorphism-based

Grothendieck institutions were defined in [185] by dualization of the morphism-based Grothendieck institutions and have been extensively used as foundations for heterogeneous specification with CASL extensions [188]. The ontology language DOL [189] (together with its tool Hets [182]) are also based on Grothendieck institutions. Heterogeneity of the institution mappings involved was also considered in [187] by a "Bi-Grothendieck" construction for an indexed structure of both institution morphisms and comorphisms. The paper [62] shows that Grothendieck institutions are just a special case of the more general concept of Grothendieck construction in an arbitrary 2-category. Cor. 14.7 extends Bénabou's result in [19] to fibred institutions.

'Globalisation' results for Grothendieck institutions have been obtained in [62] for theory co-limits, liberality, model amalgamation, and signature inclusions by following the same pattern of lifting each of these properties from the 'local' level of the indexed institution to the 'global' level of the Grothendieck institution. Although the 'globalisation' results can be immediately translated into the language of fibred institutions, the framework of indexed institutions seems to be the most appropriate for applications and for the presentations and development of these results. In the case of theory co-limits and liberality, the sufficient part of the globalisation results was first obtained in [60]. This paper had conjectured an 'if and only if' characterization of model amalgamation for extra theory morphisms, and [62] solved it. Later on, in [185] it was shown that comorphism-based Grothendieck institutions interact in a simpler and more natural way with model amalgamation. The general interpolation problem in Grothendieck institutions was solved in [67].

Chapter 15

Specification

Algebraic specification is the 'birth place' of institution theory. Of course, institution theory has also other several different scientific roots, such as universal algebra, model theory, category theory, but the motivation that triggered the theory of institutions came from algebraic specification. More precisely, it was that of an abstract uniform theory supporting the mathematical foundations for structuring software modules in logic-based computing languages, especially 'specification' languages. In some cases these can be directly executable, thus enabling formal verifications. Moreover, this executable aspect may allow for advanced forms of programming as-such.

Institution theory supports various methods for designing advanced modularisation systems, and the formal specification literature based on institution theory is rather rich and diverse. However, this was not the topic of our book, as here we developed a model theory as-such direction within institution theory. While doing this, we included and emphasised also certain topics that are relevant for computing applications. In this chapter we will show how institution-independent model theory helps with the foundations of structuring logic-based formal specifications and verifications. We will do this in a succint way that fits the size of a book chapter. The contents of the chapter is as follows:

1. We discuss the concept of logic-based formal specification and verification.
2. We discuss the relevance for logic-based specification and programming of the model theory that we have developed so far in this book.
3. Then, we define an institution-independent concept of 'structured specification' that is based on a certain specific set of building operators. These have a quite generic nature as they can express the common aggregations of specification modules, but also of programs in certain logic-based programming languages. The structured specifications can be organised as an institution in a way similar to the institution of theories. This similarity extends also to co-limits and model amalgamation, properties that are crucial in the applications. We also prove a 'normal form' result for structured specifications that provides a way to represent structured specifications as close as possible to logical theories.

4. In a subsequent section we address the modularity problem of logic-based formal verifications that can be aggregated by following the modular structure of the respective specifications. The main result is a general completeness theorem, whose applications require a number of results previously developed in this book. This provides to the theory of structured specifications a new level of flexibility and applicability. Co-limits, model amalgamation, normal forms are studies at the new level of abstraction. Moreover, the new conceptual framework allows for a study of translations of structured specifications across potentially different collections of structuring operators. This provides foundations for representations between different modularisation systems.
5. Then, we climb one more abstraction level and re-consider the concept of structured specification, this time independently of explicit collections of building operators. This means a theory of 'abstractly structured' specifications which enjoys two levels of independence, one from the base / underlying institution, and another one from particular collections of structuring operators. Much of the concepts developed previously in the context of a particular choice of structuring operators can be recovered at the new level of abstraction in a new appropriate form. In addition to that, due to its two-fold independence, the theory of 'abstractly structured' specifications supports adequate translation concepts for structured specifications, that are applicable to situations when both the base logical systems and the collections of the structuring operators differ.
6. The final section of the chapter shows how 'pre-defined types' can be approached abstractly. These are data types that are not constructed / defined by our specifications but they come with the respective system implementation, are ready-to-use, and enjoy very efficient well tailored implementations. Typical examples include various number systems, for instance the reals.

The study of this chapter requires material from Chapters 3 and 4 (excluding Sec. 4.6). Sec. 15.3 requires also knowledge of basic concepts from Chap. 9 (Craig-Robinson interpolation) and Chap. 11 (proof systems, soundness, completeness).

15.1 What is logic-based formal specification?

Specification is an important phase / stage of system development, either software or hardware.

- First, there is the requirements phase when we try to understand what we have to develop, what kind of a system, with what kind of functionality.
- Then comes the design phase, where specification plays a big role because it represents the design written in a certain language. By this representation, the design can be understood, communicated, and even reason upon. The specification language is often natural language, but this is imprecise, leads to clumsy specifications, is prone to confusion, and hardly supports any proper form of reasoning. A specification language

that is based on some formalisms may solve much of these issues. A distinguished class of formal specification languages are those based on some form of logic, and have the following general characteristics:

– The specification [of the design and functionality of a system] comes as a logical theory.

– The models of the specification are possible implementations of the design. In the case of software systems these can be programs written in a common programming language.

– The properties of the system are (semantic) consequences of the axioms in the specification. Their validity can be established if we have an adequate proof calculus. In this case, soundness is absolutely mandatory, while the completeness is highly desirable. A big advantage is when the underlying logical system has good computational properties and the proof calculus can be modelled computationally in an efficient way, such that the verification of the properties of the system appear as a form of programming.

There is a multitude of logic-based specification languages with the above characteristics, that are based on a multitude of logical systems, each of them more or less targeting specific application domains. Some of them have a higher degree of universality, other have a narrow applicability range. Often, the applicability power depends on developing good methodologies for using the respective specification language. Some of them are directly executable, thus displaying also powerful programming capabilities, others may be logically very expressive but at the cost of losing the executability. In the case of the latter, formal verifications are performed via dedicated computational tools. In all these cases, there are technical aspects that can be addressed uniformly across the multitude of logical systems at an institution-independent level, and there are technical aspects that have to be addressed specifically at the level of particular logical systems. For instance, the so-called 'specification in-the-large', meaning the programming / specification of the aggregations of specification modules, is something that can be successfully considered at an institution-independent level. Moreover, there are also 'specification in-the-small' issues – specification at the base logical level – that can be addressed abstractly. For instance, to a large extent, initial semantics falls in this category.

Structuring specifications. Modularisation is absolutely crucial in any kind of system development, from building construction to software engineering. Without a proper modular approach, the development of any complex system is doomed to failure. But it is not only that. All systems, hardware or software, have a life, they need maintenance, they are subject to evolution. At those stages, modularity appears to be even more important than at the development stage, if we can say that.

All these general considerations apply to logic-based specifications. Specification is about defining and communicating the meaning of a design accurately and with clarity. Imagine how this would be possible in the case of a list of many thousands of axioms.

This would be beyond the human capacity of understanding things. But, if a large specification is skilfully structured from components, then we can achieve the understanding of the whole specification from the understanding of its components. The components can be also reused in order to avoid duplications, and any further upgrades can be localised at the level of the relevant components. Moreover, the very structure of a specification provides an in-depth understanding of the design and of the conceptual structure of the respective system. All these arguments can be checked on the toy case of rings. In higher algebra textbooks rings are hardly introduced in a flat manner, instead we first introduce monoids, then groups, and finally rings by aggregating 'multiplicative' monoids with 'additive' commutative groups and adding the distributivity axioms. In such a way we get a good understanding of rings, which would not be achieved if we introduced rings without relying on an apriori understanding of monoids and (commutative) groups. Moreover, groups are often introduced on the basis of monoids; this is an example of reuse, as in rings the concept / structure of monoid appears twice but needs to be specified only once.

On the model-theoretic support for specification and programming

A great part of the developments in this book do support the mathematical foundations of logic-based specification and programming, even if they were not originally meant for that purpose. Even more interesting, is that the specification / programming perspective led to a more refined understanding of logic / model-theoretic concepts as-such. Let us discuss in general a few of these with the aim to get a succint view on the complex two-way relationship between model theory and specification by discussing the relevance of some concepts and topics from institution-independent model theory to specification theory and practice, on the one hand, and the imprint of the latter on the former, on the other hand.

Initial semantics. When specifying a system we begin with the specification / definition of the data types that we are going to use. The traditional way to do that is by reliance on initial semantics, which means that the semantics of the respective data type is given by the initial model of the specification. This is how (abstract) data types are defined, often through specifications in some form of equational logic. But initial semantics has another important application, that of a foundational role in logic programming (e.g. in languages such as Prolog and variants). The semantics of logic programs is given by the so-called 'Herbrand models', which are just the initial models of the logical theories represented by the respective programs. Traditionally, the underlying logic of Prolog (the classical environment for logic programming) is Horn clause logic without equality. However, the logic programming paradigm has been gradually extended also to other types of Horn clause logic. On such an initial semantics base, in Chap. 16 we will develop the foundations of logic programming in abstract institutions. The case of logic programming is emblematic for the relationship between initial semantics and good computational properties.

The issue of the existence of initial models of logical theories is not alien to mainstream model theory, but it is rather marginal there. By contrast, in this book we addressed it quite extensively, mostly at the general level by involving quasi-varieties. We showed how initial semantics is essentially a property of Horn theories, in general not to be expected beyond that. These results involved quite a lot of institution-independent model-theoretic machinery, including preservation and Birkhoff axiomatizability via categorical injectivity, diagrams, etc.

Co-limits of theories. This topic, which is recurrent in our book, is alien to mainstream model theory; we can safely say that. The aggregations of specification modules can be modelled mathematically as co-limits in a category of specifications. Most often it is about pushouts. For instance, the most common form of module aggregation is for two specification modules, let us call them SP_1 and SP_2. Very often these share some parts, let us call these SP_0. Then the aggregation SP of SP_1 and SP_2 is a pushout of the form:

$$\begin{array}{ccc} \mathrm{SP}_0 & \longrightarrow & \mathrm{SP}_1 \\ \downarrow & & \downarrow \\ \mathrm{SP}_2 & \longrightarrow & \mathrm{SP} \end{array} \tag{15.1}$$

More sophisticated is another form of module aggregation when a module SP has a a part P that acts as a parameter which can be instantiated in various ways for obtaining various more concrete instances of SP. This method, called 'parameterised specification / programming' is one of the most advanced specification / programming in-the-large method. The instantiation of parameters in specifications can also be expressed as a pushout:

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{SP} \\ {\scriptstyle v}\downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{SP}[v] \end{array} \tag{15.2}$$

($v : P \to S$ represents an instance of the parameter P.) When each specification represents a logical theory, these pushouts / co-limits represent pushouts / co-limits of theories.

Model amalgamation. This all-pervasive property of co-cones of signatures / theories, especially co-limits (most often pushouts), of signatures / theories, is implicit and invisible in conventional model theory, but explicit in institution-independent model theory. It is all-pervasive both in model theory and computing-oriented institution-theoretic developments. The awareness about it first occurred in specification studies. Only later on its role became apparent in model theory as-such developments. The specification theory meaning of model amalgamation is related to the implementations-as-models idea. If we think that the models of specifications (for the moment we think of them as models of the theory of the specification) as possible implementations of the specifications that coincide on their shared part, then they can be 'amalgamated' as an unique implementation of the aggregation of the two specifications.

Inclusion systems. Their story bears some similarity to that of model amalgamation. Mathematically they are the 'sister' of the well-established categorical concept of factorisation systems. They appeared first in specification theory out of the necessity to model imports of specification modules as morphisms $\mathrm{SP} \to \mathrm{SP}_1$ that have the characteristics of set inclusions, a crucial aspect being their *uniqueness*. There is at most one import $\mathrm{SP} \to \mathrm{SP}_1$ in the same way there is at most one set inclusion $A \subseteq B$. This is crucial in order to consider sharing of sub-modules in a proper way. Then the modules aggregation from diagram (15.1) should be more precisely considered as a pushout of inclusions:

$$\begin{array}{ccc} \mathrm{SP}_0 & \xrightarrow{\subseteq} & \mathrm{SP}_1 \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} \\ \mathrm{SP}_2 & \xrightarrow[\subseteq]{} & \mathrm{SP} \end{array}$$

Moreover, a parameterised specification $P \to \mathrm{SP}$ should be also considered as an inclusion of specifications. The result of an instantiation, such as $S \to \mathrm{SP}[v]$ in diagram (15.2) should come also as an inclusion. This explains why we have considered and studied inclusions in the categories of theories, by following the view of specifications-as-theories. Later on, authors of institution-independent model theory realised that whenever factorisation systems are necessary, the inclusion systems do the same job but in an mathematically more convenient manner. Moreover, the latter also fit better with the concrete applications. After all, the conventional concept of sub-model is not any injective homomorphism, it is an inclusive homomorphism.

Interpolation. It is difficult to find a case where the relationship between specification and model theory would be more complex or more intense than with interpolation. For the start, interpolation is a topic at the very core of logic and model theory. From the material in Chap. 9 we see that interpolation is very difficult to obtain. In a way or another, our interpolation theorems involved almost all developments preceding Chap. 9. One implication of this is that our involvement with interpolation in this chapter relies on all those developments. What does that mean? That the foundations of logic-based specification and verification are based on *much in-depth* model theory.

On the other hand, it is quite easy to see that it was precisely the specification theory that liberated the concept of interpolation from the rather primitive view characteristic to traditional concrete logic. Now we think of interpolation in terms of arbitrary signature morphisms and sets of sentences. Reasons for both aspects were already discussed in Chap. 9. In this chapter we will get an even better understanding of these aspects.

Institutions. The concept of institution in itself illustrates the great impact of specification theory on model theory. All attempts from within logic as-such to develop an axiomatic treatment of model theory had partially failed due to lack of enough abstraction and relativity. It is now clear that a proper axiomatic model theory required the broader view rooted in the practice of logic-based specification.

The argument above belongs to the sphere of abstraction. On the concrete side, the practice of specification brought a more refined understanding even of the common logical systems, including first-order logic. First of all, for us, the default variant of first-order logic is the many-sorted one and we now understand well that many-sorted first-order logic is a non-trivial extension of traditional (single sorted) first-order logic, as witnessed by interpolation (this relates to our next argument). Then, in this book, we also consider signature morphisms that collapse syntactic entities, an idea completely alien to conventional logic, but very relevant in specification logics. Moreover, the mere exercise of properly capturing first-order logic as a mathematical object – as an institution – leads to non-trivial re-considerations of fundamental concepts, such as variables. And to get the math of the variables fixed we relied on ideas from the practice of algebraic specification, actually from the practice of implementing specification languages.

15.2 Structured specifications

The aim of this section is to introduce the main concept of this chapter, namely that of a structured specification. This includes the development of their most fundamental properties as an extension of corresponding properties for theories (as unstructured specifications, theories can be seen as a particular case of structured specifications), as follows:

- The institution of the structured specifications over an institution I supersedes the concept of institution of theories I^{th}.
- Co-limits of specifications can be obtained from co-limits of signatures in a way that parallels the lifting of co-limits of signatures to co-limits of theories.
- In the same way we obtain model amalgamation for structured specifications.
- Finally, we develop a way to express structured specifications in terms of theories that are semantically equivalent to the respective structured specification.

The intersection-union square of signatures. For the purpose of this chapter we assume that the category of the signatures of the base institution I comes equipped with an inclusion system such that the partial order of its abstract inclusions, has least upper bounds called *unions*. Given two signatures Σ_1 and Σ_2 in I let $\Sigma_1 \cup \Sigma_2$ denote their union. Then their *intersection* $\Sigma_1 \cap \Sigma_2$ is defined as the unique pullback square such that $\Sigma_1 \cap \Sigma_2 \hookrightarrow \Sigma_1$ and $\Sigma_1 \cap \Sigma_2 \hookrightarrow \Sigma_2$ are inclusions.

$$\begin{array}{ccc} \Sigma_1 \cap \Sigma_2 & \xrightarrow{\subseteq} & \Sigma_1 \\ {\scriptstyle\subseteq}\big\downarrow & & \big\downarrow{\scriptstyle\subseteq} \\ \Sigma_2 & \xrightarrow[\subseteq]{} & \Sigma_1 \cup \Sigma_2 \end{array}$$

Assuming the existence of pullbacks of inclusions in the wider category of signatures, the existence and the uniqueness of this pullback square can be shown quite easily (Ex. 4.57).

Although it holds in many concrete situations of interest, it does not follow in general that such intersection-union squares are pushouts too. Since this property is necessary in the applications, in this chapter we assume it as an axiom:

The intersection-union squares of signatures are pushout squares.

Structured specifications. Given an institution I endowed with an inclusion system enjoying the intersection-union axiom above, its *structured specifications* (or just *specifications* for short) are defined from the finite theories by iteration of several specification building operators from a such fixed collection. So, the concept of structured specification is 'parameterised' by the respective collection of operators. Specifications are expressions / terms built with these operators. The semantics of each specification SP is given by its *signature* $sig[\mathrm{SP}]$ and its category of *models* $Mod[\mathrm{SP}]$. We define only the class of objects for each $Mod[\mathrm{SP}]$, the category $Mod[\mathrm{SP}]$ being the corresponding full subcategory of $Mod(sig[\mathrm{SP}])$. In software system engineering, the class $Mod[\mathrm{SP}]$ of models of a specification SP of a system is interpreted as the class of all possible implementations of that system. From the many collections of specification building operators, that make sense, here we use what appears to be the most common such collection. The concepts and results developed for this collection can be transferred easily to other collections which at least cover the effects of the former three operators below. In Sec. 15.4 we will develop an abstract approach that deals uniformly with this situation, free from any commitment to a certain concrete collection of specification building operators.

BASIC. Each finite theory (Σ, E) is a specification such that

$sig[(\Sigma, E)] = \Sigma$, and

$Mod[(\Sigma, E)] = Mod^I(\Sigma, E)$.

UNION. For any specifications SP_1 and SP_2 we can take their *union* $\mathrm{SP}_1 \cup \mathrm{SP}_2$ with

$sig[\mathrm{SP}_1 \cup \mathrm{SP}_2] = sig[\mathrm{SP}_1] \cup sig[\mathrm{SP}_2]$, and

$|Mod[\mathrm{SP}_1 \cup \mathrm{SP}_2]| = \{M \in Mod^I(sig[\mathrm{SP}_1 \cup \mathrm{SP}_2]) \mid M\upharpoonright_{sig[\mathrm{SP}_i]} \in Mod[\mathrm{SP}_i],\ i = 1, 2\}$.

TRANS. For any specification SP and signature morphism $\varphi\colon\ sig[\mathrm{SP}] \to \Sigma'$ we can take its *translation along* φ denoted by $\mathrm{SP} \star \varphi$ and such that

$sig[\mathrm{SP} \star \varphi] = \Sigma'$, and

$|Mod[\mathrm{SP} \star \varphi]| = \{M' \in Mod^I\Sigma' \mid M'\upharpoonright_{\varphi} \in Mod[\mathrm{SP}]\}$.

DERIV. For any specification SP' and any signature morphism $\varphi\colon\ \Sigma \to sig[\mathrm{SP}']$ we can take its *derivation along* φ denoted by $\varphi \Box \mathrm{SP}'$ and such that

$sig[\varphi \Box \mathrm{SP}'] = \Sigma$, and

$|Mod[\varphi \Box \mathrm{SP}']| = \{M'\upharpoonright_{\varphi} \mid M' \in Mod[\mathrm{SP}']\}$.

The significance of BASIC is that any specification structuring process starts from the unstructured specifications. The operator UNION represents the most common way to aggregate specifications. The operator TRANS does renamings of syntactic entities. This is useful to avoid name clashes, to provide new names that are more suggestive in particular contexts, to add new syntactic entities, and to interpret parameters. Translations need not be surjective or injective. The operator DERIV is mostly used to 'hide' auxiliary syntactic entities and involve a signature morphism which is an inclusion. It adds expressive power to the specification process, as the following simple example illustrates. Let MON be the $\mathcal{EQL}$ theory of monoids and φ be the inclusion of the signature containing only the binary $(+)$ into $sig[\mathrm{MON}]$. This means that φ just adds the neutral (zero) constant. Then $Mod[\varphi\Box\mathrm{MON}]$ is the class of all monoids but without a syntax for denoting the zero element. This is not axiomatizable in $\mathcal{EQL}$ because it is not a variety in $Mod(sig[\varphi\Box\mathrm{MON}])$ since the empty semigroup is a sub-model of the natural numbers (with $+$ interpreted as addition) but is outside of $Mod[\varphi\Box\mathrm{MON}]$. This is the situation in $\mathcal{EQL}$. If we climb to a more expressive logical system, such as $\mathcal{FOL}$, we get $Mod[\varphi\Box\mathrm{MON}]$ axiomatised by the associativity of $+$ and by the axiom $(\exists x)(\forall y)(x+y=y)\wedge(y+x=y)$.

Equivalent specifications. Given two specifications SP_1 and SP_2 we let $\mathrm{SP}_1 \models \mathrm{SP}_2$ denote the situation when $sig[\mathrm{SP}_1] = sig[\mathrm{SP}_2]$ and $Mod[\mathrm{SP}_1] \subseteq Mod[\mathrm{SP}_2]$. The specifications are *equivalent*, denoted $\mathrm{SP}_1 \models\!\!\dashv \mathrm{SP}_2$, when $\mathrm{SP}_1 \models \mathrm{SP}_2$ and $\mathrm{SP}_2 \models \mathrm{SP}_1$. In general, it is possible to have specifications that are different and yet equivalent, for example $\mathrm{SP} \cup \mathrm{SP} \models\!\!\dashv \mathrm{SP}$. When we are interested only in the semantics of specifications rather than in the way they are constructed, it does make sense to consider specifications modulo this equivalence relation.

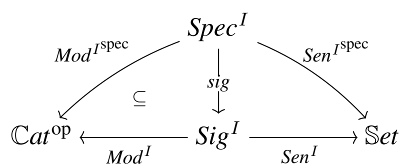
The institution of the structured specifications. The structured specifications can be organised as an institution very much in the same way of I^{th}, based on the concept of specification morphism. If we ignore the issue of the finiteness of the basic specifications as theories, then the institution of the structured specifications appears as an extension of I^{th}. A *morphism of (structured) specifications* $\varphi\colon \mathrm{SP}_1 \to \mathrm{SP}_2$ between specifications SP_1 and SP_2 is a signature morphism $\varphi\colon sig[\mathrm{SP}_1] \to sig[\mathrm{SP}_2]$ such that $M\upharpoonright_{\varphi} \in Mod[\mathrm{SP}_1]$ for each $M \in Mod[\mathrm{SP}_2]$.

Fact 15.1. *For any institution I, the specifications and their morphisms under the composition inherited from the category of the signatures, form a category, denoted $Spec^I$.*

Note that Mod can be therefore regarded as a functor $Mod\colon Spec^I \to \mathbb{C}at^{\mathrm{op}}$. This is a starting point for conceiving an institution where the structured specifications play the role of the signatures.

- An SP-*sentence* for a specification SP is any $sig[\mathrm{SP}]$-sentence; this determines a functor $Spec^I \to \mathbb{S}et$.
- A model $M \in Mod[\mathrm{SP}]$ *satisfies* a SP-sentence ρ if and only if $M \models_{sig[\mathrm{SP}]} \rho$ in the original institution.

Fact 15.2. *Given an institution I, the structured specifications together with their models, sentences, and the satisfaction between them form an institution $(Spec^I,\ sig;Sen^I,\ Mod[\_], \models)$, denoted I^{spec}. Moreover, there exists a 'forgetful' institution morphism $(sig, 1, \subseteq) : \ I^{\text{spec}} \to I$ (as illustrated by the diagram below).*



The institution I will be referred to as the 'base institution' while I^{spec} as the 'institution of the (structured) specifications'.

Co-limits of specifications. Many structuring constructs in actual specification languages rely on co-limits of specifications, especially pushouts. A typical example is that of the instantiation of parameterized specifications. Co-limits of signatures can be lifted to specifications in a manner similar to that of Prop. 4.2 lifting co-limits from signatures to theories.

Proposition 15.3. *In any institution I, the signature functor $sig : \ Spec^I \to Sig^I$ from the specifications to signatures lifts finite co-limits.*

Proof. We prove this result for the particular case of pushouts, the same argument working as well for any finite co-limit. Consider any span of specification morphisms $\phi : \ \mathrm{SP} \to \mathrm{SP}'$, $\varphi : \ \mathrm{SP} \to \mathrm{SP}_1$ and take a pushout of signatures like below

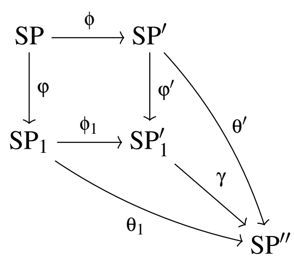
$$\begin{array}{ccc} sig[\mathrm{SP}] & \xrightarrow{\ \phi\ } & sig[\mathrm{SP}'] \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi'} \\ sig[\mathrm{SP}_1] & \xrightarrow[\ \phi_1\ ]{} & \Sigma'_1 \end{array} \tag{15.3}$$

- We define $\mathrm{SP}'_1 = (\mathrm{SP}' \star \varphi') \cup (\mathrm{SP}_1 \star \phi_1)$.
- Then $\varphi' : \ \mathrm{SP}' \to \mathrm{SP}'_1$ and $\phi_1 : \ \mathrm{SP}_1 \to \mathrm{SP}'_1$ are specification morphisms. Let us see how this works for φ', for ϕ_1 the argument being similar. Let $M'_1 \in Mod[\mathrm{SP}'_1]$. Then

1	$M'_1 \in Mod[\mathrm{SP}' \star \varphi']$	definition of SP'_1, semantics of UNION
2	$M'_1 \restriction_{\varphi'} \in Mod[\mathrm{SP}']$	1, semantics of TRANS.

- We prove that φ', ϕ_1 defines a pushout for ϕ, φ in $Spec^I$. Consider $\theta' : \ \mathrm{SP}' \to \mathrm{SP}''$ and

$\theta_1 : \mathrm{SP}_1 \to \mathrm{SP}''$ specification morphisms.



- By the pushout property for signatures there exists a unique signature morphism $\gamma\colon\ sig[\mathrm{SP}'_1] \to sig[\mathrm{SP}'']$ such that $\varphi';\gamma = \theta'$ and $\phi_1;\gamma = \theta_1$.
- It remains to show that γ is a specification morphism. Let $M'' \in Mod[\mathrm{SP}'']$. Then

1	$M''\upharpoonright_\gamma\upharpoonright_{\varphi'} = M''\upharpoonright_{\theta'} \in Mod[\mathrm{SP}']$	$\varphi';\gamma = \theta'$, $M'' \in Mod[\mathrm{SP}'']$, θ' specification morphism
2	$M''\upharpoonright_\gamma\upharpoonright_{\phi_1} = M''\upharpoonright_{\theta_1} \in Mod[\mathrm{SP}_1]$	$\phi_1;\gamma = \theta_1$, $M'' \in Mod[\mathrm{SP}'']$, θ_1 specification morphism
3	$M''\upharpoonright_\gamma \in Mod[\mathrm{SP}' \star \varphi']$	1, semantics of TRANS
4	$M''\upharpoonright_\gamma \in Mod[\mathrm{SP}_1 \star \phi_1]$	2, semantics of TRANS
5	$M''\upharpoonright_\gamma \in Mod[\mathrm{SP}'_1]$	3, 4, definition of SP'_1, semantics of UNION.

□

Note a slight difference between Prop. 15.3 and the lifting of co-limits from signatures to theories: the former lifting of co-limits is limited to the finite. The explanation for this limitation is that the structured specifications have a finitary nature because the basic specifications are finite theories, on the one hand, and the structured specifications are finitary terms / expressions, on the other hand.

Model amalgamation for specifications. An immediate but important consequence of the construction of co-limits of specifications given by Prop. 15.3 is that model amalgamation properties also lift from signatures to specifications. This works very similar to the lifting of model amalgamation from signatures to theories. Prop. 15.4 can be seen as an extension of the corresponding result for theories. Below we formulate this for ordinary model amalgamation, but it can be replicated easily to other forms of model amalgamation such as weak model amalgamation or semi-exactness.

Proposition 15.4. *The institution of specifications I^{spec} has model amalgamation whenever the base institution I has model amalgamation.*

Proof. We do this for pushout squares of specification morphisms; as for other finite co-limits it goes the same way. We may also recall from basic category that any finite co-limit can be expressed in terms of pushouts and initial objects. For any pushout of specification morphisms like below

$$\begin{array}{ccc} \mathrm{SP} & \xrightarrow{\phi} & \mathrm{SP}' \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi'} \\ \mathrm{SP}_1 & \xrightarrow[\phi_1]{} & \mathrm{SP}'_1 \end{array}$$

and any SP'-model M' and any SP_1-model M_1 such that $M' \upharpoonright_{\phi} = M_1 \upharpoonright_{\varphi}$ we consider their unique amalgamation M'_1 in $sig[\mathrm{SP}'_1]$, given by the amalgamation property of the base institution (digagram (15.3) being a pushout square). We prove that $M'_1 \in Mod[\mathrm{SP}'_1]$.

- Based on the proof of Prop. 15.3, without any loss of generality, we consider $\mathrm{SP}'_1 = (\mathrm{SP}' \star \varphi') \cup (\mathrm{SP}_1 \star \phi_1)$ (isomorphic specifications are semantic equivalent).
- Since $M'_1 \upharpoonright_{\varphi} = M' \in Mod[\mathrm{SP}']$ and $M'_1 \upharpoonright_{\phi_1} = M_1 \in Mod[\mathrm{SP}_1]$ we have that $M'_1 \in Mod[\mathrm{SP}' \star \varphi'] \cap Mod[\mathrm{SP}_1 \star \phi_1]$, hence $M'_1 \in Mod[\mathrm{SP}'_1]$.

□

The general practical meaning of this result is that any implementation of a structured specification SP can be obtained by iterative aggregations of implementations of the components of SP by following its structure.

Normal forms of structured specifications

The structuring of specifications has two main purposes. The obvious one is the modular building of specifications. The other power of the structuring of specification is often downplayed: some collections of structuring operators allow for significantly more specification power in the sense that with structured specifications we can specify classes of models that are not axiomatizable, in other words beyond what finite theories can specify. In what follows we study the relationship between specification in-the-large (with structured specifications) and in-the-small (by finite theories) strictly from the point of view of their specification power, more precisely when and to what extent the latter can substitute the former. We will do this for the collection of the four operators introduced above.

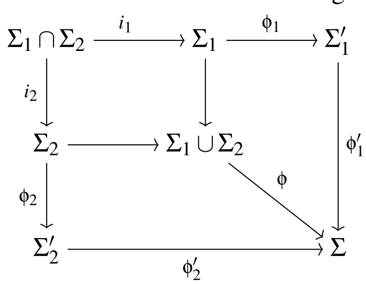
In order to keep notations simpler, and without losing the full generality of our discussion, let us assume that the base institution I is *inclusive*; recall from Sec. 4.5 that in addition to the assumptions at the beginning of this section this means also that $Sen\Sigma \subseteq Sen\Sigma'$ whenever $\Sigma \subseteq \Sigma'$ is an inclusion of signatures.

Normal forms. In general, the specifications that include also DERIV are not semantically equivalent to finite theories. We have seen already an example of this situation, when $\varphi\Box$MON could not be axiomatised in $\mathcal{EQL}$. However, the following result shows that we can get rid off all DERIV but one, which then occurs at the top of the structuring.

Theorem 15.5. *Let the base institution I have pushouts of signature morphisms and model amalgamation. Then any specification structured by* BASIC*,* UNION*,* TRANS*, and* DERIV *is semantically equivalent to a specification of the form $\phi\Box(\Sigma', E')$.*

Proof. We prove this result by induction on the structure of a specification SP.

- For each finite theory (Σ, E), $(\Sigma, E) \models\!\!\mid 1_\Sigma\Box(\Sigma, E)$.
- Consider $\mathrm{SP}_1 \cup \mathrm{SP}_2$ such that $sig[\mathrm{SP}_k] = \Sigma_k$, $k = 1,2$.
  - By the induction hypothesis $\mathrm{SP}_k \models\!\!\mid \phi_k\Box(\Sigma'_k, E'_k)$, $\phi_k : \ \Sigma_k \to \Sigma'_k$, $k = 1,2$.
  - Let ϕ'_1, ϕ'_2 be a pushout co-cone for $i_1;\phi_1$, $i_2;\phi_2$ where $i_k : \ \Sigma_1 \cap \Sigma_2 \hookrightarrow \Sigma_k$ are the inclusions of the intersection of the signatures.



(15.4)

  - By using the axiom of this chapter that the intersection-union of a square of signatures is a pushout, let $\phi : \ \Sigma_1 \cup \Sigma_2 \to \Sigma$ is the unique signature morphism making the diagram (15.4) commute.

  We prove that $\mathrm{SP}_1 \cup \mathrm{SP}_2 \ \models\!\!\mid \ \phi\Box(\Sigma, \phi'_1 E'_1 \ \cup \ \phi'_2 E'_2)$.

  - Obviously, $sig[\mathrm{SP}_1 \cup \mathrm{SP}_2] = \Sigma_1 \cup \Sigma_2 = sig[\phi\Box(\Sigma, \phi'_1 E'_1 \ \cup \ \phi'_2 E'_2)]$.
  - On the one hand, let $M' \in Mod[\phi\Box(\Sigma, \phi'_1 E'_1 \ \cup \ \phi'_2 E'_2)]$. Then

    1 there exists $M \in Mod(\Sigma, \phi'_1 E'_1 \ \cup \ \phi'_2 E'_2)$ s.th. $M\!\upharpoonright_\phi = M'$ — semantics of DERIV

    Let $M'_k = M\!\upharpoonright_{\phi'_k}$, $k = 1,2$.

    2 $M'_k \models E'_k$, $k = 1,2$ — 1, Satisfaction Condition
    3 $M'\!\upharpoonright_{\Sigma_k} = M'_k\!\upharpoonright_{\phi_k}$, $k = 1,2$ — commutativity of diagram (15.4)
    4 $M'\!\upharpoonright_{\Sigma_k} \in Mod[\mathrm{SP}_k]$, $k = 1,2$ — 2, 3, $\mathrm{SP}_k \models\!\!\mid \phi_k\Box(\Sigma'_k, E'_k)$
    5 $M' \in Mod[\mathrm{SP}_1 \cup \mathrm{SP}_2]$ — 4, semantics of UNION.

  - On the other hand, let $M' \in Mod[\mathrm{SP}_1 \cup \mathrm{SP}_2]$. Then

    6 $\exists\, M'_k \in Mod(\Sigma'_k, E'_k)$, $M'\!\upharpoonright_{\Sigma_k} = M'_k\!\upharpoonright_{\phi_k}$, $k = 1,2$ — $\mathrm{SP}_k \models\!\!\mid \phi_k\Box(\Sigma'_k, E'_k)$, sem. of DERIV
    7 $M'_k\!\upharpoonright_{(i_k;\phi_k)} = M'\!\upharpoonright_{\Sigma_1 \cap \Sigma_2}$, $k = 1,2$ — 6, commutativity of (15.4)
    8 $M'_1\!\upharpoonright_{(i_1;\phi_1)} = M'_2\!\upharpoonright_{(i_2;\phi_2)}$ — 7
    9 there exists $M \in |Mod\Sigma|$, $M\!\upharpoonright_{\phi'_k} = M'_k$, $k = 1,2$ — 8, I has model amalgamation

10 $M \models \phi'_1 E'_1 \cup \phi'_2 E'_2$ — 6, 9, Satisfaction Condition

11 $M\restriction_{\phi}\restriction_{\Sigma_k} = M\restriction_{\phi'_k}\restriction_{\phi_k} = M'_k\restriction_{\phi_k} = M'\restriction_{\Sigma_k}$, $k = 1,2$ — commutativity of (15.4), 9, 6

12 $M\restriction_{\phi} = M'$ — 11, uniqueness of model amalgamation

13 $M' \in Mod[\phi\Box(\Sigma, \phi'_1 E'_1 \cup \phi'_2 E'_2)]$ — 12, 10.

- Consider $SP \star \varphi$ such that $\varphi : \Sigma \to \Sigma'$.
  - By the induction hypothesis there exists $\phi : \Sigma \to \Sigma_1$ and finite theory (Σ_1, E_1) such that $SP \models\!\!\mid \phi\Box(\Sigma_1, E_1)$.
  - Let φ_1, ϕ' be a pushout co-cone for φ, ϕ like in the diagram below:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\phi} & \Sigma_1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\ \Sigma' & \xrightarrow[\phi']{} & \Sigma'_1 \end{array} \qquad (15.5)$$

We prove that $SP \star \varphi \models\!\!\mid \phi'\Box(\Sigma'_1, \varphi_1 E_1)$.

- We have that $sig[SP \star \varphi] = sig[\phi'\Box(\Sigma'_1, \varphi_1 E_1)] = \Sigma'$.
- On the one hand, let $M' \in Mod[\phi'\Box(\Sigma'_1, \varphi_1 E_1)]$. Then

1 there exists $M'_1 \in Mod(\Sigma'_1, \varphi_1 E_1)$, $M'_1\restriction_{\phi'} = M'$

2 $M'_1\restriction_{\varphi_1} \models E_1$ — 1, Satisfaction Condition

3 $M'_1\restriction_{\varphi_1}\restriction_{\phi} \in Mod[\phi\Box(\Sigma_1, E_1)]$ — 2, semantics of DERIV

4 $M'_1\restriction_{\varphi_1}\restriction_{\phi} \in Mod[SP]$ — 3, $SP \models\!\!\mid \phi\Box(\Sigma_1, E_1)$

5 $M'_1\restriction_{\phi'}\restriction_{\varphi} \in Mod[SP]$ — 4, commutativity of (15.5)

6 $M' = M'_1\restriction_{\phi'} \in Mod[SP \star \varphi]$ — 5, semantics of TRANS.

- On the other hand, let $M' \in Mod[SP \star \varphi]$. Then

7 $M'\restriction_{\varphi} \in Mod[SP]$ — $M' \in Mod[SP \star \varphi]$, semantics of TRANS

8 there exists $M_1 \in Mod(\Sigma_1, E_1)$, $M_1\restriction_{\phi} = M'\restriction_{\varphi}$ — 7, $SP \models\!\!\mid \phi\Box(\Sigma_1, E_1)$, sem. of DERIV.

By the lifting of co-limits from signatures to theories 4.2, the square below is a pushout square of theory morphisms:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\phi} & (\Sigma_1, E_1) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\ \Sigma' & \xrightarrow[\phi']{} & (\Sigma'_1, \varphi_1 E_1) \end{array}$$

It is also a model amalgamation square by the lifting of model amalgamation from signatures to theories (cf. Thm. 4.8). Hence

9 there exists $M'_1 \in Mod(\Sigma'_1, \varphi_1 E_1)$ such that $M'_1\restriction_{\phi'} = M'$, $M'_1\restriction_{\varphi_1} = M_1$

10 $M' \in Mod[\phi' \Box (\Sigma'_1, \varphi_1 E_1)]$ 9, semantics of DERIV.

- Consider $\varphi \Box \mathrm{SP}'$, $\varphi : \Sigma \to \Sigma'$.
  - By the induction hypothesis there exists $\phi : \Sigma' \to \Sigma''$ and finite theory (Σ'', E'') such that $\mathrm{SP}' \models\!\!\!\mid \phi \Box (\Sigma'', E'')$.

  We prove that $\varphi \Box \mathrm{SP}' \models\!\!\!\mid (\varphi;\phi) \Box (\Sigma'', E'')$. We have that

$$
\begin{aligned}
Mod[\varphi \Box \mathrm{SP}'] &= Mod[\mathrm{SP}'] \restriction_{\varphi} && \text{semantics of DERIV} \\
&= Mod[\phi \Box (\Sigma'', E'')] \restriction_{\varphi} && \mathrm{SP}' \models\!\!\!\mid \phi \Box (\Sigma'', E'') \\
&= Mod(\Sigma'', E'') \restriction_{\phi} \restriction_{\varphi} && \text{semantics of DERIV} \\
&= Mod(\Sigma'', E'') \restriction_{(\varphi;\phi)} && Mod \text{ is functor.}
\end{aligned}
$$

□

Given any structured specification SP, any specification of the form $\phi \Box (\Sigma', E')$ which is semantically equivalent to SP is called a *normal form of* SP. Note that if SP is built only from BASIC, UNION, TRANS, then its normal forms are just theories.

Exercises

15.1. Consider the following four $\mathcal{EQL}$ theories:

- MON, the theory of monoids, with the neutral constant denoted 0 and the binary operation denoted $+$.
- INV, whose signature extends *sig*[MON] with an unary operation $-$ and has the equations $(\forall x) x + -x = 0$ and $(\forall x)(-x) + x = 0$.
- COMM, with one binary operation $+$ and a commutativity equation for that.
- DIST, with two binary operations, $\cdot$ and $+$, and equations expressing the distributivity of $\cdot$ over $+$, both left and right.

Specify the category of the rings by aggregating the four theories using BASIC, UNION and TRANS.

15.2. Given specifications SP_1 and SP_2, show that the following commutative square of specification morphisms

$$
\begin{array}{ccc}
(sig[\mathrm{SP}_1] \cap sig[\mathrm{SP}_2], \emptyset) & \xrightarrow{\subseteq} & \mathrm{SP}_1 \\
{\scriptstyle \subseteq}\downarrow & & \downarrow{\scriptstyle \subseteq} \\
\mathrm{SP}_2 & \xrightarrow[\subseteq]{} & \mathrm{SP}_1 \cup \mathrm{SP}_2
\end{array}
$$

is a pushout square in $Spec^I$.

15.3. The specification building operator UNION can be replaced by its particular version where SP_1 and SP_2 have the same signature. The general union of specifications can be obtained from TRANS and the union over the same signature.

15.4. Algebraic properties of the structuring of specifications

- For any specifications SP, SP$'$ and SP$''$
  - $\mathrm{SP} \cup \mathrm{SP}' \models\!\mid \mathrm{SP}' \cup \mathrm{SP}$,
  - $\mathrm{SP} \cup \mathrm{SP} \models\!\mid \mathrm{SP}$,
  - $(\mathrm{SP} \cup \mathrm{SP}') \cup \mathrm{SP}'' \models\!\mid \mathrm{SP} \cup (\mathrm{SP}' \cup \mathrm{SP}'')$.
- For any specification SP and any signature morphisms $\varphi\colon\ sig[\mathrm{SP}] \to \Sigma'$ and $\varphi'\colon\ \Sigma' \to \Sigma''$

  $$\mathrm{SP} \star (\varphi;\varphi') \models\!\mid (\mathrm{SP} \star \varphi) \star \varphi'.$$
- For any specifications SP$_1$ and SP$_2$ and any signature morphism $\varphi\colon\ sig[\mathrm{SP}_1 \cup \mathrm{SP}_2] \to \Sigma$

  $$(\mathrm{SP}_1 \cup \mathrm{SP}_2) \star \varphi \models\!\mid (\mathrm{SP}_1 \star (i_1;\varphi)) \cup (\mathrm{SP}_2 \star (i_2;\varphi))$$

  where i_k is the inclusion $sig[\mathrm{SP}_k] \hookrightarrow sig[\mathrm{SP}_1 \cup \mathrm{SP}_2]$ for $k \in \{1,2\}$.
- For any specification SP and any signature morphisms $\varphi'\colon\ \Sigma'' \to \Sigma'$ and $\varphi\colon\ \Sigma' \to sig[\mathrm{SP}]$,

  $$(\varphi';\varphi)\Box\mathrm{SP} \models\!\mid \varphi'\Box\varphi\Box\mathrm{SP}.$$

Assume the institution has model amalgamation and consider any specifications SP$_1$ and SP$_2$. Let i be the signature inclusion $sig[\mathrm{SP}_1] \cap sig[\mathrm{SP}_2] \hookrightarrow sig[\mathrm{SP}_1] \cup sig[\mathrm{SP}_2]$, and for $k = 1,2$ let i_k be the signature inclusion $sig[\mathrm{SP}_1] \cap sig[\mathrm{SP}_2] \subseteq sig[\mathrm{SP}_k]$. Then

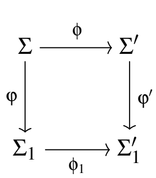
- $i\Box(\mathrm{SP}_1 \cup \mathrm{SP}_2) \models\!\mid (i_1\Box\mathrm{SP}_1) \cup (i_2\Box\mathrm{SP}_2)$.

15.5. Inclusions of specifications
Let $Spec^I/_{\models\!\mid}$ be the 'quotient' of the category of specifications under specification equivalence $\models\!\mid$. Then $Spec^I/_{\models\!\mid}$ has two inclusion systems inheriting the inclusion system of the signatures:

- a *closed* one, where the abstract inclusions of specifications $\varphi\colon\ \mathrm{SP}_1 \to \mathrm{SP}_2$ are the abstract inclusions of I-signatures such that $\mathrm{SP}_1 \models\!\mid \varphi\Box\mathrm{SP}_2$, and the abstract surjections of specifications are just the abstract surjections of I-signatures, and
- a *strong* one, where the abstract inclusions of specifications are just the abstract inclusions of I-signatures and the abstract surjections of specifications $\varphi\colon\ \mathrm{SP}_1 \to \mathrm{SP}_2$ are the abstract surjections of signatures such that $\mathrm{SP}_2 \models\!\mid \mathrm{SP}_1 \star \varphi$.

Moreover, the strong inclusion system for specifications has unions where $(\mathrm{SP}_1/_{\models\!\mid}) \cup (\mathrm{SP}_2/_{\models\!\mid}) = (\mathrm{SP}_1 \cup \mathrm{SP}_2)/_{\models\!\mid}$ for any specifications SP$_1$ and SP$_2$.

15.6. Interpolation for structured specifications
The following *Craig interpolation property* holds for specifications. For any weak amalgamation square of signatures



any Σ'-specification SP$'$ and any Σ_1-specification SP$_1$ such that $\mathrm{SP}' \star \varphi' \models \mathrm{SP}_1 \star \phi_1$, there exists a Σ-specification SP such that $\mathrm{SP}' \models \mathrm{SP} \star \phi$ and $\mathrm{SP} \star \varphi \models \mathrm{SP}_1$. (*Hint:* define $\mathrm{SP} = \phi\Box\mathrm{SP}'$.)

15.7. Intersection of structured specifications
Define the intersection $\mathrm{SP}_1 \cap \mathrm{SP}_2$ of any specifications SP_1 and SP_2 as a new specification building operation and use it for showing that the signature functor from specifications to signatures $sig : Spec^I \to Sig^I$ lifts pullbacks.

15.8. Diagrams for structured specifications
Consider an inclusive (base) institution I with diagrams ι such that for each Σ-model M, the elementary extension $\iota_\Sigma M : \Sigma \to \Sigma_M$ is an inclusion. Then I^{spec} has diagrams. (*Hint:* The diagram of a model M of a specification SP is $\mathrm{SP} \to \mathrm{SP} \cup (\Sigma_M, E_M)$ where $\Sigma = sig[\mathrm{SP}]$ and $\Sigma \to (\Sigma_M, E_M)$ is the diagram of M in I.)

15.9. Lifting comorphisms to institutions of structured specifications
Each comorphism between the base institutions $(\Phi, \alpha, \beta) : I \to I'$ such that Φ preserves unions of signatures determines a canonical comorphism between the institutions of the structured specifications $(\Phi^{\mathrm{spec}}, \alpha^{\mathrm{spec}}, \beta^{\mathrm{spec}}) : I^{\mathrm{spec}} \to (I')^{\mathrm{spec}}$.

15.3 Specifications with proofs

In the case of complex specifications, a modular approach to proofs allows for efficient formal verifications. This means that we can develop proofs at the level of the components of the structured specifications and then aggregate them by following the structure of the respective specifications. In this section we develop such a theory of structured proofs as follows:

1. Given a proof system for a base institution I we extend it to a proof system for I^{spec}, for the four specification building operators introduced in Sec. 15.2.
2. We provide general conditions under which the soundness and the completeness carry from I to I^{spec}.
3. We illustrate the applicability of the general theory in a couple of concrete situations.

$\langle \mathcal{T}, \mathcal{D} \rangle$-specifications. In the theory of structuring specifications, to consider *all* signature morphisms for TRANS and DERIV is impractical for two reasons. On the one hand, in concrete situations, for TRANS and DERIV we use specific kinds of signature morphisms, this being more obvious in the case of DERIV. On the other hand, some of the important general results of this section cannot be applied unless the structuring is parameterised by designated classes of signature morphisms for TRANS and DERIV.

Let $\mathcal{T}, \mathcal{D} \subseteq Sig$ be classes of signature morphisms in the base institution I. The specifications thus built by BASIC, UNION, TRANS by morphisms in $\mathcal{T}$, and DERIV by morphisms in $\mathcal{D}$, are called *$\langle \mathcal{T}, \mathcal{D} \rangle$-specifications*. Let us denote the category of *$\langle \mathcal{T}, \mathcal{D} \rangle$-specifications* by $Spec_{\mathcal{T},\mathcal{D}}$. The resulting institution of the $\langle \mathcal{T}, \mathcal{D} \rangle$-specifications is denoted $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$.

Extending proof systems to specifications. Given an institution with proofs $I = (Sig, Sen, Mod, \models, Pf)$ and classes of signature morphisms $\mathcal{T}, \mathcal{D} \subseteq Sig$, the institution $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ of

$\langle \mathcal{T}, \mathcal{D}\rangle$-specifications can be enhanced with a proof system by taking the initial proof system such that

(*base*) for each specification SP: $\Gamma \vdash_{\mathrm{SP}} E$ if $\Gamma \vdash_{sig[\mathrm{SP}]} E$

(*pres*) for each finite theory (Σ, E): $\emptyset \vdash_{(\Sigma,E)} E$

(*deriv*) and such that it satisfies following the (meta-)rule, for each $d \in \mathcal{D}$ with $d : \Sigma \to sig[\mathrm{SP}']$: $d\Gamma \vdash_{\mathrm{SP}'} dE$ implies $\Gamma \vdash_{d \Box \mathrm{SP}'} E$.

The rule *base* says that any proof at the structured level can be considered as a proof at the structured level. The rule *pres* is rather trivial but nevertheless necessary: any axiom of an unstructured specification is considered already proved. Although expected, *deriv* is more interesting: any proof in SP$'$ involving only sentences from the derivation $d\Box\mathrm{SP}'$ should count as a proof in $d\Box\mathrm{SP}'$. This can be regarded as a proof-theoretic conservativity property imposed on d.

Proposition 15.6. *There exists the initial proof system $Pf_{\mathcal{T},\mathcal{D}}$ for $\langle \mathcal{T}, \mathcal{D}\rangle$-specifications satisfying* base*,* pres *and* deriv *defined above.*

Proof. The formal construction can be done in the manner of constructions of the free proof systems over systems of rules (Thm. 11.3) and the construction of the free proof systems with connectives (Thm. 11.14). Instead of doing this, in order to grasp $Pf_{\mathcal{T},\mathcal{D}}$, let us describe rather informally its construction. This can be done by the following two steps:

1. for each theory (Σ, E) we add rules $P_{\Sigma,E} : \emptyset \to E$ to the Σ-proofs (from I) such that $\varphi P_{\Sigma,E} = P_{\Sigma',\varphi E}$ for each signature morphism $\varphi : \Sigma \to \Sigma'$, and we take the free proof system which preserves the horizontal and vertical composition of the original proofs of I, and
2. when $\Gamma \not\vdash_{d\Box\mathrm{SP}'} E$ for some specification SP$'$ and $(d : \Sigma \to sig[\mathrm{SP}']) \in \mathcal{D}$ such that $d\Gamma \vdash_{\mathrm{SP}'} dE$, we add a $d\Box\mathrm{SP}'$-proof $\Gamma \to E$ and take again the free proof system which preserves the horizontal and vertical composition of the existing proofs.

□

Soundness

Proposition 15.7. *For any sound institution with proofs I, the corresponding institution with proofs $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ of the structured $\langle \mathcal{T}, \mathcal{D}\rangle$-specifications is also sound.*

Proof. Soundness of $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ means that there exists a comorphism of proof systems

$$(1, 1, \gamma) : (Spec_{\mathcal{T},\mathcal{D}}, Sen, Pf_{\mathcal{T},\mathcal{D}}) \to (Spec_{\mathcal{T},\mathcal{D}}, Sen, \models)$$

where $(Spec_{\mathcal{T},\mathcal{D}}, Sen, \models)$ is the semantic proof system determined by the institution $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ of $\langle \mathcal{T}, \mathcal{D}\rangle$-specifications. By Prop. 15.6 it is enough to show that the semantic proof system $(Spec_{\mathcal{T},\mathcal{D}}, Sen, \models)$ satisfies properties *base*, *pres*, and *deriv*. Indeed these hold as follows:

- Property *base* holds for $\models$, where SP is any specification, because, if $\Gamma \vdash_{sig[\mathrm{SP}]} E$ then by the soundness of the base institution I we have that $\Gamma \models_{sig[\mathrm{SP}]} E$. Because $Mod[\mathrm{SP}] \subseteq Mod(sig[\mathrm{SP}])$, we get that $\Gamma \models_{\mathrm{SP}} E$.
- Property *pres* for $\models$ means that for each theory (Σ, E) we have that $\emptyset \models_{(\Sigma,E)} E$, which is trivial.
- Property *deriv* for $\models$ means that $d\Gamma \models_{\mathrm{SP}'} dE$ implies $\Gamma \models_{d\Box\mathrm{SP}'} E$. Consider a model $M \in Mod[d\Box\mathrm{SP}']$ such that $M \models \Gamma$. Then

1	there exists $M' \in Mod[\mathrm{SP}']$, $M = M'\upharpoonright_d$	semantics of DERIV
2	$M' \models d\Gamma$	1, $M \models \Gamma$, Satisfaction Condition
3	$M' \models dE$	1, 2, $d\Gamma \models_{\mathrm{SP}'} dE$
4	$M = M'\upharpoonright_d \models E$	1, 3, Satisfaction Condition.

□

Note that in the proof of the soundness Prop. 15.7 we have not used anything about $\mathcal{T}$ and $\mathcal{D}$, so with respect to soundness there is complete freedom about them. Also, the proofs of the soundness of the three rules *base*, *pres*, *deriv* are somehow proportional in difficulty with their degree of non-triviality. For instance, the proof of the soundness of *pres* is trivial, while the proof of the soundness of *deriv* is the most interesting from all three.

Completeness

The lifting of completeness from the base institution to the institution of $(\mathcal{T}, \mathcal{D})$-specifications, unlike the lifting of the soundness, requires some conditions. These are substantial conditions that set the boundaries of completeness in the applications. However, all these conditions come as a consequence of having DERIV as one of the structuring operators.

Theorem 15.8. *Consider a base institution with proofs $I = (Sig, Sen, Mod, \models, Pf)$ and classes of signature morphisms $\mathcal{T}$ and $\mathcal{D}$ such that*

1. *$\mathcal{T}$ and $\mathcal{D}$ satisfy the following properties:*
   - *$\mathcal{D}$ is a subcategory of Sig,*
   - *$\mathcal{D} \subseteq \mathcal{T}$ and each signature inclusion belongs to $\mathcal{T}$,*
   - *for each $t\colon\ \Sigma \to \Sigma'$ in $\mathcal{T}$ and $d\colon\ \Sigma \to \Sigma_1$ in $\mathcal{D}$ there exists a pushout square with $d' \in \mathcal{D}$,*

$$\begin{array}{ccc} \Sigma & \xrightarrow{d} & \Sigma_1 \\ {\scriptstyle t}\downarrow & & \downarrow \\ \Sigma' & \xrightarrow[d']{} & \Sigma'_1 \end{array}$$

2. *I is complete,*

3. *I has model amalgamation, and*

4. *I has Craig-Robinson $(\mathcal{D},\mathcal{T})$-interpolation.*

Then the institution $I^{\text{spec}}_{\mathcal{T},\mathcal{D}}$ of $\langle \mathcal{T}, \mathcal{D}\rangle$-specifications endowed with the proof system $Pf_{\mathcal{T},\mathcal{D}}$ is complete.

Proof. We prove by induction on the structure of the specification SP that $\Gamma \vdash_{SP} E$ if $\Gamma \models_{\text{SP}} E$. For this proof we will systematically use the existence and uniqueness of normal forms for specifications formed with BASIC, UNION, TRANS and DERIV given by the straightforward adaptation of Thm. 15.5 to $\langle \mathcal{T}, \mathcal{D}\rangle$-specifications that uses the conditions on $\mathcal{T}$ and $\mathcal{D}$ from the statement of the theorem.

- The base step BASIC. We prove that for each finite theory (Σ, E_1), if $\Gamma \models_{(\Sigma,E_1)} E$ then $\Gamma \vdash_{(\Sigma,E_1)} E$. So, we consider a semantic consequence $\Gamma \models_{(\Sigma,E_1)} E$. We have that:

1	$\Gamma \cup E_1 \models_{\Sigma} E$	$\Gamma \models_{(\Sigma,E_1)} E$
2	$\Gamma \cup E_1 \vdash_{\Sigma} E$	1, completeness of I
3	$\Gamma \cup E_1 \vdash_{(\Sigma,E_1)} E$	2, *base*
4	$\vdash_{(\Sigma,E_1)} E_1$	*pres*
5	$\Gamma \vdash_{(\Sigma,E_1)} \emptyset$	monotonicity proof
6	$\Gamma \vdash_{(\Sigma,E_1)} E_1$	4, 5, horizontal composition of proofs
7	$\Gamma \vdash_{(\Sigma,E_1)} \Gamma$	monotonicity proof
8	$\Gamma \vdash_{(\Sigma,E_1)} \Gamma \cup E_1$	6, 7, vertical composition
9	$\Gamma \vdash_{(\Sigma,E_1)} E$	8, 3, horizontal composition.

- The induction step for TRANS. Consider a translation of specifications $\text{SP}\star t$ with $t \in \mathcal{T}$ such that $t: sig[\text{SP}] \to \Sigma'$. Let us assume $\Gamma' \models_{\text{SP}\star t} E'$ and prove that $\Gamma' \vdash_{\text{SP}\star t} E'$.
  - According to the proof of Thm. 15.5, there exists a normal form $d'\square(\Sigma'_1, t_1\Gamma_1)$ for $\text{SP}\star t$ such that $d\square(\Sigma_1,\Gamma_1)$ is a normal form for SP and the square below is a pushout:

$$\begin{array}{ccc} \Sigma & \xrightarrow{d} & \Sigma_1 \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t_1} \\ \Sigma' & \xrightarrow[d']{} & \Sigma'_1 \end{array} \qquad (15.6)$$

  - We show that $d'\Gamma' \cup t_1\Gamma_1 \models d'E'$. Let $M'_1 \in Mod\Sigma'_1$ such that $M'_1 \models d'\Gamma' \cup t_1\Gamma_1$. We have that:

10	$M'_1\restriction_{d'} \in Mod[\text{SP}\star t]$	$\text{SP}\star t \models\!\mid d'\square(\Sigma'_1, t_1\Gamma_1)$, $M'_1 \models t_1\Gamma_1$
11	$M'_1\restriction_{d'} \models \Gamma'$	$M'_1 \models d'\Gamma'$, Satisfaction Condition

12 $M'_1\restriction_{d'} \models E'$ — 10, 11, $\Gamma' \models_{SP\star t} E'$

13 $M'_1 \models d'E'$ — 12, Satisfaction Condition.

- By the CRi property for the square (15.6), there exists an interpolant $I \subseteq Sen\Sigma$ such that $\Gamma_1 \models dI$ and $\Gamma' \cup tI \models E'$. Then:

14 $dI \subseteq Mod(\Sigma_1,\Gamma_1)^*$ — $\Gamma_1 \models dI$

15 $I \subseteq (Mod(\Sigma_1,\Gamma_1)\restriction_d)^*$ — 14, Satisfaction Condition

16 $Mod[\text{SP}] = Mod(\Sigma_1,\Gamma_1)\restriction_d$ — $\text{SP} \models\!\!\mid d\Box(\Sigma_1,\Gamma_1)$

17 $\models_{\text{SP}} I$ — 15, 16

18 $\vdash_{\text{SP}} I$ — 17, induction hypothesis

19 $\vdash_{\text{SP}\star t} tI$ — 18, translation along specification morphism $t:\ \text{SP} \to \text{SP}\star t$ $(Pf_{\mathcal{T},\mathcal{D}}t)$

20 $\Gamma' \vdash_{\text{SP}\star t} tI$ — 19, monotonicity and horizontal composition of proofs

21 $\Gamma' \vdash_{\text{SP}\star t} \Gamma'$ — monotonicity

22 $\Gamma' \vdash_{\text{SP}\star t} \Gamma' \cup tI$ — 20, 21, vertical composition of proofs

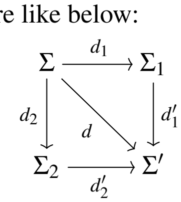
23 $\Gamma' \cup tI \vdash_{\Sigma'} E'$ — $\Gamma' \cup tI \models_{\Sigma'} E'$, completeness of I

24 $\Gamma' \cup tI \vdash_{\text{SP}\star t} E'$ — 23, *base*

25 $\Gamma' \vdash_{\text{SP}\star t} E'$ — 22, 24, horizontal composition of proofs.

- The induction step for UNION. Consider a union of specifications $\text{SP}_1 \cup \text{SP}_2$. Because arbitrary union of specifications can be (denotationally) reduced to translations and union of specifications having the same signature, and because all signature inclusions are in $\mathcal{T}$, in order to simplify our discussion and without any loss of generality we may assume that $sig[\text{SP}_1] = sig[\text{SP}_2] = \Sigma$. Let us assume $\Gamma \models_{\text{SP}_1\cup\text{SP}_2} E$ and prove that $\Gamma \vdash_{\text{SP}_1\cup\text{SP}_2} E$.

  - According to the proof of Thm. 15.5, there exists normal forms $d_1\Box(\Sigma_1,\Gamma_1)$ for SP_1, $d_2\Box(\Sigma_2,\Gamma_2)$ for SP_2, and $d\Box(\Sigma', d'_1\Gamma_1 \cup d'_2\Gamma_2)$ for $\text{SP}_1 \cup \text{SP}_2$ for a pushout square like below:



(15.7)

  The existence of such pushout square such that $d \in \mathcal{D}$ follows from the properties of $\mathcal{T}$ and $\mathcal{D}$. First we get a pushout with $d'_1 \in \mathcal{D}$, and then we obtain that $d \in \mathcal{D}$ as the composition $d_1;d'_1$.

  - We show that $d\Gamma \cup d'_1\Gamma_1 \cup d'_2\Gamma_2 \models dE$. Let M' be a Σ'-model such that $M' \models d\Gamma \cup d'_1\Gamma_1 \cup d'_2\Gamma_2$. Then:

26 $M'\restriction_d \models \Gamma$ — $M' \models d\Gamma$, Satisfaction Condition

27 $M'\restriction_{d'_k} \models \Gamma_k$, $k = 1,2$ — $M' \models d'_k\Gamma_k$, Satisfaction Condition

28 $M'\upharpoonright_d \in Mod[\mathrm{SP}_k],\ k=1,2$ — 27, $\mathrm{SP}_k \models\!\!| \ d_k\Box(\Sigma_k,\Gamma_k)$, $d = d_k;d'_k$, $k=1,2$

29 $M'\upharpoonright_d \in Mod[\mathrm{SP}_1 \cup \mathrm{SP}_2]$ — 28, semantics of UNION

30 $M'\upharpoonright_d \models E$ — $\Gamma \models_{\mathrm{SP}_1\cup\mathrm{SP}_2} E$, 26, 29

31 $M' \models dE$ — 30, Satisfaction Condition

– Since $d_2 \in \mathcal{T}$ (because $\mathcal{D} \subseteq \mathcal{T}$), the pushout square (15.7) is a CRi square. Therefore, there exists an interpolant $I \subseteq Sen\Sigma$ such that $\Gamma_1 \models d_1 I$ and $d_2\Gamma \cup \Gamma_2 \cup d_2 I \models d_2 E$. Then

32 $\vdash_{\mathrm{SP}_1} I$ — from $\Gamma_1 \models d_1 I$, similar to the proof of 18

33 $\vdash_{\mathrm{SP}_1\cup\mathrm{SP}_2} I$ — 32, translation along specification morphisms $\mathrm{SP}_1 \to \mathrm{SP}_1 \cup \mathrm{SP}_2$

34 $\Gamma \cup I \vdash_{\mathrm{SP}_1\cup\mathrm{SP}_2} E$ — from $d_2\Gamma \cup d_2 I \cup \Gamma_2 \models d_2 E$, similar to the proof of 33

35 $\Gamma \vdash_{\mathrm{SP}_1\cup\mathrm{SP}_2} E$ — 33, 34, by using the general properties of entailment (like in the proof of 25).

- The induction step for DERIV. Consider a derived specification $d\Box\mathrm{SP}'$ and assume $\Gamma \models_{d\Box\mathrm{SP}'} E$. We prove that $\Gamma \vdash_{d\Box\mathrm{SP}'} E$.
  – We prove that $d\Gamma \models_{\mathrm{SP}'} dE$. Let $M' \in Mod[\mathrm{SP}']$ such that $M' \models d\Gamma$. Then:

36 $M'\upharpoonright_d \models \Gamma$ — $M' \models d\Gamma$, Satisfaction Condition

37 $M'\upharpoonright_d \in Mod[d\Box\mathrm{SP}']$ — $M \in Mod[\mathrm{SP}']$, semantics of DERIV

38 $M'\upharpoonright_d \models E$ — 36, 37, $\Gamma \models_{d\Box\mathrm{SP}'} E$

39 $M' \models dE$ — 38, Satisfaction Condition.

– Consequently,

40 $d\Gamma \vdash_{\mathrm{SP}'} dE$ — $d\Gamma \models_{\mathrm{SP}'} dE$, induction hypothesis

41 $\Gamma \vdash_{d\Box\mathrm{SP}'} E$ — 40, *deriv*.

$\Box$

Note that the proof of the completeness Thm. 15.8 uses only that the proof system for the structured specifications fullfils properties *base*, *pres*, and *deriv*, it does not require the initiality of $Pf_{\mathcal{T},\mathcal{D}}$. The initiality is necessary only for the soundness of $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ (Prop. 15.7). Also, the model amalgamation condition, although not used in the proof of Thm. 15.8, is necessary when involving the normal forms Thm. 15.5. Because we are using a version of Thm. 15.5 relativised to classes $\mathcal{T}$, $\mathcal{D}$ of signature morphisms, it is technically sufficient to have model amalgamation for pushout squares based on $\langle\mathcal{T}, \mathcal{D}\rangle$-spans of signature morphisms. However, in the applications this has little meaning as model amalgamation, unlike interpolation, is in general a uniform property.

Now, we shift our discussion to examples of how Thm. 15.8 can be applied.

What happens in the absence of DERIV? Things get much simpler. In Thm. 15.8 we get $\mathcal{D}$ to be the class of the identity morphisms. Then all conditions apart of the completeness of I hold trivially. A special note should be made for the model amalgamation. This is trivialised in the light of our above comment on the sufficiency of model amalgamation for pushout squares based on $\langle\mathcal{T}, \mathcal{D}\rangle$-spans of signature morphisms.

The $\mathcal{FOL}$ case. This means the base institution I is $\mathcal{FOL}$. For the base proof system, the literature on first-order logic provides several sound and complete proof systems for $\mathcal{FOL}$. The chief condition that dictates the choice of $\mathcal{D}$ and $\mathcal{T}$ is the interpolation condition. By virtue of Cor. 9.17, a maximal choice for $\mathcal{D}$ and $\mathcal{T}$ would be the class of the signature morphisms that are injective on the sorts, and the class of all signature morphisms, respectively. Recall that in $\mathcal{FOL}$, CRi is obtained from Ci, compactness, and implications. The former two properties came as consequences of extensive developments.

The $\mathcal{HCL}$ case. Here the base institution I is $\mathcal{HCL}$. In Sec. 11.7 (last paragraph) we obtained a sound and complete proof system for $\mathcal{HCL}$. Like in the $\mathcal{FOL}$ case, the CRi condition dictates the choice of $\mathcal{D}$ and $\mathcal{T}$. According to Cor. 14.18 we can take $\mathcal{D}$ to be the class of the $(ie*)$-morphisms and $\mathcal{T}$ to be the class of all signature morphisms. In the $\mathcal{HCL}$ case the route to CRi is very different than in $\mathcal{FOL}$, actually it is more difficult. The interpolation results of Cor. 14.18 are obtained from corresponding Ci results plus Grothendieck interpolation. And the relevant Ci result is obtained from Birkhoff axiomatizability. Moreover, $\mathcal{EQL}$ and $\mathcal{UNIV}$ can be treated similarly.

15.4 Abstractly structured specifications

In this section we go more abstract about the structuring in the sense of developing a theory of structuring specifications that is independent not only from the base institution I, but also from any collection of specification structuring operators. The second independence resides in the possibility to apply this theory to any collection of structuring operators. The four operators introduced in Sec. 15.2 are powerful and can express a lot of modularisation constructs, but still there are other useful collections of structuring operators. Moreover, this abstraction has also the potential to accomodate other structuring frameworks beyond the tradition of formal specification. We do as follows:

1. We define the concept of abstractly structured specification mentioned above.
2. We extend the model amalgamation property from concretely to abstractly structured specifications.
3. We show how concrete structuring operators can still have a presence in the context of abstract structuring.
4. In the same way we introduce a concept of normal form for abstractly structured specifications.
5. The two independencies characteristic to the abstract structuring allow for a proper approach to translations of structured specifications. We develop the fundamental mathematical concepts and results for that.

Abstractly structured specifications. Recall the 'forgetful' institution morphism $(sig, 1, \subseteq) : I^{\mathrm{spec}} \to I$ of Fact 15.2. The mathematical idea of abstractly structured specifications is to consider such a morphism with I^{spec} replaced with an abstract institution

I'. If we do that, then I' can be interpreted as institutions of concretely structured specification over I, but for various different collections of structuring operators. Thus, we say that an institution I' is (I, sig)-*structured* if and only if there exists an institution morphism $(sig, 1, \subseteq)\colon\ I' \to I$ such that for I'-signature Σ', each $\subseteq_{\Sigma'}$ is a full subcategory inclusion. Let us make this definition more explicit:

- $sig\colon\ Sig' \to Sig$ is a functor,
- $Sen' = sig\ ;\ Sen$,
- for each I'-signature Σ' we have that $Mod'[\Sigma']$ is a full subcategory of $Mod(sig[\Sigma'])$ such that for each I'-signature morphism $\varphi'\colon\ \Sigma'_1 \to \Sigma'_2$ the square below commutes,

$$\begin{array}{ccc} \Sigma'_1 & \quad & Mod'[\Sigma'_1] \xrightarrow{\ \subseteq\ } Mod(sig[\Sigma'_1]) \\ {\scriptstyle \varphi'}\big\downarrow & & {\scriptstyle Mod'\varphi'}\big\uparrow \qquad\qquad\qquad \big\uparrow{\scriptstyle Mod(sig\varphi')} \\ \Sigma'_2 & & Mod'[\Sigma'_2] \xrightarrow[\ \subseteq\ ]{} Mod(sig[\Sigma'_2]) \end{array} \tag{15.8}$$

  and

- for each I'-signature Σ', each Σ'-model M', and each $sig[\Sigma']$-sentence ρ we have that

$$M' \models'_{\Sigma'} \rho \quad \text{if and only if} \quad M' \models_{sig[\Sigma']} \rho. \tag{15.9}$$

From a specification theoretic perspective, the I'-signatures may be refereed to as (I, sig)-*specifications*.

One note on the (I, sig)-specification morphisms. Consider the canonical mappings

$$Sig'(\Sigma'_1, \Sigma'_2) \to \{\varphi \in Sig(sig[\Sigma'_1], sig[\Sigma'_2]) \mid (Mod'[\Sigma'_2])\restriction_{\varphi} \subseteq Mod'[\Sigma'_1]\}.$$

Very often in the applications, and this is also the case of I^{spec} of Sec. 15.2, these mappings are bijections. When so, we say that I' *inherits the signature morphisms*.

Model amalgamation. We aim to extend the result of Prop. 15.4 lifting model amalgamation from a base institution I to the institution of its specifications I^{spec}, to the more general situation of abstractly structured specifications. For this we need the following concept. A (I, sig)-structured institution I' is *compositional* when for each pushout in Sig' like below

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\ \varphi'_1\ } & \Sigma'_1 \\ {\scriptstyle \varphi'_2}\big\downarrow & & \big\downarrow{\scriptstyle \theta'_1} \\ \Sigma'_2 & \xrightarrow[\ \theta'_2\ ]{} & \Omega' \end{array}$$

for any model $M' \in Mod(sig[\Omega'])$, if $M'\restriction_{sig[\theta_1]} \in Mod'[\Sigma'_1]$ and $M'\restriction_{sig[\theta_2]} \in Mod'[\Sigma'_2]$ then $M' \in Mod'[\Omega']$.

Proposition 15.9. *In any (I, sig)-structured institution I' that is compositional and such that sig preserves pushouts, if I has model amalgamation (resp. weak model amalgamation, semi-exactness) then I' has model amalgamation (resp. weak model amalgamation, semi-exactness).*

Proof. Let us assume the model amalgamation property for I and prove it for I'. Let the square below be a pushout square of I'-signature morphisms.

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\varphi'_1} & \Sigma'_1 \\ {\scriptstyle \varphi'_2}\downarrow & & \downarrow{\scriptstyle \theta'_1} \\ \Sigma'_2 & \xrightarrow[\theta'_2]{} & \Omega' \end{array}$$

Let $M'_k \in Mod'[\Sigma'_k]$, $k = 1,2$ such that $M'_1 \upharpoonright_{\varphi'_1} = M'_2 \upharpoonright_{\varphi'_2}$.

- By the condition on the preservation of pushouts we have that the following square is a pushout of I-signature morphisms.

$$\begin{array}{ccc} sig[\Sigma'] & \xrightarrow{sig\varphi'_1} & sig[\Sigma'_1] \\ {\scriptstyle sig\varphi'_2}\downarrow & & \downarrow{\scriptstyle sig\theta'_1} \\ sig[\Sigma'_2] & \xrightarrow[sig\theta'_2]{} & sig[\Omega'] \end{array}$$

- By the model amalgamation property of I there exists an unique amalgamation $M' \in Mod(sig[\Omega'])$ of M'_1 and M'_2.
- Since $M'_k \in Mod'[\Sigma'_k]$ for $k = 1,2$, by the compositionality condition we obtain that $M' \in Mod'[\Omega']$.
- The uniqueness of the amalgamation in I' follows directly from the corresponding property in I.
- Similar arguments may be employed for establishing the weak model amalgamation and semi-exactness properties, respectively.

□

Note how the amalgamation result of Prop. 15.4 arises as an instance of the result of Prop. 15.9.

- The condition that *sig* preserves pushouts is fulfilled by the result of Prop. 15.3.
- The compositionality condition is also fulfilled immediately from the form taken by the pushouts of structured specifications (see the last paragraph in the proof of Prop. 15.4).

Concrete structuring operators for abstractly structured specifications. Although one of the main points of the theory of abstractly structured specifications is the liberation from concrete structuring operators, in some situations it is useful to talk about concrete structuring operators in an abstract context. With this, we can eat the cake and still have it. We can have certain concrete specification building operators, but in a semantic form rather than in an explicit building as-such form, while still allowing for 'specifications' that are not necessarily expressible as terms built with a respective collection of building operators. We can still discuss about some specific specification building operators but without committing to a specific set of such operators. Below we re-introduce the structuring operators BASIC, UNION, TRANS, DERIV within the more general context of abstractly structured specifications.

Let I' be an institution that is (I, sig)-structured.

- We say that I' has *semantic basic specifications* when for each finite I-presentation (Σ, E) there exists a I'-signature $\mathrm{SP}(\Sigma, E)$ such that
  - $sig[\mathrm{SP}(\Sigma, E)] = \Sigma$, and
  - $Mod'[\mathrm{SP}(\Sigma, E)] = Mod(\Sigma, E)$.
- If Sig^I is endowed with an inclusion system, the we say that I' *has semantic unions* when for any I'-signatures Σ'_1 and Σ'_2 there exists a designated I'-signature, denoted $\Sigma'_1 \cup \Sigma'_2$, such that
  - $sig[\Sigma'_1 \cup \Sigma'_2] = sig[\Sigma'_1] \cup sig[\Sigma'_2]$, and
  - $Mod'[\Sigma'_1 \cup \Sigma'_2] = \{M' \in Mod(sig[\Sigma'_1 \cup sig[\Sigma'_2]) \mid M' \upharpoonright_{sig[\Sigma'_k]} \in Mod'[\Sigma'_k],\ k = 1,2\}$.
- For any I-signature morphism $\varphi\colon\ \Sigma \to \Omega$, we say that I' has *semantic* φ*-translations* when for any I'-signature Σ' such that $sig[\Sigma'] = \Sigma$ there exists a designated I'-signature, denoted $\Sigma' \star \varphi$, such that
  - $sig[\Sigma' \star \varphi] = \Omega$, and
  - $Mod'[\Sigma' \star \varphi] = \{M' \in Mod\Omega \mid M' \upharpoonright_{\varphi} \in Mod'[\Sigma']\}$.
- For any I-signature morphism $\varphi\colon\ \Omega \to \Sigma$, we say that I' has *semantic* φ*-derivations* when for any I'-signature Σ' such that $sig[\Sigma'] = \Sigma$ there exists a designated I'-signature, denoted $\varphi \Box \Sigma'$, such that
  - $sig[\varphi \Box \Sigma'] = \Omega$, and
  - $Mod'[\varphi \Box \Sigma'] = \{M' \upharpoonright_{\varphi} \mid M' \in Mod'[\Sigma']\}$.

It is easy to see that I^{spec} of Sec. 15.2 (and also its more refined version $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ of Sect. 15.3) does indeed have semantic basic specifications, unions, translations, and derivations in the sense of the definitions above. Moreover, in actual situations, while $\mathrm{SP}(\Sigma, E) = (\Sigma, E)$ is an obvious choice, there can be also other choices for $\mathrm{SP}(\Sigma, E)$ that are not necessarily flat.

Semantic normal forms. The concept of the normal form of a structured specification of Sec. 15.2 can be easily reflected to the level of abstractly structured specifications but in a semantic rather than in a syntactic form. Given a (I, sig)-structured institution I', a pair (φ, E) consisting of

- a signature morphism $\varphi : sig[\Sigma'] \to \Sigma$ and
- a set of sentences $E \subseteq Sen\Sigma$

is a *semantic normal form* for an I'-signature Σ' when $Mod'[\Sigma'] = Mod(\Sigma, E)\restriction_{\varphi}$. When E is finite we say that the normal form is *finitary*. We say that I' *admits (finitary) semantic normal forms* when each I'-signature has at least a (finitary) normal form.

The semantic normal forms contrast those of Sec. 15.2 that are *syntactic* in the sense of being terms formed with specification building operators. Because $Mod[\varphi\Box(\Sigma, E)] = Mod(\Sigma, E)\restriction_{\varphi}$, it follows that the existence of syntactic normals forms imply the existence of semantic normal forms. However as Ex. 15.18 below shows, the concept of semantic normal form is significantly more general than its syntactic counterpart as the former may occur even in the absence of the latter or of derivations.

Comorphisms of structured institutions. A recent important trend in formal specification is that of heterogeneously logical environments in which translations between the institutions underlying different specification formalisms play a crucial role. But in the presence of specification structuring mechanisms these translations are more than logical interpretations, they have to interpret the structured specifications in the source formalism as structured specifications in the target formalism. Of course, such translations should 'extend' corresponding translations between the underlying logics. This calls for a concept of comorphism between structured institutions that is 'structured' by a comorphism between the base institutions.

Let I'_k be (I_k, sig_k)-structured institutions, $k = 1, 2$. We say that a comorphism $(\Phi', \alpha', \beta') : I'_1 \to I'_2$ is $((\Phi, \alpha, \beta), sig_1, sig_2)$-*structured*, when

- $(\Phi, \alpha, \beta) : I_1 \to I_2$ is a comorphism,
- $sig_1 : Sig'_1 \to Sig_1$ and $sig_2 : Sig'_2 \to Sig_2$ are functors such that the following diagram commutes,

$$\begin{array}{ccc} Sig'_1 & \xrightarrow{\Phi'} & Sig'_2 \\ {\scriptstyle sig_1}\downarrow & & \downarrow{\scriptstyle sig_2} \\ Sig_1 & \xrightarrow[\Phi]{} & Sig_2 \end{array} \tag{15.10}$$

- for each I_1'-signature Σ_1', $\alpha'_{\Sigma_1'} = \alpha_{sig_1[\Sigma_1']}$:

$$\begin{array}{ccccc}
Sen_1'\Sigma_1' & & \xrightarrow{\quad\alpha'_{\Sigma_1'}\quad} & & Sen_2'(\Phi'\Sigma_1') \\
= \Big| & & & & \Big| = \\
Sen_1(sig[\Sigma_1']) & \xrightarrow[\alpha_{sig_1[\Sigma_1']}]{} & Sen_2(\Phi(sig_1[\Sigma_1'])) & \overset{}{\underset{=}{\text{———}}} & Sen_2(sig_2[\Phi'\Sigma_1'])
\end{array}$$

- $\beta'_{\Sigma_1'}$ is the restriction of $\beta_{sig_1[\Sigma_1']}$:

$$\begin{array}{ccccc}
Mod_2'[\Phi'\Sigma_1'] & & \xrightarrow{\quad\beta'_{\Sigma_1'}\quad} & & Mod_1'[\Sigma_1'] \\
\subseteq \Big\downarrow & & & & \Big\downarrow \subseteq \\
Mod_2(sig_2[\Phi'\Sigma_1']) & \underset{=}{\text{———}} & Mod_2(\Phi(sig_1[\Sigma_1'])) & \xrightarrow[\beta_{sig_1[\Sigma_1']}]{} & Mod_1(sig_1[\Sigma_1'])
\end{array} \tag{15.11}$$

A simple exercise that provides some insight into the concept of structured comorphism, is given by the structuring with BASIC as the only building operator. Given an institution comorphism $(\Phi,\alpha,\beta): I_1 \to I_2$ we consider the institutions of the theories I_1^{th} and I_2^{th} as being structured by the respective forgetful functors from theories to signatures. Then $(\Phi,\alpha,\beta): I_1 \to I_2$ can be lifted canonically to a comorphism $(\Phi',\alpha',\beta'): I_1^{\text{th}} \to I_2^{\text{th}}$ defined essentially by $\Phi'(\Sigma_1,E_1) = (\Phi\Sigma, \alpha_{\Sigma_1}E_1)$. A straightforward check reveals that in this case, the Satisfaction Condition of (Φ,α,β) causes that each $\beta'_{(\Sigma_1,E_1)}$ exists as a restriction of β_{Σ_1}, on the one hand, and the Satisfaction Condition for (Φ',α',β'), on the other hand. As we will see from Prop. 15.10 below, from these two, only the latter causality can be expected in general, the former being a conjunctional one which in the general case has to be postulated.

The following shows that in general, the concept of structured comorphism can be given with significantly fewer data.

Proposition 15.10. *Given (I_k, sig_k)-structured institutions I_k', $k=1,2$, and functors Φ,Φ' such that the square (15.10) commutes, the following are equivalent:*

1. *There exists a comorphism $(\Phi',\alpha',\beta'): I_1' \to I_2'$ that is $((\Phi,\alpha,\beta), sig_1, sig_2)$-structured.*
2. *For each I_1'-signature Σ_1', we have that*

$$\beta_{sig_1[\Sigma_1']}(Mod_2'[\Phi'\Sigma_1']) \subseteq Mod_1'[\Sigma_1']. \tag{15.12}$$

Proof. The implication 1. $\Rightarrow$ 2. is immediate from (15.11); we therefore focus on the other implication. Since α' is determined uniquely from the definition of structured comorphisms, it remains to define β'. By (15.12) we may indeed define each $\beta'_{\Sigma_1'}$ as the restriction of $\beta_{sig_1[\Sigma'_1]}$. It remains to prove the naturality of β' and the Satisfaction Condition of (Φ', α', β').

- The naturality property of β' holds by the following calculation (for each I'-signature morphism $\theta'_1 : \Sigma'_1 \to \Omega'_1$ and each $N'_2 \in Mod'_2[\Phi'\Omega'_1]$):

$$\begin{array}{ccc} Mod'_2[\Phi'\Sigma'_1] & \xrightarrow{\beta'_{\Sigma'_1}} & Mod'_1[\Sigma'_1] \\ {\scriptstyle Mod'_2[\Phi'\theta'_1]}\uparrow & & \uparrow{\scriptstyle Mod'_1\theta'_1} \\ Mod'_2[\Phi'\Omega'_1] & \xrightarrow[\beta'_{\Omega'_1}]{} & Mod'_1[\Omega'_1] \end{array}$$

$$\begin{array}{rcll} \beta'_{\Sigma'_1}(N'_2\restriction_{\Phi'\theta'_1}) & = & \beta_{sig_1[\Sigma'_1]}(N'_2\restriction_{\Phi'\theta'_1}) & \text{definition of } \beta' \\ & = & \beta_{sig_1[\Sigma'_1]}(N'_2\restriction_{sig_2[\Phi'\theta'_1]}) & (15.8) \\ & = & \beta_{sig_1[\Sigma'_1]}(N'_2\restriction_{\Phi(sig_1\theta'_1)}) & (15.10) \\ & = & (\beta_{sig_1[\Omega'_1]}N'_2)\restriction_{sig_1\theta'_1} & \text{naturality of } \beta \\ & = & (\beta'_{\Omega'_1}N'_2)\restriction_{sig_1\theta'_1} & \text{definition of } \beta' \\ & = & (\beta'_{\Omega'_1}N'_2)\restriction_{\theta'_1} & (15.8). \end{array}$$

- The Satisfaction Condition for the comorphism $(\Phi', \alpha', \beta') : I'_1 \to I'_2$ is established by the following sequence of equivalent satisfaction relations (for each I'_1-signature Σ'_1, each $M'_2 \in Mod'_2[\Phi'\Sigma'_1]$ and each $\rho'_1 \in Sen'_1\Sigma'_1$):

$$\begin{array}{ll} \beta'_{\Sigma'_1}M'_2 \models^{I'_1}_{\Sigma'_1} \rho'_1 & \\ \beta'_{\Sigma'_1}M'_2 \models^{I_1}_{sig_1[\Sigma'_1]} \rho'_1 & (15.9) \\ \beta_{sig_1[\Sigma'_1]}M'_2 \models^{I_1}_{sig_1[\Sigma'_1]} \rho'_1 & \beta' \text{ restriction of } \beta \\ M'_2 \models^{I_2}_{\Phi(sig_1[\Sigma'_1])} \alpha_{sig_1[\Sigma'_1]}\rho'_1 & \text{Satisfaction Condition for } (\Phi, \alpha, \beta) \\ M'_2 \models^{I_2}_{sig_2[\Phi'\Sigma'_1]} \alpha_{sig_1[\Sigma'_1]}\rho'_1 & (15.10) \\ M'_2 \models^{I'_2}_{\Phi'\Sigma'_1} \alpha_{sig_1[\Sigma'_1]}\rho'_1 & (15.9) \\ M'_2 \models^{I'_2}_{\Phi'\Sigma'_1} \alpha'_{\Sigma'_1}\rho'_1 & \text{definition of } \alpha'. \end{array}$$

□

The existence of translations of abstractly structured specifications. The result of Prop. 15.11 answers this problem. It is clear that the basis for a solution is the apriori existence of a translation at the underlying logical level, i.e. a comorphism between the base institutions. For lifting this to the upper level representing the abstractly structured specifications, we rely on normal forms.

Proposition 15.11. *Let I'_k be (I_k, sig_k)-structured institutions, $k = 1,2$, such that*

1. *I'_1 has finitary semantic normal forms (φ, E) with $\varphi \in \mathcal{D}$ for $\mathcal{D}$ a class of I_1-signature morphisms,*
2. *I'_2 inherits the signature morphisms,*
3. *I'_2 has basic specifications.*

Then any comorphism $(\Phi, \alpha, \beta) : \ I_1 \to I_2$ such that

4. *any morphism in $\mathcal{D}$ has weak (Φ, β)-amalgamation, and*
5. *I'_2 has semantic $\Phi\varphi$-derivations for any $\varphi \in \mathcal{D}$,*

determines a $((\Phi, \alpha, \beta), sig_1, sig_2)$-structured comorphism $(\Phi', \alpha', \beta') : \ I'_1 \to I'_2$.

Proof. We rely on the characterisation of $((\Phi, \alpha, \beta), sig_1, sig_2)$-structured comorphisms given by Prop. 15.10. Then we have to do two main things: to define the functor Φ' satisfying the commutativity property of (15.10), and then to prove the relation (15.12).

- The definition of Φ'.
  - For any I'_1-signature Σ'_1, $\Phi'\Sigma'_1 \ = \ \Phi\sigma_1 \ \square \ \mathrm{SP}(\Phi\Sigma_1, \alpha E_1)$, where $(\sigma_1 : \ sig_1[\Sigma'_1] \to \Sigma_1, E_1)$ is a finitary semantic normal form for Σ'_1.
  - For any I'_1-signature morphism $\theta'_1 : \ \Sigma'_1 \to \Omega'_1$ we define $\Phi'_1\theta'_1$ as the lifting of $\Phi(sig_1\theta'_1)$ by virtue of I'_2 inheriting the signature morphisms of I_2. Then the functoriality of Φ and sig_1 imply the functoriality of Φ'. But in order to define $\Phi'\theta'_1$ like that, by inheritance, we need to prove:

    1 $\quad Mod'_2[\Phi'\Omega'_1]\restriction_{\Phi(sig_1\theta'_1)} \subseteq Mod'_2(\Phi'\Sigma'_1).$
  - With these definitions, the commutativity of the square (15.10) is straightforward.
  - We complete the definition of Φ' by proving 1. Let $\Phi'\Omega'_1 \ = \ \Phi\omega_1 \ \square \ \mathrm{SP}(\Phi\Omega_1, \alpha\Gamma_1)$, where $(\omega_1 : \ sig_1[\Omega'_1] \to \Omega_1, \Gamma_1)$ is a finitary normal form for Ω'_1.

$$\begin{array}{ccccccc} sig_1[\Sigma'_1] & & \Sigma'_1 & & \Phi'\Sigma'_1 & = & \Phi\Sigma_1 \square \mathrm{SP}(\Phi\Sigma_1, \alpha E_1) \\ {\scriptstyle sig_1\theta'_1}\downarrow & & {\scriptstyle \theta'_1}\downarrow & & {\scriptstyle \Phi'\theta'_1}\downarrow & & \\ sig_1[\Omega'_1] & & \Omega'_1 & & \Phi'\Omega'_1 & = & \Phi\Omega'_1 \square \mathrm{SP}(\Phi\Sigma_1, \alpha E_1) \end{array}$$

Then 1 means that for each $M_2 \in |Mod_2(\Phi\Omega_1, \alpha\Gamma_1)|$

2 $\quad M_2\restriction_{\Phi\omega_1}\restriction_{\Phi(sig_1[\theta'_1])} \in Mod_2(\Phi\Sigma_1, \alpha E_1)\restriction_{\Phi\sigma_1}.$

– So, we have to prove 2. We do as follows:

$$
\begin{array}{lll}
\beta_{sig_1[\Sigma'_1]}(M_2\!\upharpoonright_{\Phi\omega_1}\upharpoonright_{\Phi(sig_1\theta'_1)}) = (\beta_{\Omega_1}M_2)\!\upharpoonright_{\omega_1}\upharpoonright_{sig_1\theta'_1} & & \text{naturality of } \beta \\
\qquad \in Mod_1(\Omega_1,\Gamma_1)\!\upharpoonright_{\omega_1}\upharpoonright_{sig_1\theta'_1} & & \text{Satisfaction Condition of } (\Phi,\alpha,\beta) \\
\qquad = Mod'_1[\Omega'_1]\!\upharpoonright_{sig_1\theta'_1} & & (\omega_1,\Gamma_1) \text{ normal form for } \Omega'_1 \\
\qquad \subseteq Mod'_1[\Sigma'_1] & & \theta'_1 : \Sigma'_1 \to \Omega'_1, (15.8) \\
\qquad = Mod_1(\Sigma_1,E_1)\!\upharpoonright_{\sigma_1} & & (\sigma_1,E_1) \text{ normal form for } \Sigma'_1.
\end{array}
$$

Hence there exists $M_1 \in Mod_1(\Sigma_1,E_1)$ such that $M_1\!\upharpoonright_{\sigma_1} = \beta_{sig_1[\Sigma'_1]}(M_2\!\upharpoonright_{\Phi\omega_1}\upharpoonright_{\Phi(sig_1\theta'_1)})$.

– By the amalgamation property of the following naturality square

$$
\begin{array}{ccc}
Mod_2(\Phi(sig_1[\Sigma'_1])) & \xleftarrow{Mod_2(\Phi\sigma_1)} & Mod_2(\Phi\Sigma_1) \\
\Big\downarrow \beta_{sig_1[\Sigma'_1]} & & \Big\downarrow \beta_{\Sigma_1} \\
Mod_1(sig_1[\Sigma'_1]) & \xleftarrow[Mod_1\sigma_1]{} & Mod_1\Sigma_1
\end{array}
$$

there exists $N_2 \in Mod_2(\Phi\Sigma_1)$ such that $N_2\!\upharpoonright_{\Phi\sigma_1} = M_2\!\upharpoonright_{\Phi\omega_1}\upharpoonright_{\Phi(sig_1\theta'_1)}$ and $\beta_{\Sigma_1}N_2 = M_1$.

– By the Satisfaction Condition of (Φ,α,β), since $M_1 \models E_1$, it follows that $N_2 \models \alpha E_1$. This completes the proof of 2.

• The proof of the relation (15.12). This holds by the following calculations.

$$
\begin{array}{lll}
\beta_{sig_1[\Sigma'_1]}(Mod'_2[\Phi'\Sigma'_1]) = \beta_{sig_1[\Sigma'_1]}(Mod_2(\Phi\Sigma_1,\alpha E_1)\!\upharpoonright_{\Phi\sigma_1}) & & (\sigma_1,E_1) \text{ is a normal form for } \Sigma'_1 \\
\qquad \subseteq \beta_{\Sigma_1}(Mod_2(\Phi\Sigma_1,\alpha E_1))\!\upharpoonright_{\sigma_1} & & \text{naturality of } \beta \\
\qquad \subseteq Mod_1(\Sigma_1,E_1)\!\upharpoonright_{\sigma_1} & & \text{Satisfaction Condition of } (\Phi,\alpha,\beta) \\
\qquad = Mod'_1\Sigma'_1 & & (\sigma_1,E_1) \text{ is a normal form for } \Sigma'_1.
\end{array}
$$

□

From the conditions of Prop. 15.11 only 1. and 5. are substantial, the rest are just technical conditions. The third condition underlying Prop. 15.11 can be fulfilled in various ways, leading to various different translations. In actual situations that support the BASIC building operator, the most straightforward way is $\mathrm{SP}(\Sigma,E) = (\Sigma,E)$, however this corresponds to having the result of the translation always in normal form. Alternatively $\mathrm{SP}(\Sigma,E)$ can be chosen to be a properly structured specification, for example in concrete situations that support such building operators, a specification structured by BASIC, UNION, and TRANS.

Exercises

15.10. For any institutions I, I', I'', if I' is (I, sig)-structured and I'' is (I', sig')-structured then I'' is $(I,\ sig'; sig)$-structured.

15.11. [75] **Quotienting structured institutions**
Let I' be an institution that is (I, sig)-structured. A congruence $\equiv$ on Sig' is a *structuring congruence* when for any I'-signatures Σ', Ω', we have that $\Sigma' \equiv \Omega'$ implies $sig[\Sigma'] = sig[\Omega']$ and $Mod'[\Sigma'] = Mod'[\Omega']$ (and similarly for I'-signature morphisms).

1. Any structuring congruence determines a canonical quotient $I'/_{\equiv}$ of I' that is $(I, sig/_{\equiv})$-structured where $sig_{\equiv} : Sig'/_{\equiv} \to Sig$ is the canonical quotient of sig.
2. If sig is faithful and whenever $\Sigma'_1 \equiv \Sigma'_2$ we have that there exists $i : \Sigma'_1 \to \Sigma'_2$ such that $i/_{\equiv} = 1_{\Sigma'_k/_{\equiv}}$, then
   - $sig/_{\equiv}$ lifts whatever co-limits are lifted by sig, and
   - $I'/_{\equiv}$ is compositional is I' is compositional.
3. If I' has semantic basic specifications / unions / φ-translations / φ-derivations then $I'/_{\equiv}$ has them too.

15.12. Derive the result on amalgamation for the institution of theories of Thm. 4.8 as an instance of the result of Prop. 15.9. (*Hint:* Consider the institution I^{th} as (I, sig)-structured where sig is the forgetful functor $Th^I \to Sig^I$.)

15.13. Extend the algebraic rules of Ex. 15.4 to the more general case of abstractly structured specifications that have unions, translations, and derivations.

15.14. [75] **Compactness for abstractly structured specifications**
Let I' be a (I, sig)-structured institution that admits normal forms. If I is compact then I' is compact too.

15.15. [75] **Interpolation for abstractly structured specifications**
Let I' be a (I, sig)-structured institution and let $\mathcal{D}$, $\mathcal{L}'$ and $\mathcal{R}'$ be classes of I-signature morphisms such that

1. sig preserves pushouts,
2. the structuring of I' is compositional,
3. I' admits semantic normal forms (φ, E) with $\varphi \in \mathcal{D}$,
4. $sig\mathcal{L}' \,;\, \mathcal{D} \subseteq \mathcal{L}$ and $sig\mathcal{R}' \,;\, \mathcal{D} \subseteq \mathcal{R}$.

If I has Craig-Robinson $(\mathcal{L}, \mathcal{R})$-interpolation then I' has Craig-Robinson $(\mathcal{L}', \mathcal{R}')$-interpolation too.

15.16. [50] Let (Φ', α', β') be comorphism that is $((\Phi, \alpha, \beta), sig_1, sig_2)$-structured.

- If (Φ, α, β) is persistently liberal then (Φ', α', β') is persistently liberal too.
- If (Φ, α, β) has (weak) model amalgamation then (Φ', α', β') has (weak) model amalgamation too.

15.17. For any institution I let I^{fth} be the sub-institution of I^{th} determined by the finite theories. Define a comorphism $(\Phi, \alpha, \beta) : (I^{\text{fth}})^{\text{spec}} \to I^{\text{spec}}$ such that the components of β are identities.

15.18. Let us consider a base institutions I with two designated classes of signature morphisms $\mathcal{D}, \mathcal{T} \subseteq Sig^I$ such that

1. I has disjunctions,
2. $D \subseteq Sig^I$ is a subcategory, $\mathcal{D} \subseteq \mathcal{T}$ and each signature inclusion belongs to $\mathcal{D}$,

3. for any $t: \Sigma \to \Sigma'$ in $\mathcal{T}$ and $d_1: \Sigma \to \Sigma_1, d_2: \Sigma \to \Sigma_2$ in $\mathcal{D}$ there are pushout squares like below with $d_1', d_2' \in \mathcal{D}$

$$\begin{array}{ccc} \Sigma & \xrightarrow{d_1} & \Sigma_1 \\ {\scriptstyle t}\downarrow & & \downarrow \\ \Sigma' & \xrightarrow[d_1']{} & \Sigma_1' \end{array} \qquad\qquad \begin{array}{ccc} \Sigma & \xrightarrow{d_1} & \Sigma_1 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_2'} \\ \Sigma_2 & \xrightarrow[d_1']{} & \Sigma'' \end{array}$$

4. each morphism in $\mathcal{D}$ has the model expansion property.

Let I' be the institution of the structured specifications over I that are constructed by BASIC, UNION, TRANS with morphisms from $\mathcal{T} \subseteq Sig^I$, and intersections (in the sense of Ex. 15.7). Then I' has finitary semantic normal forms (φ, E) with $\varphi \in \mathcal{D}$. (Note that I' does not have syntactic normal forms in the sense of the definitions given in Sec. 15.2.)

15.5 Pre-defined types

In this section we touch a different specification topic. We do as follows.

1. We discuss briefly the concept of pre-defined types in general terms.
2. We illustrate the concept by an example.
3. We give an institution-theoretic semantics for pre-defined types.

This means that we limit our discussion only to the very basics, without any substantial development. However, in Sec. 16.4 we will come back to pre-defined types for developments in the context of constraint logic programming.

What are pre-defined types?

In programming and specification, data types play a very basic role as they represent the structure of the data we are using. There is no programming or specification without data types. There are two kinds of data types: user-defined and pre-defined. As the names suggest, in the former case the user has to write a specification, while in the latter case he just imports it from a respective system library. Languages that have full user-defined data types capabilities enjoy a high expressivity power. However, this is a rare feature that can be found only among some logic-based languages, usually specification languages. On the other hand, pre-defined types, although available only for a limited set of data types (such as numbers, lists, arrays, etc.) have the obvious benefits that they save specification effort and they may have rather efficient implementations. In principle, computations with pre-defined types should run more efficiently than if we specified the respective data types by ourselves. Here we have to clarify an important distinction between pre-defined data types that are implemented in a lower-level language or even in the underlying hardware platform, and those who are just specifications in the respective language that have been developed previously and stored as libraries. The latter just save our specification effort, in substance being no different from the user-defined types. In this section we are concerned only with the former kind, which we consider to be the proper pre-defined types. The aspect of being implemented at a lower-level is strongly related to computations efficiency and defines the main aspect of the mathematical semantics of pre-defined types, namely that the respective implementation is a particular model which is present in all logical entities, including also the presumed 'syntactic' ones, such as signatures and sentences. Although conventionally this may seem to be an outrageous idea, in the context of the pre-defined types is very natural, and the concept of institution, due to its abstract nature, accommodates it without problems. Let us see a simple concrete exemple of how this works.

The Euclidean plane. We consider the problem of the first-order specification of the 2-dimensional Euclidean plane $\mathbb{R}^2$ as a real vector space. Let us start from the following rather standard $\mathcal{FOL}$ signature $\Sigma = (S, F, P)$ for this problem:

- $S = \{\texttt{Real}, \texttt{Vect}\}$, where `Real` designates the sort of the scalars and `Vect` the sort of the vectors,
- F consists of the usual ring operations for the real numbers plus $F_{\to\texttt{Vect}} = \{0\}$, $F_{\texttt{RealReal}\to\texttt{Vect}} = \{\langle\_,\_\rangle\}$, $F_{\texttt{VectVect}\to\texttt{Vect}} = \{\_+\_\}$, $F_{\texttt{Vect}\to\texttt{Vect}} = \{-\_\}$, $F_{\texttt{RealVect}\to\texttt{Vect}} = \{\_*\_\}$, and $F_{w\to s} = \emptyset$ otherwise,
- P consists of empty sets. Of course, we could consider here a 'less than or equal' binary relation $\leq$, or other common relations on the reals, but they will not make a difference for what we want to illustrate.

There are several issues about this specification that we have to address:

1. The sort `Real` and the ring operations on it, should always be interpreted as the model of the real numbers or something like that. If we think of real numbers as-such, it is impossible to specify them in $\mathcal{FOL}$. Though their model can be specified in second-order logic as a complete ordered field.
2. We should be able to use the real numbers as syntactic entities. More clearly, we should be able to write terms such as $3.14 * \langle a, b\rangle$ or even $\pi * \langle\sqrt{2}, \sqrt{3}\rangle$ where $\pi, \sqrt{2}, \sqrt{3}$ represent / denote the respective real numbers. Though the latter term would require new operations such as π and $\sqrt{\_}$.
3. We should be able to use the specification of $\mathbb{R}^2$ to compute. From a pure specification perspective, computation is optional, but then what we get would be just some logical axiomatisation of $\mathbb{R}^2$ without any computational capabilities.

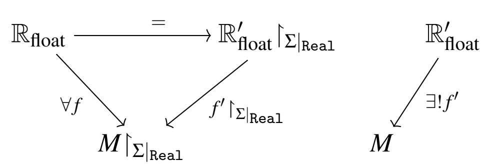
The solution to these issues involves a quite radical paradigm shift when a model becomes a component of a signature. Conventionally, this is utter non-sense, but the concept of institution and its associated way of thinking can take us safely beyond conventionality. In the case of $\mathbb{R}^2$ we can do as follows.

- In order to have a proper computational side for our specification we replace $\mathbb{R}$ by a model called $\mathbb{R}_{\text{float}}$ that implements operations for manipulating floating-point numbers. The use of floating-point reals is common in programming and specification whenever reals are involved. Besides the ring operations, $\mathbb{R}_{\text{float}}$ also implements other operations such as constants $1, 2, \ldots$, etc., but very notably division (which means $\mathbb{R}_{\text{float}}$ is a field), but in order to stay within $\mathcal{FOL}$ it is better to avoid having division as an operation. At this point it is helpful to consider that Σ contains also at least all constants that makes the writing of floating-point reals possible.
- $\mathbb{R}_{\text{float}}$ is a model of the sub-signature of Σ that is determined by the sort `Real`. We can get it as a model of Σ by considering its free extension $\mathbb{R}'_{\text{float}}$ to Σ, which means that in $\mathbb{R}'_{\text{float}}$ the operations involving `Vect` are interpreted freely. Thus the elements of $\mathbb{R}'_{\text{float}}$ are terms formed with those operations plus the elements of $\mathbb{R}_{\text{float}}$.
- Then the vector space $\mathbb{R}^2$ is obtained as the quotient of $\mathbb{R}'_{\text{float}}$ by the vector space equations E:

$$
\begin{aligned}
&0 = \langle 0, 0\rangle \\
&(\forall a, b, a', b')\ \langle a, b\rangle + \langle a', b'\rangle = \langle a + a', b + b'\rangle \\
&(\forall k, a, b)\ k * \langle a, b\rangle = \langle k * a, k * b\rangle \\
&(\forall a, b)\ -\langle a, b\rangle = \langle -a, -b\rangle.
\end{aligned}
$$

With the help of $\mathcal{FOL}$ diagrams, let us present this from a more institution-theoretic perspective.

- The signature is the pair $(\Sigma, \mathbb{R}'_{\text{float}})$.
- A $(\Sigma, \mathbb{R}'_{\text{float}})$-model is a Σ-model M together with an interpretation of the floating-point reals into M, which means a model homomorphism $h\colon\ \mathbb{R}'_{\text{float}} \to M$. Because of the universal property of $\mathbb{R}'_{\text{float}}$, this is the same with the $(\Sigma|_{\texttt{Real}}, \mathbb{R}_{\text{float}})$-model $h\upharpoonright_{\Sigma|_{\texttt{Real}}}$, where $\Sigma|_{\texttt{Real}}$ is the sub-signature of Σ for the real numbers.



- A $(\Sigma, \mathbb{R}'_{\text{float}})$-sentence is just a $\Sigma_{\mathbb{R}'_{\text{float}}}$-sentence, where $\Sigma \hookrightarrow \Sigma_{\mathbb{R}'_{\text{float}}}$ is the elementary extension of Σ via $\mathbb{R}'_{\text{float}}$. The $(\Sigma, \mathbb{R}'_{\text{float}})$-terms are thus formed from the floating-point reals and the vector space operations. Strictly speaking, a term such as $\langle 1, 2\rangle$ carries a level ambiguity because, on the one hand, it can be regarded as an element of $\mathbb{R}'_{\text{float}}$, and on the other hand, it can be thought as a term formed from the operation $\langle \_, \_\rangle$ and the constants 1 and 2. But this ambiguity is vacuous because both ways yield the same interpretation in the models M as their equality belongs to the diagram of $\mathbb{R}'_{\text{float}}$.

- The satisfaction relation $(h\colon\ \mathbb{R}'_{\text{float}} \to M) \models \rho$ is defined as $M_h \models \rho$, where M_h is the expansion of M that interprets the floating-point reals according to h.

Note that there are many models of the pre-defined theory $((\Sigma, \mathbb{R}'_{\text{float}}), E)$, for example the model $\mathbb{R}+$ interpreting `Vect` as the set of real numbers and $\langle \_, \_ \rangle$ as addition of real numbers. The 'intended' model of this theory, the Euclidean plane $\mathbb{R}^2$, is in fact the initial model of the theory.

The institution of pre-defined types

The ideas of the specification $\mathbb{R}^2$ presented above can be formalised as an institution-independent construction. We consider a base institution with diagrams $I = (Sig, Sen, Mod, \models, \iota)$ and define an institution $I^{\wr} = (Sig^{\wr}, Sen^{\wr}, Mod^{\wr}, \models^{\wr})$, called the *institution of pre-defined types* as follows.

- $Sig^{\wr}$ is the Grothendieck category $Mod^{\sharp}$.
- For each $I^{\wr}$-signature (Σ, A), $Sen^{\wr}(\Sigma, A) = Sen(\Sigma_A)$. For each $I^{\wr}$-signature morphism (φ, h), $Sen^{\wr}(\varphi, h) = Sen\ \iota_{\varphi} h$.
- For each $I^{\wr}$-signature (Σ, A), $Mod^{\wr}(\Sigma, A) = Mod(\Sigma_A, E_A)$. For each $I^{\wr}$-signature morphism (φ, h), $Mod^{\wr}(\varphi, h) = Mod\ \iota_{\varphi} h$.
- for each $I^{\wr}$-signature (Σ, A), $M' \models^{\wr}_{(\Sigma, A)} \rho$ if and only if $M' \models_{\Sigma_A} \rho$.

Because of the natural isomorphism $Mod(\Sigma_a, E_A) \cong A/Mod\Sigma$ we can have an alternative equivalent definition for $Mod^{\wr}$ as $Mod^{\wr}(\Sigma, A) = A/Mod\Sigma$. In the example of $\mathbb{R}^2$ we have discussed the models in this style. Depending on how we involve the models of $I^{\wr}$ we can use the most convenient variant.

Proposition 15.12. *$I^{\wr} = (Sig^{\wr}, Sen^{\wr}, Mod^{\wr}, \models^{\wr})$ is an institution indeed.*

Proof. The functoriality of $Sen^{\wr}$ and $Mod^{\wr}$ follows directly from the general categorical properties of institution-theoretic diagrams. For the proof of the Satisfaction Condition for $I^{\wr}$, we consider a $I^{\wr}$-signature morphism $(\varphi, h)\colon\ (\Sigma, A) \to (\Sigma', A')$, a (Σ', A')-model M', and a (Σ, A)-sentence ρ. We have that:

$$
\begin{aligned}
M \upharpoonright_{(\varphi,h)} \models^{\wr} \rho \ &= \ M' \upharpoonright_{\iota_{\varphi} h} \models \rho && \text{definitions of } Mod^{\wr} \text{ and } \models^{\wr} \\
&= \ M' \models (\iota_{\varphi} h)\rho && \text{Satisfaction Condition in } I \\
&= \ M' \models^{\wr} (\varphi, h)\rho && \text{definitions of } Sen^{\wr} \text{ and } \models^{\wr}.
\end{aligned}
$$

□

Exercises

15.19. For any base institution $(Sig, Sen, Mod, \models, \iota)$ with diagrams:

- define a canonical institution morphism $(Sig^{\wr}, Sen^{\wr}, Mod^{\wr}, \models^{\wr}) \to (Sig, Sen, Mod, \models)$ from the institution of pre-defined types to the base institution; moreover this is an adjoint institution morphism whenever the categories of models have initial models, and

- define an institution comorphism $(Sig^{\wr}, Sen^{\wr}, Mod^{\wr}, \models^{\wr}) \to (Th, Sen^{\mathrm{th}}, Mod^{\mathrm{th}}, \models^{\mathrm{th}})$ from the institution of pre-defined types to the institution of theories over the base institution. (*Hint:* Each signature of pre-defined types (Σ, A) gets mapped to the diagram (Σ_A, E_A).)

15.20. Sound / complete proof system for predefined types
Let $(Sig, Sen, Mod, \models, \vdash, \iota)$ be an institution with entailments and with diagrams ι.

1. The institution with pre-defined types $(Sig^{\wr}, Sen^{\wr}, Mod^{\wr}, \models^{\wr}, \vdash^{\wr})$ admits an entailment system defined by $\Gamma \vdash^{\wr}_{(\Sigma,A)} E$ if and only if $\Gamma \cup E_A \vdash_{\Sigma_A} E$ (where $\iota_\Sigma A : \Sigma \to (\Sigma_A, E_A)$ is the diagram of A).
2. $I^{\wr}$ is sound / complete, whenever the base institution I is sound / complete.
3. The entailment system of $I^{\wr}$ is precisely the free entailment system generated by the entailments of I plus the rules $\emptyset \vdash_{\Sigma_A} E_A$ for each Σ-model A.

15.21. Model amalgamation for pre-defined types
For any base institution with diagrams $(Sig, Sen, Mod, \models, \iota)$ such that

- its category of signatures *Sig* has pushouts, and
- it is semi-exact,
- all its signature morphisms are liberal,
- for each signature $\Sigma \in |Sig|$, the category $Mod\Sigma$ of Σ-models has pushouts,

the corresponding institution of pre-defined types $(Sig^{\wr}, Sen^{\wr}, Mod^{\wr}, \models^{\wr})$ has pushouts of signature morphisms and is semi-exact.

15.22. Diagrams for pre-defined types
Each institution of pre-defined types has 'empty' diagrams. (*Hint:* For any signature with predefined type (Σ, A) and any (Σ, A)-model with pre-defined types M', the elementary extension $(\Sigma, A) \to (\Sigma, A)_{M'}$ is defined as $(1_\Sigma, i_{\Sigma,A} M') : (\Sigma, A) \to (\Sigma, M' \upharpoonright_{\iota_\Sigma A})$.)

15.23. Initial semantics for pre-defined types
An institution of pre-defined types is liberal whenever its base institution is liberal.

15.24. $I^{\wr}$ as a Grothendieck institution
Present any institution $I^{\wr}$ of pre-defined types as a (morphism-based) Grothendieck institution of an indexed institution $Sig^{\mathrm{op}} \to \mathbb{I}ns$, where *Sig* is the category of the signature of the base institution I.

15.25. [77] **Interpolation in $I^{\wr}$**
Let $I = (Sig, Sen, Mod, \models, \iota)$ be an institution with diagrams such that:

- *Sig* has pushouts,
- it has model amalgmation, and
- all its signature morphisms are liberal,
- for each signature $\Sigma \in Sig$, the category $Mod\Sigma$ of Σ-models has pushouts.

For any class $\mathcal{S} \subseteq Sig$ of signature morphisms, let $\mathcal{S}^{\wr} = \{(\varphi, h) \in Sig^{\wr} \mid \iota_\varphi h \in \mathcal{S}\}$. If I has Craig-Robinson $\langle \mathcal{L}, \mathcal{R} \rangle$-interpolation then $I^{\wr}$ has Craig-Robinson $\langle \mathcal{L}^{\wr}, \mathcal{R}^{\wr} \rangle$-interpolation. (*Hint:* Translate the interpolation problem from $I^{\wr}$ to I^{th} by the comorphism of Ex. 15.19. The resulting square of theory morphisms has model amalgamation by the model amalgamation property of $I^{\wr}$ (see Ex. 15.21). The pushout of the first two morphisms of this square is CRi square by Ex. 9.13. By model amalgamation, the unique mediating morphism between these two squares has the model expansion property, which allows for the transfer of the CRi property from the inner pushout square to the original outer square.)

15.26. Basic sentences in $I^{\wr}$
In any institution $I^{\wr}$ of pre-defined types, a (Σ, A)-sentence ρ is (epic) basic when it is (epic) basic as a sentence of the base institution I. (*Hint:* Use the combined results of Exercises 15.19, 5.31 and 5.32.)

Notes. The material of this chapter belongs to the area known as 'algebraic specification'. Originally, that was concerned only with specifications based on some forms of universal algebra, which explains the name of the area. Over time, many new specification logics began to lack algebraic characteristics, hence the terminology 'logic-based specification'. However, at a higher level, the general role played by category theory (itself an algebraic structure), brings another meaning to the concept of 'algebra specification'. Algebraic specification is a vast scientific area where theory and practice complement and support each other. Here we gave only a very theoretical taste of this area. Books such as [219, 193, 46, 94] are a good source of practical examples of algebra specification 'in action'.

The kernel language for structuring specifications presented here has been introduced in [218] but with the union restricted to the situation when the specifications have the same signature; it constitutes the most common set of specification building operators in the literature. This is a special case of our more general union when we consider the trivial inclusion system for the signatures with inclusions being the identities. Modern algebraic specification languages provide more sophisticated structuring constructs, however it is possible to translate many them, but not all, to this kernel language (see [184] for CASL). To fill this gap other specification building operators have been considered (see [219, 92], etc.). A very important operator that we did not include in our discussion is the free construction / initial semantics operator. The normal forms for specifications formed only with unions, translations and derivations are well known from [104, 20, 33]. The importance of normal forms is that it allows us to replace any specification by its appropriate normal form, for which some basic properties are more easily available.

The extension of the entailment from a base institution to the institution of its specifications was originally defined in [218]. The idea of $(\mathcal{T}, \mathcal{D})$-specifications was introduced in [33], which under assumptions similar to the conditions of Thm. 15.8 proved the lifting of entailment completeness from the base institution to specifications. However the completeness result of [33] is obtained in a framework assuming implications, conjunctions, and Craig interpolation for the base institution which is significantly narrower in terms of applications than our framework which assumes just Craig-Robinson interpolation. For example, the completeness results of [33] cannot be used in important computing logics such as $\mathcal{EQL}$ or $\mathcal{HCL}$, which, on the other hand, support the applications of Thm. 15.8.

The theory of abstractly structured specifications has been introduced in [75], the concept of comorphism for structured institutions in [50], and the existence of translations between abstractly structured specifications has been studied in [80]. Specification building operators for abstractly structured specifications have been introduced in [93]. The theory of abstractly structured specifications constitutes the underlying theoretical framework for a series of works on specification structuring including [75, 236, 93, 52].

The institutions of pre-defined types were first introduced in [61] under the name of 'constraint institutions' in a slightly different form and in the context of the so-called 'category-based equational logic'. In their current form they were introduced [77].

Chapter 16

Logic Programming

In this chapter we discuss institution-independent foundations for what is probably the most eccentric major existing programming paradigm, namely logic programming. This programming paradigm is related to traditional algebraic specification, though this relationship is best understood at the foundational level. The institution-independent approach to logic programming liberates it from its traditional context and thus provides the opportunity to develop it easily and uniformly over various different structures. Consequently, we can enhance logic programming with other computing paradigms, such combinations having the potential of new powerful domain specific programming paradigms. The contents of this chapter is as follows.

1. We explain logic programming briefly, from the traditional perspective. This means that engagement with this chapter does not require any previous familiarity with logic programming.
2. We develop the fundamental concepts of logic programming in an institution-independent framework. They include programs, queries, solutions, and solution forms. Herbrand theorems do constitute the model-theoretic foundations of logic programming. We prove two institution-independent versions of a Herbrand theorem. These represent two gradual steps that bridge the model-theoretic to the computational semantics of logic programming.
3. Then we touch the topic of modularisation for logic programs. Much of that is shared with specification structuring. This means that, on the one hand, we can rely on concepts and results from Chap. 15. On the other hand, we will address modularisation issues that are specific to logic programming.
4. In another section we extend the logic programming paradigm to 'constraint' logic programming, an extension of great practical importance. The institution-independent view on logic programming allows us to regard constraint logic programming just as ordinary logic programming over institutions with pre-defined types. Consequently, we obtain for free general versions of Herbrand theorems for constraint logic pro-

gramming just as instances of the general institution-independent Herbrand theorems for ordinary logic programming.

5. In our discussion we do not include the full standard operational / computational semantics of logic programming (i.e. how logic programs are executed), such as the so-called ‘resolution’ algorithm and variants (paramodulation, narrowing, etc.). But we dedicate the final section to unification, which lies at the core of this operational semantics. At the abstract level, unification can be regarded as a co-limit construction problem. This allows for a category-theoretic analysis of unification which includes also the development of a generic categorical unification algorithm applicable to various institutions.

The study of this chapter requires familiarity with material from the first part of the book (until Chap. 5 included) and from Chap. 15.

16.1 What is logic programming?

The logic programming paradigm emerged from an understanding that some important fragment of $\mathcal{FOL}$, with good computational properties, can be turned into a programming paradigm. In essence, this fragment is $\mathcal{HCL}$. Thus, a logic program is a finite theory in $\mathcal{HCL}$, and its model-theoretic semantics is its initial model, called ‘Herbrand model’ in the specific logic programming terminology. The initial semantics is the point of convergence with traditional algebraic specification of abstract data types, both paradigms being limited to Horn theories. However, in the traditional contexts this is also the point from where the two paradigms begin to diverge. While traditional algebraic specification is based on the many-sorted equational version of $\mathcal{HCL}$ (i.e. $\mathcal{CEQL}$), logic programming is traditionally based on the Horn clause sub-institution of $\mathcal{REL}^1$, this being the single-sorted equation-free variant of $\mathcal{HCL}$. From a broader perspective, both paradigms can be supported by $\mathcal{HCL}$ in its full form.

Logic programming programs. Logic programming is a *declarative* programming paradigm, which means that, in principle, a program is a specification. Let us introduce a very simple example. First, let us consider a $\mathcal{CEQL}$ specification of the natural numbers with succesor and addition operations. The signature consists of one sort, a constant 0, an unary operation $s\_$ for the succesor, and a binary operation $\_+\_$. Then the initial model of the theory formed by the following two equations

$$(\forall x)\ x+0=x,\ \ (\forall x,y)\ x+sy=s(x+y) \tag{16.1}$$

is the model of the natural numbers with addition. This is the equational approach to the problem. We can have a $\mathcal{REL}^1$ (relational) alternative to this by introducing a ternary relation symbol *add* where $add(x,y,z)$ encodes $x+y=z$:

$$(\forall x)\ add(x,0,x),\ \ (\forall x,y,z)\ add(x,y,z) \Rightarrow add(x,sy,sz). \tag{16.2}$$

This is a Horn clause theory in $\mathcal{REL}^1$. This encoding cannot be explained as a comorphism, because we cannot specify by relations the equality of elements. There is no way to do this due to the very nature of equality. This explains why first-order logic with equality is more than first-order logic without equality. However, both (16.1) and (16.2) yield essentially the same initial models, i.e. the models of the natural numbers with addition. Since logic programming semantics is confined essentially to initial models of programs, we can always simulate safely equations in the manner of (16.2).

Running programs. The main difference between logic programming and traditional specification of abstract data types resides in what it computes rather than in what it specifies. For instance, in equational logic programming, a program is hardly different from an algebraic specification, but these two have different purposes. While the aim of specification is to define axiomatically classes of models, the purpose of logic programming is a computational one. From a logical viewpoint, given a logic programming program P, the aim of logic programming computations is to prove that

$$P \models (\exists Y)q \tag{16.3}$$

where q is an atomic formula. From a computation viewpoint, the aim is to prove (16.3) constructively, which means to actually find its solutions. In logic programming terminology, a consequence problem like (16.3) is called *query*. The foundations guarantee (we will see how later on in the chapter) that this is a achieved by finding valuations $\theta : Y \to 0_\Sigma$ (where Σ is the signature of P and 0_Σ is its term model) such that $(0_P)^\theta \models q$, where $(0_P)^\theta$ is the expansion of 0_P, the initial model of P, given by θ. When P is clear from the context we may call just $(\exists Y)q$ a query.

A simple example. Let us consider the program P to be the theory (16.1) and try to solve $(\exists a)\ a+1=3$. Of course, here 1 abbreviates $s0$ and 3 abbreviates $sss0$. To do this in the most traditional form, we rewrite P in the relational form (16.2) and so we do with the query, which gets encoded as $(\exists a)\ add(a, s0, sss0)$. The computation follows the so-called *resolution algorithm*, which means that:

- We set the initial 'goal' to $add(a, s0, sss0)$.
- At each step we generate another set of goals by 'unifying' one of the goals in the list with the conclusion of a Horn clause from the program P. Unification means that we find a substitution of the variables such that the respective goal and the conclusion become equal; this is called a 'unifier'. If fact we have to find a 'minimal' unifier, usually called 'most general'. Then we add to the list of goals the hypothesis of the Horn clause, instantiated by the unifier.
- We stop when the list of the goals gets empty. The result is obtained by composing all substitutions obtained in the process.

It can be proved that this general algorithm produces all possible solutions. It has a non-deterministic nature because, in principle, it is possible that in some moments of running it, we can have different several choice about what Horn clause from the program to use. In the case of our current problem, it just happens that the execution of this algorithm is deterministic and consists of the following two steps. In order to avoid clashes of names of variables, at each step we use fresh variables in the clauses.

1	$add(a, s0, sss0)$	the initial goal is set
2	$add(x_1, 0, ss0)$	second clause of (16.2), substitution: $a \mapsto x_1$, $y_1 \mapsto 0$, $z_1 \mapsto ss0$
3	$\emptyset$	first clause of (16.2), substitution: $x_1 \mapsto x_2$, $x_2 \mapsto ss0$.

The list of goals is now empty, so we get the result by composing the substitutions:

$$(a \mapsto x_1) \ ; \ (x_1 \mapsto x_2) \ ; \ (x_2 \mapsto ss0) \ = \ (a \mapsto ss0).$$

This resolution process corresponds to the following sequence of logical consequences, read in reverse order.

4	$P \models add(ss0, 0, ss0)$	monotonicity, Substitutivity $(x \mapsto ss0)$
5	$P \models add(ss0, s0, sss0)$	4, Substitutivity ($x \mapsto ss0$, $y \mapsto 0$, $z \mapsto ss0$), Modus Ponens
6	there exists θ substitution such that $P \models \theta \ add(a, s0, sss0)$	5, $\theta a = ss0$
7	$P \models (\exists a) \ add(a, s0, sss0)$.	

Resolution is a typical example of the *proof-as-computation* paradigm that is based on ‘proof goals’. In this case, in order to establish 7, we had to establish 6, then 5, and finally 4. The step from 6 to 7 is a general one, but the rest corresponds to the resolution steps 1, 2, 3.

Resolution is the standard operational semantics for logic programming based on relations, in equational logics the corresponding operational semantics is called ‘paramodulation’. In the context of the encoding of relations as operations from Sec. 3.3, it is possible to simulate resolution by paramodulation. Here, we gave only a glimpse of the operational semantics of logic programming, other fundamental issues about resolution and paramodulation being their general / abstract definitions and their soundness and completeness properties. In this chapter we will not do all these. From the operational semantics, here we will study only unification, which represents the core of the logic programming operational semantics.

Exercises

16.1. Run the program P above for the query $(\exists a, b) \ a + b = ss0$. Do this in two ways, first by hand and then install a Prolog system on your computer and run this problem with the system.

16.2 Herbrand theorems

A full understanding of logic programming requires both an understanding of its model-theoretic meaning and its computational / operational side, and also of the bridge between them. Logic programming means the aggregation of all these three aspects. The model-theoretic semantics and the bridge to the operational semantics can be done nicely at an abstract institution-independent level. The operational semantics can be also approached to a certain extent at that level of abstraction but in a more complicated way. The bridge consists of two 'Herbrand theorems' representing the two stages of the move from model-theoretic to operational semantics. In this section we do as follows:

1. We define logic programming in the most general model-theoretic terms.
2. We prove an institution-independent first Herbrand theorem that gives logic programming its initial semantics meaning.
3. We prove an institution-independent second Herbrand theorem, that builds on the first one, and that reformulates logic programming in a way that makes it ready for the operational semantics.

The definition of logic programming in abstract institutions

Let I be an arbitrary institution with a designated class $\mathcal{D}$ of quasi-representable signature morphisms. Then a *$\mathcal{D}$-query* is a question of the form $E \overset{?}{\models}_\Sigma (\exists\chi)q$, where (Σ, E) is a finite theory that admits an initial model (Σ, E), and $(\exists\chi)q$ is any existential $\mathcal{D}$-quantification of a basic sentence q. In this context, (Σ, E) is called a *program*. When the program is fixed then we may call $(\exists\chi)q$ the query.

Now, a couple of reflections on these definitions. concrete traditional logic programming:

- At the concrete level, the quasi-representable quantifications are near first-order extensions of signatures. We will see that full representability is required by the Second Herbrand Theorem, so we can say that the concrete scope of the Herbrand Theorems requires first-order quantifications.
- Basic sentences represent only a loose abstraction of the atomic sentences. This is another reason why abstract logic programming has the potential of a wider scope than traditional logic programming.
- The $\mathcal{D}$-queries $(\exists\chi)q$ need not represent actual sentences in the institution I, their satisfaction can be considered outside of the satisfaction relation $\models^I$. This means that $E \models (\exists\chi)q$ is an abbreviation for "for each model M such that $M \models E$ there exists a χ-expansion M' of M such that $M' \models q$".
- Our definition of programs as theories correspond to *unstructured* programs. In Sec. 16.3 we will study structured programs, which, like structured specifications, are not theories anymore. However, in logic programming, each structured program admit theories as normal forms, so the development in this section apply to the structured level as well.

First Herbrand theorem

The condition that a logic programming program admits an initial model defines to a great extent the scope of logic programming. In the logic programming culture, $0_{\Sigma,E}$ is called a 'Herbrand model' of (Σ, E), and it serves as the universe for the computation by resolution or by paramodulation. What this means is that the initial semantics allows for the computational side. The initial semantics also tells us much about what to expect in terms of the sentences used in the programs. Results from Sections 8.3 and 8.4 show that initial semantics is caused by quasi-varieties (Thm. 8.14), which are axiomatizable by Horn sentences (Thm. 8.18). This means that in any concrete institution, a logic program (Σ, E) appears as some kind of Horn (clause) theory.

The First Herbrand Theorem reduces the problem of checking the satisfaction of a query by a program / theory P from all possible models, to the initial model of P only. Moreover, this is an equivalence.

Theorem 16.1 (Herbrand theorem I)**.** *In any institution consider a theory* (Σ, E) *which has an initial model* $0_{\Sigma,E}$*. Then, for each query* $E \models (\exists\chi)q$*,*

$$E \models (\exists\chi)q \quad \textit{if and only if} \quad 0_{\Sigma,E} \models (\exists\chi)q.$$

Proof. The implication from left to right is trivial, hence we focus on the other implication. Let $\chi : \Sigma \to \Sigma'$. Assume that $0_{\Sigma,E} \models (\exists\chi)q$ and consider a Σ-model M such that $M \models E$. We have to prove that $M \models (\exists\chi)q$. Let M_q be a basic model for q.

1	$M_q\restriction_\chi$ basic model for $(\exists\chi)q$	M_q basic model for q, Fact 5.22
2	there exists homomorphism $M_q\restriction_\chi \to 0_{\Sigma,E}$	1, $0_{\Sigma,E} \models (\exists\chi)q$
3	there exists homomorphism $0_{\Sigma,E} \to M$	$M \models E$
4	there exists homomorphism $M_q\restriction_\chi \to M$	2, 3
5	$M \models (\exists\chi)q$	4, 1.

□

Solutions. In the proof of Thm. 16.1, each χ-expansion N' of $0_{\Sigma,E}$ such that $N' \models q$ is called a *solution* for the query $E \models (\exists\chi)q$. Because in concrete situations χ is often a signature extension with first-order variables, in these cases N' is represented by a valuation of the variables by terms. For instance, in the context of the example about addition of natural numbers, the query $(\exists a)\ add(a, s0, sss0)$ has one solution given by $N'_a = ss0$.

Second Herbrand theorem

Solution forms. In the same simple context of the addition of the natural numbers, the query $(\exists a, b)\ add(a, ss0, sb)$ has an infinite number of solutions, $\{N'_k \mid k \in \omega\}$ where $(N'_k)_a = k$ and $(N'_k)_b = k+1$. (Here 1 abbreviates $s0$, 2 abbreviates $ss0$, etc.) Obviously, we cannot get all these solutions by computing them, as the result of a computation process is always finite. However, they can be presented in a generic form by the substitution

$\theta : \{a,b\} \to \{a\}$ defined by $\theta a = a$ and $\theta b = sa$. Any valuation for a in the initial model $0_{\Sigma,E}$, which means any χ-expansion N of $0_{\Sigma,E}$, provides a solution $(Mod\theta)N$ for the query. Moreover, all solutions can be obtained like this. The substitution θ is called a *solution form* for the query. Commonly, solution forms rather than solutions are the results of logic programming computations. Solution forms are computationally more friendly than solutions, not only because they are fewer (sometimes this means finite rather than infinite), but also because for the former we have better algorithms to compute. This is the sense in which the result of Thm. 16.2 below represents a decisive step towards the computational semantics of logic programming.

The institution-independent concept of solution form is as follows. For any theory (Σ,E), a *solution form* for a $\mathcal{D}$-query $(\exists\chi_1)q$ is any $\mathcal{D}$-substitution $\theta : \chi_1 \to \chi_2$ such that $E \models (\forall\chi_2)\theta q$.

Theorem 16.2 (Herbrand Theorem II)**.** *Consider an institution with representable $\mathcal{D}$-substitutions for a class $\mathcal{D}$ of representable signature morphisms such that for each theory (Σ,E) with initial models*

- *the signature Σ also admits an initial model 0_Σ, and*
- *each signature morphism $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ has its representation M_χ projective with respect to the 'quotient' homomorphism $p_{\Sigma,E} : 0_\Sigma \to 0_{\Sigma,E}$.*

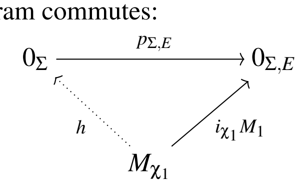
Then, for each theory (Σ,E) having an initial model, and for any $\mathcal{D}$-query $E \models (\exists\chi_1)q$, the following are equivalent:

1. *$E \models (\exists\chi_1)q$.*
2. *There exists a solution form $\theta : \chi_1 \to \chi_2$ such that χ_2 has the model expansion property.*

Proof. • *1. implies 2.* Assume $E \models (\exists\chi_1)q$ and let $\chi_1 : \Sigma \to \Sigma_1$. The signature morphism χ_2 will be just the identity morphism 1_Σ. We have to find the substitution $\theta : \chi_1 \to 1_\Sigma$.

– Since $0_{\Sigma,E} \models E \models (\exists\chi_1)q$, there exists a χ_1-expansion M_1 of $0_{\Sigma,E}$ such that $M_1 \models q$.

– By the projectivity of M_{χ_1}, there exists a homomorphism h such that the following diagram commutes:



(16.4)

– χ_1 is represented by M_{χ_1}, and 1_Σ by 0_Σ. Thus, let $\theta = \Psi_h : \chi_1 \to 1_\Sigma$ be the substitution represented by h. We have to show that $E \models \theta q$. This follows from the First Herbrand Theorem 16.1 if we can show that $0_{\Sigma,E} \models \theta q$ and that θq is basic.

– Let us prove the first thing, that $0_{\Sigma,E} \models \theta q$:

1	$(Mod\theta)(M_1\upharpoonright_{\chi_1}) = M_1$	semantic property of substitution $\theta : \chi_1 \to 1_\Sigma$

2	$(Mod\theta)(M_1\restriction_{\chi_1}) \models q$	1, $M_1 \models q$
3	$0_{\Sigma,E} \models \theta q$	2, Satisfaction Condition for θ, $0_{\Sigma,E} = M_1\restriction_{\chi_1}$.

– Finally, we show that θq is basic. We prove that $M_q\restriction_{\chi_1}$ is a basic model for θq when M_q is a basic model for q. This is achieved by the following sequence of equivalent facts, where N is any Σ-model:

4	$N \models \theta q$	
5	$(Mod\theta)N \models q$	Satisfaction Condition for θ
6	there exists a homomorphism $M_q \to (Mod\theta)N$	M_q basic model for q
7	there exists a homomorphism $M_q\restriction_{\chi_1} \to N$.	

The equivalence between 6 and 7 requires justification. On the one hand, for any homomorphism $M_q \to (Mod\theta)N$ we have that:

8	there exists homomorphism $M_q\restriction_{\chi_1} \to ((Mod\theta)N)\restriction_{\chi_1}$	application of $Mod\chi_1$
9	$N = ((Mod\theta)N)\restriction_{\chi_1}$	$\theta : \chi_1 \to \chi_2$ substitution
10	there exists homomorphism $M_q\restriction_{\chi_1} \to N$	8, 10.

On the other hand, for any homomorphism $M_q\restriction_{\chi_1} \to N$:

11	there exists homomorphism $(Mod\theta)(M_q\restriction_{\chi_1}) \to (Mod\theta)N$	application of $Mod\theta$
12	$M_q \models q$	M_q basic model for q
13	$\chi_1(\theta q) = q$	$\theta : \chi_1 \to 1_\Sigma$ substitution, syntactic property of substitutions
14	$(Mod\theta)(M_q\restriction_{\chi_1}) \models q$	12, 13
15	there exists homomorphism $M_q \to (Mod\theta)(M_q\restriction_{\chi_1})$	14, M_q basic model for q
16	there exists homomorphism $M_q \to (Mod\theta)N$	11, 15.

- *2. implies 1.* Let $\chi_k : \Sigma \to \Sigma_k$, $k = 1, 2$. Then

1	there exists a χ_2-expansion M_2 of $0_{\Sigma,E}$	χ_2 has the model expansion property
2	$M_2\restriction_{\chi_2} \models E$	1, $0_{\Sigma,E} \models E$, Satisfaction Condition for χ_2
3	$M_2\restriction_{\chi_2} \models (\forall\chi_2)\theta q$	2, $E \models (\forall\chi_2)\theta q$
4	$M_2 \models \theta q$	3
5	$(Mod\theta)M_2 \models q$	4, Satisfaction Condition for θ
6	$((Mod\theta)M_2)\restriction_{\chi_1} = M_2\restriction_{\chi_2} = 0_{\Sigma,E}$	semantic property of θ, definition of M_2
7	$0_{\Sigma,E} \models (\exists\chi_1)q$	5, 6
8	$E \models (\exists\chi_1)q$	7, First Herbrand Theorem.

□

The following remarks, that emerge from a close examination of both the result and the proof of the Second Herbrand Theorem, help towards a deeper mathematical understanding of logic programming.

- On the one hand, the proof that 1. implies 2. shows that each solution (M_1) of a query is an instance of a solution form (θ) of that query. This is one of the meanings of diagram (16.4) (with M_1 being represented by $i_{\chi_1} M_1$ and θ by h). This has the flavour of a completeness property. On the other hand, the proof that 2. implies 1. can be interpreted in a reverse way, that any solution form (θ) provides solutions ($(Mod\theta)M_2$) of the query. Each particular choice of M_2 yields a solution. This has the flavour of a soundness property.
- The 'Herbrand model' (aka initial model) $0_{\Sigma,E}$ lies at the heart of the Second Herbrand Theorem through the involvement of the First, which is bi-directional in the sense that it is involved in both implications of the former theorem.
- The representability premise is used only in the first implication of the Second Herbrand Theorem. Apparently, the second implication does not require any additional properties for χ_1 and χ_2. However, the involvement of the First Herbrand Theorem does require the quasi-representability property for χ_1.
- The Second Herbrand Theorem requires premises that are technically stronger than those of the First, but in most applications, the premises of both theorems lead to identical concrete frameworks. These applications include the common frameworks for logic programming. The actual meaning of the premises of the Second Herbrand Theorem have already been discussed at several places in this book, so we are familiar with this. In relation to this, instances of both Herbrand Theorems are straightforward to formulate in $\mathcal{HCL}$. However, in the applications, the model expansion condition on χ_2 makes a notable difference between the two Herbrand theorems. In the context of $\mathcal{HCL}$, according to Fact 5.6, the non-empty sorts condition on Σ gets involved (i.e. that 0_Σ does not have empty sorts).

Exercises

16.2. Consider the theory of the addition of the natural numbers of (16.1) and the query $(\exists a,b,c)\ a+ssb=sc$.

1. Find a solution form for this query.
2. Consider the same problem in the relational form of (16.2). Then the corresponding query is $(\exists a,b,c)\ add(a,ssb,sc)$. What about its solution forms?

By comparing the answers to the questions above draw a conclusion about the difference between relational and equational logic programming.

16.3. Formulate valid instances of both Herbrand Theorems in $\mathcal{PA}$ (the institution of partial algebras).

16.3 Modularization

In practice, the structuring of logic programming programs can be done by using the specification building operators BASIC, TRANS, and UNION. Other building operators can be involved as long as each program has a normal form which is a theory; this assumption underlies the developments in this section, so we can just represent programs by their normal forms. For instance, DERIV breaks this assumption, but in the context of logic programming DERIV does not have meaning. The components of a structured program are inter-connected by morphisms, which can be assimilated to theory morphisms between corresponding normal forms. For instance UNION and TRANS lead to such morphisms. The most fundamental logic programming modularisation issue is the interaction between morphisms and queries and their solutions. The main idea is the solutions of queries should not be affected by translations along morphisms. We do as follows:

1. We define the mathematical framework for structured logic programs.
2. We clarify how queries and their solutions translate along morphisms.
3. We show the soundness of translation of queries and their solutions. This means that a translated solution remains a solution.
4. We give sufficient conditions for the completeness of translations of queries and of their solutions. This means that a translated query should not have solutions besides those of the original query. The main condition for this is that the morphisms 'protect' the Herbrand models of the source theory.
5. Then we show how this condition is fulfilled by two kinds of pushouts of program morphisms. One of them covers the union of programs, and the other one instantiations of program parameters (in the context of generic / parameterised programming).

Abstractly structured logic programs. The concept of (I, sig)-structured institution of Sec. 15.4 can be employed for defining axiomatically an abstract concept of structured programs in logic programming that does not commit to any particular set of structuring operators. This axiomatisation goes as follows. Let I be a (base) institution and I' be a (I, sig)-structured institution such that

- $sig : \ Sig' \to Sig$ preserves pushouts.
- I' inherits the signature morphisms.
- I' has semantic basic specifications.

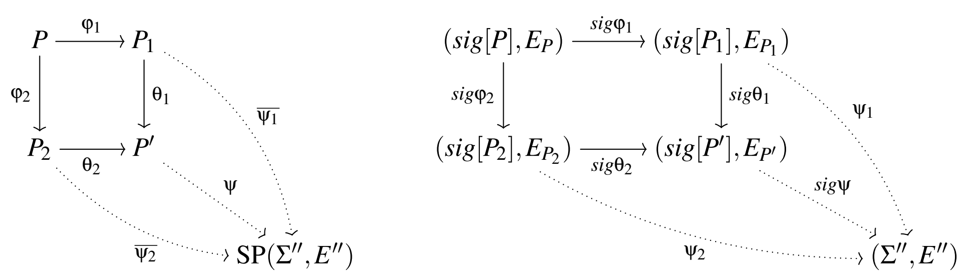
Each I'-signature P that has a designated normal form $(1_{sig[P]}, E_P)$ such that the theory $(sig[P], E_P)$ admits initial models, denoted 0_P, is called a *(structured) logic program*. The initial models 0_P are called *Herbrand models of P*. The following expected result, which will be technically useful, interprets pushouts of program morphisms as pushouts of corresponding normal forms.

Proposition 16.3. *For any pushout of program morphisms like in the left-hand side* *square below, we have that the the right-hand square below is a pushout in I^{th}.*

$$\begin{array}{ccc} P & \xrightarrow{\varphi_1} & P_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ P_2 & \xrightarrow[\theta_2]{} & P' \end{array} \qquad\qquad \begin{array}{ccc} (sig[P], E_P) & \xrightarrow{sig\varphi_1} & (sig[P_1], E_{P_1}) \\ {\scriptstyle sig\varphi_2}\downarrow & & \downarrow{\scriptstyle sig\theta_1} \\ (sig[P_1], E_{P_1}) & \xrightarrow[sig\theta_2]{} & (sig[P'], E_{P'}) \end{array} \tag{16.5}$$

Proof. Directly from the definition of the (I, sig)-structured institutions, we have that the left-hand side square of (16.5) is indeed a commutative square of morphisms in I^{th}. We prove it has the pushout property. Let $\psi_k : (sig[P_k], E_{P_k}) \to (\Sigma'', E'')$, $k = 1,2$, be I^{th}-morphisms such that $sig\varphi_1 ; \psi_1 = sig\varphi_2 ; \psi_2$.

- For $k = 1,2$, we consider morphisms $\overline{\psi_k} : P_k \to \mathrm{SP}(\Sigma'', E'')$ such that $sig\overline{\psi_k} = \psi_k$.
- It follows that $\varphi_1 ; \overline{\psi_1} = \varphi_2 ; \overline{\psi_2}$. Let $\psi : P' \to \mathrm{SP}(\Sigma'', E'')$ be the unique morphism such that $\theta_k;\psi = \overline{\psi_k}$, $k = 1,2$.



- Then $sig\psi : (sig[P'], E_{P'}) \to (\Sigma'', E'')$ is the unique mediating morphism in I^{th} making the right-hand side diagram above commute.

□

Translations

Now we establish the concepts of translations of queries and of their solutions along program morphisms. Then we prove that a soundness-like property, that by translating solutions we obtain solutions of the translated query.

Translation of queries along morphisms. Consider that the base institution I comes with a designated class $\mathcal{D}$ of quasi-representable signature morphisms. Let $\varphi : P \to P'$ be a program morphism. A *translation of a $\mathcal{D}$-query $E_P \models (\exists\chi)q$ along* φ is any $\mathcal{D}$-query

$E_{P'} \models (\exists\chi')\varphi_1 q$ where

$$\begin{array}{ccc} sig[P] & \xrightarrow{\chi} & \Sigma_1 \\ {\scriptstyle sig\varphi}\downarrow & & \downarrow{\scriptstyle \varphi_1} \\ sig[P'] & \xrightarrow[\chi']{} & \Sigma'_1 \end{array} \tag{16.6}$$

is a pushout square of signature morphisms. Query translations are not unique, but are unique up to semantic equivalence because pushout squares are unique up to isomorphisms.

Proposition 16.4. *If $\mathcal{D}$ is stable under pushouts then any $\mathcal{D}$-query admits translations along morphisms φ that are liberal as signature morphisms.*

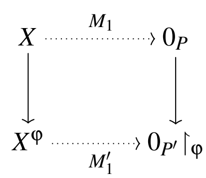
Proof. Let $(\exists\chi)q$ be a $\mathcal{D}$-query. We consider any pushout square of signature morphisms like (16.6). Then any 'sentence' $(\exists\chi')\varphi_1 q$ is a query because $\varphi_1 q$ is basic since the translations of basic sentences along liberal signature morphisms are still basic (cf. Ex. 5.28). □

The liberality condition on the signature morphisms is mild in the applications, as can be noticed for example in the case of $\mathcal{FOL}$, where each signature morphism is liberal (see Ex. 4.78).

Translations of solutions along morphisms. Consider a program morphism $\varphi : P \to P'$. Let M_1 be a solution for a $\mathcal{D}$-query $E_P \models (\exists\chi)q$. A *translation of* M_1 *along* φ is any χ'-expansion M'_1 of $0_{P'}$, for any pushout square like (16.6) such that there exists a Σ_1-homomorphism $h_1 : M_1 \to M'_1\restriction_{\varphi_1}$. Let us try to understand this abstract concept through a concrete situation.

- Let $\chi : sig[P] \to sig[P] + X$, $\chi' : sig[P'] \to sig[P'] + X^\varphi$ be $\mathcal{FOL}$ signature extensions with blocks of first-order variables, such that X^φ represents a 're-sorting' of X according to φ, like we did when we defined the translations of the quantified sentences in $\mathcal{FOL}$ (Sec. 3.1).
- Then M_1 means a valuation $X \to 0_P$ while M'_1 means a valuation $X^\varphi \to 0_{P'}$. Keep in the mind that X and X^φ are essentially the same, only the sorts of the variables are renamed according to φ.
- The homomorphism h_1 provides the crucial link between M_1 and M'_1, which allows us to speak of M'_1 as a translation of M_1. Its reduct is a homomorphism $0_P \to 0_{P'}\restriction_\varphi$. Since $E_{P'} \models \varphi E_P$, we have that $0_{P'}\restriction_\varphi \models E$, hence by the initiality of 0_P there exists exactly one such homomorphism. By the quasi-representability of χ, there exists only a χ-expansion of the homomorphism $0_P \to 0_{P'}\restriction_\varphi$ to a homomorphism $M_1 \to N_1$. This means that either a homomorphism $M_1 \to M'_1\restriction_{\varphi_1}$ does not exist, or else there is only one.

- Then, for each variable x in X, h_1 maps each interpretation $(M_1)_x$ to $(M'_1)_x$. In other words, the valuation $X^{\varphi} \to 0_{P'}$ (that represents M'_1) is the mere translation of the valuation $X \to 0_P$ (that represents M_1) according to the unique homomorphism $0_P \to 0_{P'}\!\upharpoonright_{\varphi}$.



It is important to establish two things:

1. The existence of the translations of solutions of queries.
2. The translation of a solution of a query is a solution for any translation of the respective query.

The former fact is established as follows, under a very mild condition.

Proposition 16.5. *If the base institution I has weak model amalgamation then any solution of a $\mathcal{D}$-query for P has a translation along any program morphism $\varphi\colon\ P \to P'$.*

Proof. Let M_1 be a solution for a $\mathcal{D}$-query $E_P \models (\exists\chi)q$.

- We consider any pushout square like (16.6).
- We consider the unique model homomorphism $0_P \to 0_{P'}\!\upharpoonright_{\varphi}$, which, by using the quasi-representability of χ, we χ-expand to a Σ_1-homomorphism $h_1\colon\ M_1 \to N_1$.
- By the weak amalgamation property there exists a Σ'_1-model M'_1 which is the amalgamation of N_1 and $0_{P'}$.

□

The latter fact is established as follows under the same condition of Prop. 16.5.

Proposition 16.6. *If the base institution I has weak model amalgamation, then for any program morphism $\varphi\colon\ P \to P'$ we consider a*

- *a $\mathcal{D}$-query $E_P \models (\exists\chi)q$ and one of its translations $E_{P'} \models (\exists\chi')\ \varphi_1 q$ along φ, and*
- *a solution M_1 for the former query and one of its translations M'_1 along φ.*

Then M'_1 is a solution for the translated query $E_{P'} \models (\exists\chi')\ \varphi_1 q$.

Proof. We have to prove that $M'_1 \models \varphi_1 q$. We consider the Σ_1-homomorphism $h_1\colon\ M_1 \to M'_1\!\upharpoonright_{\varphi_1}$. Then

1 there exists a basic model M_q for q — q basic

2 there exists a homomorphism $M_q \to M_1$ — 1, $M_1 \models q$ (M_1 solution)

3	there exists a homomorphism $M_q \to M'_1\restriction_{\varphi_1}$	2, $h_1 : M_1 \to M'_1\restriction_{\varphi_1}$
4	$M'_1\restriction_{\varphi_1} \models q$	3, q basic
5	$M'_1 \models \varphi_1 q$	4, Satisfaction Condition

Hence M'_1 is a solution for the translation $E_{P'} \models (\exists\chi')\varphi_1 q$. □

Protections

Now we develop a completeness-like result for the translations of queries and solutions along program morphisms. We also show how the main condition for this result gets fulfilled in general for two of the most relevant modularisation situations.

Protecting solutions. Consider a translation of a query along a theory morphism φ like above. Think of φ as representing a module import $P \to P'$ of logic programs. What we definitely do not want is that $E_P \models (\exists\chi)q$ acquires some new solutions in P', in other words that $E_{P'} \models (\exists\chi')\ \varphi_1 q$ has some solutions M'_1 that are not translations of some solutions M_1 of $E_P \models (\exists\chi)q$. The most fundamental principle of program structuring is that the structuring should not affect the components. For instance, an import of a module should not affect in any way the imported module. In the case of logic programming, one way to express this is that queries at the level of the imported module do not get new solutions at the level of the importing module. We will prove that this property is caused by the 'protection' of the Herbrand / initial model of the imported module. So, given a morphism $\varphi : \ P \to P'$, we say that it is *protecting* [the Herbrand model of P] when $0_{P'}\restriction_\varphi = 0_P$.

Proposition 16.7. *For any protecting morphism $\varphi : \ P \to P'$, each solution for the query $E_{P'} \models (\exists\chi')\ \varphi_1 q$ is a translation along φ of a solution of the original query $E_P \models (\exists\chi)q$.*

Proof. Let M'_1 be a solution for $E_{P'} \models (\exists\chi')\ \varphi_1 q$, the context being a pushout square of signature morphisms like (16.6).

- We show that $M'_1\restriction_{\varphi_1}$ is a solution for $E_P \models (\exists\chi)q$. We have that

$$
\begin{aligned}
M'_1\restriction_{\varphi_1}\restriction_\chi &= M'_1\restriction_{\chi'}\restriction_\varphi && \chi;\varphi_1 = \varphi;\chi',\ \text{functoriality of } Mod \\
&= 0_{P'}\restriction_\varphi && M'_1\restriction_{\chi'} = 0_{P'}\ (M'_1 \text{ solution}) \\
&= 0_P && \varphi \text{ protects the Herbrand model.}
\end{aligned}
$$

  It remains to prove that $M'_1\restriction_{\varphi_1} \models q$. But this follows from $M'_1 \models \varphi_1 q$ by the Satisfaction Condition.

- That M'_1 is translation of $M'_1\restriction_{\varphi_1}$ along φ is obvious as $M'_1\restriction_{\chi'} = 0_{P'}$, while the homomorphism h_1 is the identity Σ_1-homomorphism on $M'_1\restriction_{\varphi_1}$.

□

Protecting Herbrand models. The condition on the program morphism that it protects the Herbrand model of the source program, which is the basis for protecting solutions, is apparently quite stringent. However, this should be a methodological norm when developing logic programming software, and, in fact, we almost always follow it by common sense. Take the following example. We want to enhance the equational logic program (16.1) with a multiplication operation on the naturals (denoted $\_ * \_$). The obvious way to do this is to add the equations that define the multiplication by recursion, very much how we defined the addition.

$$(\forall x)\ x * 0 = 0,\ \ (\forall x, y)\ x * sy = (x * y) + x. \tag{16.7}$$

Let P be the program represented by (16.1) and P' the program represented by both (16.1) and (16.7). The inclusive program morphism $\varphi : P \to P'$ does protect the Herbrand model of P, which simply means that P' does not produce 'new' natural numbers, it does not collapse natural numbers, and it does not change the succesor and the addition operations. These preservations are not optional, as any such change would be a disaster.

The following result shows how, in logic programming, the protection of Herbrand models is preserved by pushouts of morphisms of programs.

Proposition 16.8. *Assume that the (I, sig)-structured institution I' is semi-exact. Then for any pushout of program morphisms like below*

$$\begin{array}{ccc} P & \xrightarrow{\varphi_1} & P_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ P_2 & \xrightarrow[\theta_2]{} & P' \end{array} \tag{16.8}$$

if some initial models of P_k, $k = 1,2$, share their φ_k-reducts as an initial model of P, then the morphisms θ_k, $k = 1,2$, protect the initial models.

Proof. By the semi-exactness assumption, the square below is a pullback in $\mathbb{C}at$:

$$\begin{array}{ccc} Mod'P & \xleftarrow{Mod'\varphi_1} & Mod'P_1 \\ {\scriptstyle Mod'\varphi_2}\uparrow & & \uparrow{\scriptstyle Mod'\theta_1} \\ Mod'P_2 & \xleftarrow[Mod'\theta_2]{} & Mod'P' \end{array}$$

- Let 0_{P_k}, $k = 1,2$, be initial models of P_k, $k = 1,2$, whose reducts are shared to an initial model for P, i.e. $0_{P_1}\restriction_{\varphi_1} = 0_{P_2}\restriction_{\varphi_2} = 0_P$.
- Let M' be the unique model amalgamation of 0_{P_1} and 0_{P_2}. We prove that this is an initial model of P'.
- For any P'-model N', for $k = 1,2$, let N_k be its θ_k-reduct.
  - For $k = 1,2$, let $h_k : 0_{P_k} \to N_k$ be the unique P_k-homomorphism.

– By the initiality of 0_P, we have that $h_1\upharpoonright_{\varphi_1} = h_2\upharpoonright_{\varphi_2} : 0_P \to N'\upharpoonright_{\theta_k}\upharpoonright_{\varphi_k}$. This means we can amalgamate h_1 and h_2 to a P'-homomorphism $h' : M' \to N'$.

– By the uniquenesses of h_k, $k = 1,2$ (from the initiality of 0_{P_k}) and by the uniqueness of model amalgamation, we obtain the uniqueness of h'.

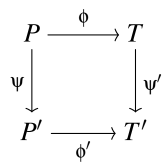
□

Some remarks on the conditions and on the applicability of this result:

- The normal forms of programs help also with the semi-exactness condition, which can be obtained easily from the semi-exactness of the base institution I via Prop. 15.9 by and Prop. 16.5.
- The most obvious application of Prop. 16.8 is when the square (16.8) corresponds to a union of programs, where all morphisms are inclusive and $P' = P_1 \cup P_2$. For instance, imagine that P is a program for natural numbers with addition, φ_1 extends P with multiplication, while P_2 is a program for lists of natural numbers with addition. Then $P_1 \cup P_2$ is a program for lists of natural numbers with addition and multiplication. You can check what it means that the Herbrand models are protected by the union $P_1 \cup P_2$ in this case.
- Note that our proof of Prop. 16.8 does not use that P' has a Herbrand model, it actually *proves* it. This means that this result can be reformulated by assuming Herbrand models only for P, P_1 and P_2, but this slightly higher generality does not have much practical meaning since if we are doing things in the realm of logic programming then that all programs have Herbrand models is always there.

The case of parameterised / generic programming. Pushout squares of logic programs such as (16.8) appear also in the context of parameterised / generic programming, where φ_1 is a parameterised program with P as parameter. Then φ_2 is an instantiation of the parameter, and θ_2 is the result of this instantiation for the respective program. The following is a simple example. Let P consist of one sort `Elt` and let P_1 have two sorts `Elt` and `List`, an operation $\_.\_ :$ `Elt List` $\to$ `List` that constructs lists, a *nil* constant for the empty lists, a relation *car* : `List Elt` giving the head of the list, specified by $(\forall X,L)\ car(X.L,\ X)$, and a relation *cdr* : `List List` giving the tail of a list, specified by $(\forall X,L)\ cdr(X.L,\ L)$. Then φ_1 is just the inclusion $P \to P_1$. Let P_2 be a program about natural numbers, and φ_2 mapping `Elt` to the sort `Nat` of the natural numbers. Then P' is a program for lists of natural numbers. If we desire a program for lists of something else the, in the pushout square, we just change φ_2. We make the following important point. θ_2 protects the [Herbrand model of the] natural numbers, but this cannot be obtained from Prop. 16.8 because the sharing hypothesis fails. The φ_1-reduct of the Herbrand model of P_1 is empty while the φ_2-reduct of the Herbrand model of P_2 is the set of the natural numbers. The following general result covers such instantiations of parameters.

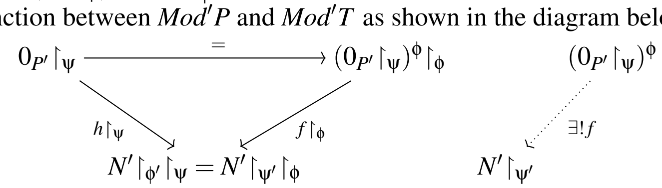
Proposition 16.9. *Under the same semi-exactness condition like in Prop. 16.8, consider a pushout of program morphisms like below*



such that ϕ *is persistently liberal, i.e. it has a left-inverse left-adjoint. Then* ϕ' *protects the initial models of* P'.

Proof. Let $0_{P'}$ be an initial model of P', and let $(0_{P'}\restriction_{\psi})^{\phi}$ be the free expansion of $0_{P'}\restriction_{\psi}$ along ϕ (according to the persistently liberal property of ϕ).

- Since $(0_{P'}\restriction_{\psi})^{\phi}\restriction_{\phi} = 0_{P'}\restriction_{\psi}$, let M' be the unique amalgamation of $(0_{P'}\restriction_{\psi})^{\phi}$ and $0_{P'}$ in I'. We prove that M' is an initial model of T'.
- Consider any model N' of T'.
  - Let $h' :\ 0_{P'} \to N'\restriction_{\phi'}$ be the unique P'-homomorphism.
  - Let $f :\ (0_{P'}\restriction_{\psi})^{\phi} \to N'\restriction_{\psi'}$ be the unique T-homomorphism given by the persistent adjunction between $Mod'P$ and $Mod'T$ as shown in the diagram below:



- Then the amalgamation of h and$f$ gives a homomorphism $M' \to N'$. Moreover, the homomorphism $M' \to N'$ is unique by the uniqueness of h and f and of the homomorphism amalgamation (by the semi-exactness assumption).

□

Like with Prop. 16.8, the Herbrand model of T' need not be assumed, its existence is actually proved through the proof itself. The condition on ϕ is fundamental for the concept of parameter, being stronger than the protection of the Herbrand model of P.

Exercises

16.4. Relational multiplication program
Develop a relational correspondent to the equational multiplication program given by (16.1) and (16.7). Does the multiplication program P' protects the Herbrand model of the relational addition program P given by (16.1)? Run Prolog to solve the query $P' \models (\exists a,b)\ a * b = 6$.

16.5. Translating solution forms
Consider an institution with representable $\mathcal{D}$-substitutions. For any liberal signature morphism $\varphi :\ \Sigma \to \Sigma'$ let $(-)^{\varphi} :\ Mod\Sigma \to Mod\Sigma'$ be the left adjoint to the model reduct functor $Mod\varphi :\ Mod\Sigma' \to$

$Mod\Sigma$. For any theory morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ if θ is a solution form for a $\mathcal{D}$-query in (Σ, E) then any $\mathcal{D}$-substitution ψ determined by $(M_\theta)^\varphi$ is a solution form for any translation of the respective query in (Σ', E').

16.4 Constraints

Pure symbolic logic programming may be sometimes impractical. In the context of the examples about natural numbers, think of situations when bigger numbers are involved. How would it be to write something like 11,537 as $ss\ldots s0$? And there is the issue of arithmetic calculations, at the purely symbolic level they are inefficient. Whenever number systems are involved, it is realistic to involve them as pre-defined types, which also means involvement of specific efficient computational tools for them. Not only number systems, but also other common data types can be involved in a pre-defined built-in form. When we do this in the context of logic programming, we speak of *constraint logic programming*, and two levels require our attention:

1. The denotational semantics, or the model theory level. Here, our abstract approach to logic programming pays off, we can just instantiate it to an institution of pre-defined types. We can do that even at a general level, and everything carries on smoothly, including the basic logic programming concepts and the Herbrand theorems. The scope of this section is to show in some detail how this happens.

2. The operational semantics is just another story, being not only institution-dependent but also type-dependent. In very general terms, the computation in constraint logic programming integrates execution of built-in packages with general symbolic computations based on resolution, or paramodulation, or something else. We will not touch this matter.

The theory of institution-independent constraint logic programming is based on a hierarchy of three abstract institutions. There is a base institution I with diagrams for the underlying logic, for the unstructured programming an institution of pre-defined types $I^\wr$ like in Sec. 15.5, and for the structured programming an $(I^\wr, sig)$-structured institution I'. The latter level is obtained just by applying the theory Sec. 16.3 to $I^\wr$, which is a straightforward enterprise. Also, because of this, in this section we focus on showing how the institution-independent semantics of unstructured logic programming gets interpreted in $I^\wr$ to obtain an institution-independent semantics for constraint logic programming. In this context, the key concepts that require more clarification effort are those of constraint query and constraint substitution. Programs, solutions, and solution forms are more straightforward.

Linear inequations with real numbers. Before defining constraint queries at the general level, let us look into a concrete situation. Let us recall the Euclidean plane example $\mathbb{R}^2$ given in Sec. 15.5. In $\mathbb{R}$ we may consider also a ‘less than of equal’ relation $\leq$. This extends to $\mathbb{R}'_{\text{float}}$ and further on to $\mathbb{R}^2$. The system of inequalities

$$\begin{cases} 3.14 * x + \sqrt{2} * y \leq 2, \\ 5.79 * x + 7.13 * y \leq -1.65 \end{cases}$$

can be regarded as a 'constraint query' in the Euclidean plane $\mathbb{R}^2$:

$$(\exists\{x,y\})\ x * \langle 3.14, 5.79\rangle + y * \langle\sqrt{2}, 7.13\rangle \leq \langle 2, -1.65\rangle. \tag{16.9}$$

We note the following aspects:

- The variables x, y are just $\mathcal{FOL}$ variables.
- The terms of (16.9) are obtained from the elements of $\mathbb{R}'_{\text{float}}$ and the $\mathcal{FOL}$ variables.

Constraint queries in general. How can we express the two remarks above about the nature of constraint queries in $\mathbb{R}^2$ at the general level? We consider an $I^{\wr}$-signature (Σ, A) (an example being $(\Sigma, \mathbb{R}'_{\text{float}})$). Then

- The 'variables' are I-signature morphisms $\chi\colon\ \Sigma \to \Sigma'$ from a designated class $\mathcal{D}$.
- The sentences in the role of q are considered from $Sen^{\wr}(\Sigma, A + M_\chi) = Sen(\Sigma_{A+M_\chi})$, where M_χ represents χ and $A + M_\chi$ is a co-product in $Mod\Sigma$. This also means we assume that $\mathcal{D}$ contains only representable signature morphisms. What the co-product does in concrete situations is that is provide the 'constraint terms' formed from the elements of A and the variables provided by χ.

We can write such a constraint query as $(\exists\chi)q$, which matches (16.9) well. But, technically this does not yet correspond to a query in $I^{\wr}$ because χ is not $I^{\wr}$-signature morphism. The solution is to consider $(1_\Sigma, j_A)$ where j_A is the component of the co-product co-cone as shown below:

$$A \xrightarrow{j_A} A + M_\chi \xleftarrow{j_\chi} M_\chi.$$

Thus, given $\mathcal{D}$, we can define $\mathcal{D}^{\wr}$ to be the class of the $I^{\wr}$-signature morphisms of the form $(1_\Sigma, j_A)$, when $\chi \in \mathcal{D}$. Then $(\exists\chi)q$ would be an abbreviation for $(\exists(1_\Sigma, j_A))q$.

Properties of $\mathcal{D}^{\wr}$. In order to apply the general theory of abstract logic programming to the constraint framework of $I^{\wr}$ we have to take care that $\mathcal{D}^{\wr}$ enjoys the following properties:

- It contains only representable signature morphisms; necessary for the Herbrand Theorems.
- It is stable under pushouts; necessary for translating queries.

Both properties can be transferred from $\mathcal{D}$.

Proposition 16.10. $(1_\Sigma, j_A)$ *is representable. Moreover, in the comma-category variant of* $Mod^{\wr}$, $M_{(1_\Sigma, j_A)} = j_A$.

Proof. The conclusion follows by the following commutative square of natural isomorphisms:

$$\begin{array}{ccc}
Mod^{\wr}(\Sigma, A+M_\chi) & \xrightarrow{\cong} & j_A/Mod^{\wr}(\Sigma, A) \\
\cong\big\downarrow & & \big\downarrow\cong \\
(A+M_\chi)/Mod\Sigma & \xrightarrow[\cong]{} & j_A/(A/Mod\Sigma)
\end{array}$$

□

The stability of $\mathcal{D}^{\wr}$ under pushouts can be established by intricate but rather canonical constructions. We exile this problem to the exercises part of this section.

First Herbrand theorem for constraint logic programming. By instantiating Herbrand Theorem 16.1 to $I^{\wr}$ we obtain the following Herbrand theorem for constraint logic programming over arbitrary institutions.

Theorem 16.11 (First Herbrand theorem for constraint logic programming)**.** *Let I be an institution with diagrams and with binary co-products of models. Let $((\Sigma, A), E)$ be an $I^{\wr}$-theory such that $(\Sigma_A, E_A \cup E)$ has initial models. Then $((\Sigma, A), E)$ has initial models and for each constraint (Σ, A)-$\mathcal{D}^{\wr}$-query $E \models^{\wr} (\exists\chi)q$,*

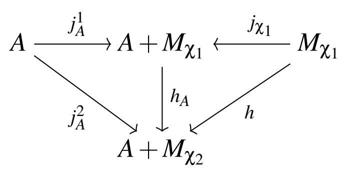
$$E \models^{\wr} (\exists\chi)q \ \text{ if and only if } \ 0_{(\Sigma,A),E} \models^{\wr} (\exists\chi)q.$$

Often, the sentences E of a constraint logic program do not involve the elements of the pre-defined model A, which means they are Σ-sentences rather than Σ_A-sentences. For instance, this happens in the Euclidean plane $\mathbb{R}^2$ example, when only the queries involve elements of $\mathbb{R}_{\text{float}}$. In this case, the Herbrand model $0_{(\Sigma,A),E}$ is just the 'quotient' of A by E as shown by the following fact.

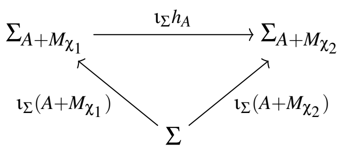
Fact 16.12. *For any set E of Σ-sentences such that the forgetful functor $Mod(\Sigma, E) \to Mod\Sigma$ admits a left-adjoint, the initial model of $((\Sigma, A), (\iota_\Sigma A)E)$ is $q_E : A \to A_E$, the universal 'quotient' model homomorphism from A to the free (Σ, E)-model over A.*

The left-adjoint condition is widely satisfied in the concrete applications; for this we can refer to general results from Sec. 4.6 showing that this is just a consequence of ordinary initial semantics (Prop. 4.29). Anyway, initial semantics is a pre-condition for any form of logic programming. Concerning the example of the Euclidean plane, Fact 16.12 explains at the general level how $\mathbb{R}^2$ arises as the initial model of the vector space specification of Sec. 15.5.

Constraint substitutions and solution forms. For the Second Herbrand Theorem for constraint logic programming we have to clarify the concept of 'constraint substitution'. In the (base) institution I, given representable signature morphisms in $\mathcal{D}$, $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$, for any Σ-model A, we consider the $\mathcal{D}^{\wr}$-morphisms $(1_\Sigma, j_A^1)$ and $(1_\Sigma, j_A^2)$. Then any homomorphism $h : M_{\chi_1} \to A + M_{\chi_2}$ can be regarded as a 'substitution' of the 'variables' χ_1 with '$(\Sigma, A+M_{\chi_2})$-terms'. By considering the tuple $h_A = \langle j_A^2, h\rangle$ given by the co-product property of $A+M_{\chi_1}$, we get a homomorphism in $Mod^{\wr}(\Sigma,A)$ between the representations of $(1_\Sigma, j_A^1)$ and $(1_\Sigma, j_A^2)$, $h_A : \ j_A^1 \to j_A^2$.



Fact 16.13. $\Psi_{h_A} : \ (1_\Sigma, j_A^1) \to (1_\Sigma, j_A^2)$ *is a* $\mathcal{D}^{\wr}$*-substitution (in* $I^{\wr}$*) defined by* $Sen^{\wr}\Psi_{h_A} = Sen(\iota_\Sigma h_A)$ *and* $Mod^{\wr}\Psi_{h_A} = Mod(\iota_\Sigma h_A)$.



Then Ψ_{h_A} is called a *constraint* (Σ,A)*-*$\mathcal{D}^{\wr}$*-substitution* $\chi_1 \to \chi_2$. A *constraint* (Σ,A)*-*$\mathcal{D}^{\wr}$*-solution form* for a $\mathcal{D}^{\wr}$-query $(\exists\chi_1)q$ is any (Σ,A)-$\mathcal{D}^{\wr}$-substitution $\theta : \ \chi_1 \to \chi_2$ such that $E \models (\forall\chi_1)\ \theta q$.

Second Herbrand theorem for constraint logic programming. We can now instantiate Herbrand Theorem 16.2 to the institution $I^{\wr}$ of pre-defined types.

Theorem 16.14 (Second Herbrand theorem for constraint logic programming)**.** *In the context of Thm. 16.11 we further assume that*

- *each representation* M_χ *of any* $\chi : \ \Sigma \to \Sigma'$ *in* $\mathcal{D}$ *is projective with respect to* $0_{(\Sigma,A),E} : \ A \to A_E$*, the initial* $((\Sigma,A),E)$*-model.*

Then for any $\mathcal{D}^{\wr}$*-query* $E \models (\exists\chi_1)q$*, the following are equivalent:*

1. $E \models (\exists\chi_1)q$*.*
2. *There exists a constraint* (Σ,A)*-*$\mathcal{D}^{\wr}$*-solution form* $\theta : \ \chi_1 \to \chi_2$ *such that* χ_2 *has the model expansion property.*

Similarly to the applications of Thm. 16.2, the projectivity condition of Thm. 16.14 can be established easily in the actual examples, since the initial models $0_{((\Sigma,A),E)} : \ A \to A_E$ are 'quotients' of the pre-defined model A (see Fact 16.12), and hence they are surjective.

Exercises

16.6. Representable signature morphisms in $I^{\wr}$

1. Any signature morphism $(\chi, f): (\Sigma, A) \to (\Sigma', A')$ is representable in the institution $I^{\wr}$ of predefined types if $\chi: \Sigma \to \Sigma'$ is quasi-representable in the base institution I.
2. If the base institution I has binary co-products of models and a signature morphism $\chi: \Sigma \to \Sigma'$ is represented by M_χ (in I), then $(\chi, j_A): (\Sigma, A) \to (\Sigma', i_\chi^{-1}(j_\chi))$ is represented by $j_A: A \to A + M_\chi$ (in $I^{\wr}$).

16.7. In any semi-exact institution with diagrams the co-product $M_\chi + A$ can be obtained as $0_{\Sigma'(A),\chi' E_A} \upharpoonright_\Sigma$ when the square below is a pushout square of signature morphisms.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\iota_\Sigma A} & \Sigma_A \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi'} \\ \Sigma' & \xrightarrow[\iota']{} & \Sigma'(A) \end{array}$$

16.8. Stability of $\mathcal{D}^{\wr}$ under pushouts
Based on the stability of $\mathcal{D}$ under pushouts, develop a result establishing the stability of $\mathcal{D}^{\wr}$ under pushouts.

16.5 Unification

The operational / computational semantics of concrete forms of logic programming is an interplay of several algorithms, with the unification algorithm at its core. In this section we develop a general category-theoretic approach to unification as concept, and at the same level of abstraction, an analysis of the unification algorithm.

What is unification?

We have already seen a bit of what this means when we discussed the resolution procedure on the simple example of addition of natural numbers. The short answer to this question is "the endeavour to find a substitution that equalizes a pair of terms". But we can do better than this in some ways. First of all, it is useful to liberate this concept from its traditional context and to formulate it abstractly in order to achieve a higher understanding and make it available in an uniform way to a multitude of various concrete frameworks. In order to do this, we start with looking at the traditional concept of unification, and then we project a categorical perspective.

Unification of $\mathcal{FOL}$ terms. Let Σ be a $\mathcal{FOL}$ signature, X a block of variables for Σ, and t, t' be two $(\Sigma + X)$-terms of the same sort. A *unifier* for t and t' is any substitution $\theta: X \to Y$ (i.e., a function $X \to 0_{\Sigma+Y}$) such that $\theta t = \theta t'$. For instance, when $X = \{x, y\}$ and $Y = \emptyset$, $\theta x = b$, $\theta y = a$ define a unifier for $(x * a, b * y)$. Moroever, this is the only unifier for this problem.

We can present unifiers in a conceptually uniform way in terms of substitutions only. For this we regard terms as substitutions. For each term $t \in 0_{\Sigma+X}$ by $\bar{t}$ we denote the Σ-substitution $\{*\} \to X$ defined by $\bar{t}(*) = t$.

Fact 16.15. *A unifier* θ *for* t *and* t' *is precisely a co-cone for the parallel pair of substitutions* $(\bar{t}, \overline{t'})$*,*

$$\{*\} \underset{\overline{t'}}{\overset{\bar{t}}{\rightrightarrows}} X \xrightarrow{\theta} Y.$$

Unifiers may exist or not. For instance, when x, y are variables, $x * a$ and $b * y$ have unifiers, but $x * a$ and $(x * a) * a$ do not. Also, $x * a$ and $b \circ y$ do not, but for a different reason, even more obvious than in the previous example.

Categorical unification. The view of terms-as-substitutions, as put forward by Fact 16.15, allows for a fully abstract categorical view on unification. A *(categorical) unifier* θ for a parallel pair of arrows t, t' is just a co-cone for t, t', i.e. $t; \theta = t'; \theta$. Now, let us assume an epi inclusion system $(\mathcal{I}, \mathcal{E})$ for the category. A $\mathcal{E}$*-unifier* is any unifier that belongs to to $\mathcal{E}$. A *most general unifier* (abbreviated *mgu*) for a parallel pair of arrows t, t' is any unifier θ that is not a proper instance of any $\mathcal{E}$-unifier, i.e. for any $\mathcal{E}$-unifier θ' for t, t' such that $\theta = \theta'; \gamma$ we have that γ is isomorphism. Note that any *mgu* is an $\mathcal{E}$-unifier.

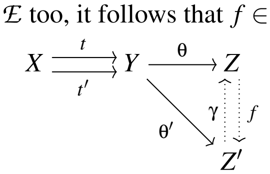
In the classical context of unification, that of $\mathcal{FOL}$ terms, the category is that of first-order substitutions. The inclusion system is similar to the standard inclusion system in $\mathbb{S}et$. A substitution $\theta\colon\ X \to Y$ is an abstract inclusion if it is a set inclusion, and is an abstract surjection when for many $y \in Y$ there exists $x \in X$ such that y occurs in the term θx. Later on in this section we will establish that in $\mathcal{FOL}$, the *mgu*'s are unique up to isomorphisms, which in categorical terms means a co-equaliser.

In general, for any pair of parallel arrows (t, t') in a category, let us denote by $coeq(t, t')$ the class of its co-equalisers, and by $mgu(t, t')$ the class of its *mgu*'s. We have the following simple relationship between co-equalisers and *mgu*'s.

Proposition 16.16. *If* $coeq(t, t') \neq \emptyset$ *then* $coeq(t, t') = mgu(t, t')$*.*

Proof. We assume $coeq(t, t') \neq \emptyset$.

- Consider $\theta \in coeq(t, t')$. Suppose there exists an $\mathcal{E}$-unifier θ' and an arrow γ such that $\theta = \theta'; \gamma$. We prove that γ is isomorphism.
  - Since θ is co-equaliser and θ' is unifier, there exists f such that $\theta; f = \theta'$. Then $\theta \in \mathcal{E}$ as co-equaliser (in general, any co-equaliser is an abstract surjection). Since $\theta' \in \mathcal{E}$ too, it follows that $f \in \mathcal{E}$.



  - On the one hand, $\theta; f; \gamma = \theta'; \gamma = \theta$. Since θ is co-equalizer, it is epi, hence $f; \gamma = 1_Z$.
  - On the other hand, $f; \gamma = 1_Z$ implies $f; \gamma; f = f$. By the epi property of f (epi inclusion system, $f \in \mathcal{E}$), it follows that $\gamma; f = 1_{Z'}$.

- Now, consider $\theta \in mgu(t,t')$. Let $\theta' \in coeq(t,t')$. Hence there exists γ such that $\theta';\gamma = \theta$. Since $\theta' \in \mathcal{E}$ (as co-equaliser) and θ is *mgu*, by definition it follows that γ is isomorphism. This means θ is co-equaliser too.

□

There are many different unification contexts that fit general categorical unification. In some of them *mgu*'s are not necessarily co-equalisers. Here is one such example. A practically important kind of unification is that of unification modulo some equational theory, where the substitutions map variables to equivalence classes of terms modulo that theory. For instance, consider the signature of the natural numbers with addition, with the usual operations 0, $s\_$, $\_+\_$. We consider terms modulo the associativity of $\_+\_$. Then $mgu(x+0, 0+x) = \{\theta_k : \{x\} \to \emptyset \mid k \in \omega\}$ where $\theta_k x = \underbrace{0+\ldots+0}_{\times(k+1)}$.

Unification in institutions. The category-theoretic view of unification can be interpreted in the institution theory setting by considering categories of substitutions. For any signature Σ in an arbitrary institution with a designated class $\mathcal{D}$ of signature morphisms, a *$\mathcal{D}$-unifier* for any Σ-$\mathcal{D}$-substitutions $\psi_1, \psi_2 : \chi \to \chi'$ is any Σ-$\mathcal{D}$-substitution $\theta : \chi' \to \chi''$ such that $\psi_1;\theta$ and $\psi_2;\theta$ are equivalent (i.e., $Mod(\psi_1;\theta) = Mod(\psi_2;\theta)$).

$$\chi \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} \chi' \xrightarrow{\theta} \chi''$$

About this definition, we note several things.

- In the case of first order substitutions in $\mathcal{FOL}$, two substitutions are equivalent if and only if they are equal. This means that this institution-independent definition covers precisely the standard traditional $\mathcal{FOL}$ concept of unifier.
- In $\mathcal{FOL}$, χ can be interpreted as any block of variables, which means that the concept of unifier applies also to sets of pairs of terms, not only to single pairs.
- The definition does not commit to any particular type of substitutions, such as representable ones. One of the consequences of this generality is that this allows for higher-order unification also.

A categorical unification algorithm

In the context of categorical unification, now we develop an unification algorithm that generalises the $\mathcal{FOL}$ classical algorithm. As one of the main characteristics of $\mathcal{FOL}$ unification is the equality $coeq(t,t') = mgu(t,t')$, our categorical unification algorithm will be about computing co-equalisers. This algorithm can be applied as-such to other unification contexts, and sometimes it can be applied under some modifications. An algorithm finding co-equalisers consists essentially of repeatedly reducing the original problem to 'simpler' problems, until we end up with trivial problems. By 'simpler' we mean 'less symbols'. For instance, in the case of the unification of the pair $(x*a,\ b*y)$, we do as follows:

1. We eliminate the topmost symbol $*$, that is shared by both terms. Then, we get two pairs of terms to be unified, $(x,b),(a,y)$. If the topmost symbols were different operation symbols, then the unification algorithm fails in the sense that there is no unifier to be found.

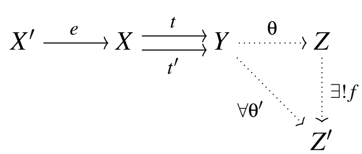
2. We solve each of the above problems separately and then compose the results. What exactly 'compose' means will be clarified later on in this section.

3. Both (x,b) and (a,y) constitute irreducible cases as we cannot eliminate symbols or split problems anymore. We get the unifiers $x\mapsto b$ and $y\mapsto a$, which by composition yield the unifier for $(x*a,\ b*y)$.

Both reduction steps and the irreducible states of unification can be expressed at the abstract categorical level. We do this in what follows.

Reducing the 'operation symbols'. The elimination of a shared topmost symbol in the unification process is explained in category-theoretic terms by the following general proposition. We omit its straightforward proof.

Proposition 16.17. *In any category, if* $e:\ X'\to X$ *is epi and* $t,t':\ X\to Y$, *then* $coeq(t,\ t') = coeq(e;t,\ e;t')$.



It may not be immediately clear how Prop. 16.17 covers the elimination of $*$ in the example above. The following $\mathcal{FOL}$ instance of Prop. 16.17 provides the general elimination of a shared topmost operation symbol.

Corollary 16.18. *For any operation symbol* σ, *the pair of terms* $(\sigma(t_1,\ldots,t_n),\sigma(t'_1,\ldots,t'_n))$ *has the same set of most general unifiers as the set of pairs of terms* $\{(t_1,t'_1),\ldots,(t_n,t'_n)\}$.

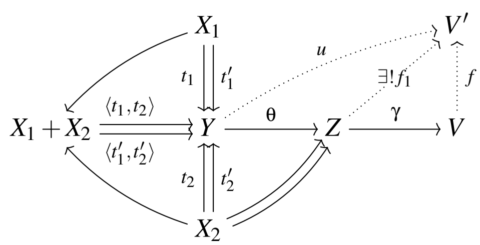
Proof. The conclusion is achieved by interpreting the entities of Prop. 16.17 in the following way:

$$X' = \{*\} \xrightarrow{e(*)=\sigma(x_1,\ldots,x_n)} X = \{x_1,\ldots,x_n\} \underset{t'x_i=t'_i}{\overset{tx_i=t_i}{\rightrightarrows}} Y$$

□

Splitting unification. By applying the result of Cor. 16.18 that reduces the total number of symbols by eliminating the symbol σ, we transform the original problem, formulated for a single pair of terms, with a problem for a set of pairs of terms. Then we can split the new unification problem into smaller parts that can be solved sequentially and their results can be composed in order to obtain the result for the big problem. The general result of Prop. 16.19 just does this in category-theoretic terms.

Proposition 16.19. *In any category, consider the commutative diagram below*



(16.10)

where $X_1 + X_2$ *is the co-product of* X_1 *and* X_2*, and* $\langle t_1, t_2\rangle, \langle t'_1, t'_2\rangle \colon\ X_1 + X_2 \to Y$ *are the tuplings of* t_1 *with* t_2 *and of* t'_1 *with* t'_2*, respectively, given by the co-product property. If* $(\theta\colon\ Y \to Z) \in coeq(t_1,\ t'_1)$ *and* $(\gamma\colon\ Z \to V) \in coeq(t_2;\theta,\ t'_2;\theta)$ *then* $\theta;\gamma \in coeq(\langle t_1, t_2\rangle,\ \langle t'_1, t'_2\rangle)$*.*

The proof of Prop. 16.19 can be 'seen' easily by inspecting diagram (16.10), where u is a co-cone for $\langle t_1, t_2\rangle, \langle t'_1, t'_2\rangle$. We can also understand that the choice of the splitting is immaterial. The following $\mathcal{FOL}$ interpretation of Prop. 16.19 shows how the splitting step works in the case of the $\mathcal{FOL}$ unification algorithm.

Corollary 16.20. *When it exists, the most general unifier for* $\{(t_1,\ t'_1), \dots, (t_n,\ t'_n)\}$*, a set of pairs of* Σ*-terms with variables* Y*, can be obtained as the substitution* θ_n*, where* $\theta_0 = 1_Y$*, and for* $k = \overline{1,n}$*,* $\theta_k = \theta_{k-1};\gamma_k$ *and* γ_k *is the most general unifier of* $(\theta_{k-1}t_k,\ \theta_{k-1}t'_k)$*.*

Proof. Follows immediately from Prop. 16.19 by noting that for any $\mathcal{FOL}$ signature, the disjoint union of blocks of variables is a co-product in the category of first order substitutions. □

The irreducible cases. The unification algorithms consist of alternations (which can be non-deterministic) of the reduction step given by Prop. 16.17 and of the splitting step given by Prop. 16.19. This process leads eventually to a finite number of irreducible unification problems. In the case of $\mathcal{FOL}$ unification, these can be of the following three kinds:

1. $(\sigma(t_1,\dots,t_n),\ \sigma'(t'_1,\dots,t'_{n'}))$ where σ and σ' are different operation symbols,
2. (x,t) where x is a variable occurring in t, and
3. (x,t) where x is a variable *not* occurring in t.

While in the former two situations there are no unifiers, the latter case has the substitution $\theta x = t$ as a most general unifier. In what follows we will express these irreducible situations in general category-theoretic terms.

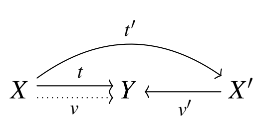
Different topmost operation symbols. This situation can be simply expressed by the negation of the premise of Prop. 16.17. Thus, a parallel pair arrows $t_0, t_0' : X' \to Y$ is *epi-irreducible* when there is no proper epi e and arrows t, t' such that $t_0 = e;t$ and $t_0' = e;t'$.

Terms that are variables. In order to express occurence / non-occurence of variables in terms at a general categorical level, we need to capture categorically the situation of a term consisting only of a variable. Within the terms-as-substitutions view, a variables-as-substitutions view means a component of a co-product co-cone. This goes for any category. If we have a co-product like below

$$X \xrightarrow{v} X + X' \xleftarrow{v'} X'$$

then v, v' can be thought as 'variables'. If we apply this to institutions, in any institution with a designated class $\mathcal{D}$ of signature morphisms, a $\mathcal{D}$-substitution $v : \chi \to \varphi$ is a $\mathcal{D}$-*variable* when φ is a co-product $\chi + \chi'$ and v is the component of the co-product co-cone corresponding to χ. For example, in the category of $\mathcal{FOL}$ $\mathcal{D}$-substitutions (with $\mathcal{D}$ being the class of the injective signature extensions with a finite number of constants), we may note immediately that the $\mathcal{D}$-variables are just injections between sets (of $\mathcal{FOL}$ constants), co-products of $\mathcal{FOL}$ $\mathcal{D}$-substitutions being disjoint unions.

Occurrence of variables in terms. Next, we express at the level of abstract categories that a variable v does *not* occur in a term t. The idea is that t can be expressed as a term t' whose variables belong to the complement of v. In any category, given a 'variable' $v : X \to Y$ and an arrow $t : X \to Y$ we say that v *does not occur in* t when $t = t';v'$ where v, v' represent a co-product co-cone.

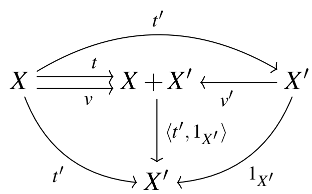


This can be interpreted immediately to categories of $\mathcal{D}$-substitutions in institutions.

The simple categorical proposition below expresses the fact that the co-equaliser (the unique *mgu*) of x and t, where x is a variable that does not occur in t, is given by the substitution mapping x to t.

Proposition 16.21. *In any category, for any 'variable'* $v : X \to X + X'$ *and for any*

$t' : X \to X'$ *we have that* $\langle t', 1_{X'} \rangle \in coeq(v, t';v')$.



Proof. We leave some of the details as exercise. First, note that $\langle t', 1_{X'} \rangle$ is a (split) epi. Then, for any u such that $t;u = v;u$, we can prove that $\langle t', 1_{X'} \rangle;(v';u) = u$. □

Termination of the unification algorithm. This issue has to be dealt with at the level of particular situations. For the first order substitution in $\mathcal{FOL}$, the algorithm determined by Prop.s 16.17, 16.19, and 16.21 terminates because the preorder on sets of pairs of terms defined by the following three criteria (in the order of their priority)

1. the number of variables in the set of pairs of terms,
2. the number of occurrences of operation symbols, and
3. the number of pairs of terms,

is well founded (i.e., does not have infinite strictly decreasing sequences) and that each application of each step given by Propositions 16.17, 16.19, and 16.21 represents a move downwards in this preorder.

We can thus formulate the following $\mathcal{FOL}$ consequence of our study of the unification algorithm.

Corollary 16.22. *In $\mathcal{FOL}$, any parallel pair of finitary first order substitutions has a coequaliser if and only if it has a unifier. Moreover, this can be computed by alternating the reduction steps given by Cor. 16.18 and the splitting step given by Cor. 16.20 and by applying the unification of a variable with a term.*

Exercises

16.9. In general, can we prove that any unifier is an instance of an *mgu*? What does it take to have this?

16.10. Develop all details of the proofs of Propositions 16.17, 16.19, and 16.21.

16.11. Prove a more general variant of Prop. 16.19, namely that $\theta \in coeq(t_1, t_1')$ and $\gamma \in mgu(t_2;\theta, t_2';\theta)$ implies that $\theta;\gamma \in mgu(\langle t_1, t_2\rangle, \langle t_1', t_2'\rangle)$. How useful is this in the applications?

16.12. [162]
Let u, v, x, y, w, z be first-order variables in $\mathcal{FOL}$. For the following pairs of terms, determine whether most general unifiers exist or not and find them when they exist.

1. $p(fy,w,gz)$, $p(u,u,v)$.
2. $p(fy,w,gz)$, $p(v,u,v)$.

3. $p(a,x,f(gy))$, $p(z,h(z,w),fw)$.

16.13. [56] **Unification of infinite terms**
Consider the institution $\mathcal{CA}$ of contraction algebras of Ex. 3.3. From there we know that for each $\mathcal{CA}$ signature Σ, there exists an initial Σ-model 0_Σ. The underlying (many-sorted) set of 0_Σ is the (many-sorted) set infinite terms, T_Σ^ω. For each signature Σ define a category of Σ-substitutions with infinite terms. Prove that for any parallel pair of such substitutions, if it has an unique unifier then it has an *mgu* (aka co-equaliser). (*Hint:* Adapt the $\mathcal{FOL}$ unification algorithm by considering the $coeq(x,t)$ where x is a variable and t is a term, even when x occurs in t.)

16.14. Unification with constraints

1. Let (Σ,E) be a theory for $\mathbb{R}$-modules, where $\mathbb{R}$ is the ring of the real numbers. Let $\mathbb{R}'$ be the free extension of $\mathbb{R}$ to Σ. Show that systems of linear equations are just parallel pairs of constraint $((\Sigma,\mathbb{R}'),E)$-substitutions, and solutions of systems of linear equations are just $((\Sigma,\mathbb{R}'),E)$-unifiers.
2. By applying Propositions 16.19 and 16.21 show that any system having solutions has a 'most general' solution. Can you recognise this as a classic result from linear algebra?

Notes. Logic programming began in the early 1970s as a direct outgrowth of earlier work in automatic theorem proving and artificial intelligence. The theory of clausal-form [first order] logic, and an important theorem by the logician Jacques Herbrand constituted the foundation for most activity in theorem proving in the early 1960s. The discovery of resolution — a major step in the mechanization of clausal-form theorem proving — was due to J. Alan Robinson [210]. In 1972, Robert Kowalski and Alain Colmerauer were led to the crucial idea that *logic could be used as a programming language* [239]. A year later the first Prolog system was implemented; now there are several quite advanced Prolog systems available. A good reference for foundations of conventional logic programming is [162]. The equational logic programming paradigm unifies logic programming based on Horn clause logic and functional programming based on equational logic. One of the earliest contributions to this field was [198]. Later Goguen and Meseguer provided a definition of equational logic programming as logic programming over classical conventional specification based on (order sorted) equational logic [128, 129]. Diaconescu generalized it to logic programming over 'category-based equational logic' in [57], and Goguen and Kemp extended it to logic programming over behavioral logic in [126]. In [57] it was shown how resolution can be simulated by paramodulation in the context of the encoding of relations as equations presented in Sec. 3.3.

The conventional Herbrand Theorem from [162] has been extended to many-sorted first-order logic with equality in [129] and generalized to category-based equational logic in [57, 58]. The non-empty sorts condition in the many-sorted version of the Second Herbrand Theorem poped-up for the first time in [129]. The latter included as its instance a Herbrand theorem for 'category-based' constraint logic [61].

Our approach to the institution-independent foundations of logic programming based on (quasi-)representable signature morphisms was developed in [65] which also introduced the concept of institution-independent substitution. This approach was further developed in [237, 51].

The earliest algorithm for computing most general unifiers in first-order logic was given by Herbrand [146], and later in [210] it was applied to automated inference. Following the observation of Goguen that most general unifiers are just co-equalizers in categories of substitutions [120], Rydeheard and Burstall developed a generic categorical approach to unification algorithms in [215]. A 2-categorical approach on unification modulo equational theories has been investigated by [216].

Our basic result on modularization for logic programming was first developed in [57] and [59] within the context of category-based equational logic programming. The results of Propositions 16.8 and 16.9 were first proved in [96].

Appendix A

Table of Notations

Sets

ω	the set of the natural numbers (non-negative integers)
S^*	the set of the strings with elements from S
$\mathbb{R}$	the set of the real numbers
${}^*\mathbb{R}$	the set of the hyperreal numbers
$\mathcal{P}A$	the set of the subsets of A
$\mathcal{P}_\omega A$	the set of the finite subsets of A
$A \setminus B$	the difference between sets A and B, $\{x \in A \mid x \notin B\}$
$A \uplus B$	$A \cup B$ when $A \cap B = \emptyset$
$card\ A$	the cardinality of the set A
λ^+	the last cardinal strictly greater than the ordinal λ
$F\vert_J$	the reduction of the filter F over I to a subset $J \in F$

Categories

$\mathbb{S}et$	the category of sets as objects and functions as arrows
$\mathbb{C}lass$	the ‘category’ of classes
$\mathbb{C}at$	the ‘category’ of categories as objects and functors as arrows
$\mathbb{CC}at$	the ‘category’ of concrete categories
$\mathbb{G}rp$	the category of groups
$\vert\mathbb{C}\vert$	the class of objects of the category $\mathbb{C}$
$\mathbb{C}(A,B)$	the set of arrows between objects A and B
$dom(f)$	the domain (source) of the arrow f
$cod(f)$	the codomain (target) of the arrow f
$f;g$	the composition of arrows f and g
$\mathbb{C}^{\mathrm{op}}$	the opposite of the category $\mathbb{C}$

$\mathbb{C}^*$	the 2-opposite of a 2-category $\mathbb{C}$
$A \cong B$	the objects A and B are isomorphic
$A \times B$	a direct product of objects A and B
$\prod_{i\in I} A_i$ / A_I	a direct product of the family of objects $(A_i)_{i\in I}$
A_F	the F-product of a family $(A_i)_{i\in I}$ of objects in a category
$A+B$	the co-product (direct sum) of the objects A and B
$0_{\mathbb{C}}$	the initial object of the category $\mathbb{C}$
$Lim(D)$	vertex of the limiting cone for the diagram D
$Colim(D)$	vertex of the co-limiting co-cone for the diagram D
$A/\mathcal{U}$	comma category
(f,B)	object of comma category $A/\mathcal{U}$ where $f: A \to \mathcal{U}B$
$B^\sharp$	the Grothendieck category of the indexed category B

Institutions and proof systems

Sig^I	the category of the signatures of institution I
Sen^I	the sentence functor of institution I
Mod^I	the model functor of institution I
$M \models^I_\Sigma \rho$	the Σ-model M satisfies the Σ-sentence ρ in the institution I
E^*	the class of models satisfying the set of sentences E
$\mathbb{M}^*$	the set of sentences satisfied by the class of models $\mathbb{M}$
$\mathbb{M} \equiv \mathbb{M}'$	$\mathbb{M}$ and $\mathbb{M}'$ are elementarily equivalent classes of models (i.e. $\mathbb{M}^* = \mathbb{M}'^*$)
$\Gamma \models E$	semantic consequence (i.e. $E^* \subseteq \Gamma^*$)
$E \models\!\mid E'$	$E \models E'$ and $E' \models E$
$\Gamma \vdash E$	entailment (i.e., there exists a proof) from Γ to E
$E \vdash\!\dashv E'$	$E \vdash E'$ and $E' \vdash E$
Th^I	the category of theories of institution I
I^{th}	the institution of the theories of the institution I
Mod^{th}	the model functor $Mod^{I^{\text{th}}}$ of I^{th}
$E^\bullet$	the closure of the theory
CTh^I	the category of closed theories of institution I
Pf^I	the proof functor of proof system I
Rl^I	the proof rule functor of system-of-proof rules I
h^I	hypotheses of proof rules in a system of proof rules I
c^I	conclusions of proof rules in a system of proof rules I
0_Σ / $0_{\Sigma,E}$	the initial [Σ / (Σ,E)]-model
$\iota_\Sigma M$	the elementary extension of the signature Σ via the model M
(Σ_M, E_M)	the diagram of the Σ-model M
$i_{\Sigma,M}$	the natural isomorphism determined by the diagram of the Σ-model M
M_M	the initial model $0_{\Sigma_M,E_M}$ of the diagram of a model M

N_h	$i^{-1}_{\Sigma,M}h$ for $h: M \to N$ model homomorphism
$E(I)$	the elementary sub-institution of I
$I^\sharp$	the institution of the 'local' satisfaction for a stratified institution I
I^*	the institution of the 'global' satisfaction for a stratified institution I
$\mathcal{K}(\mathcal{S})$	'modalisation' of a stratified institution $\mathcal{S}$
$I^\sharp$	the binary flattening of an $\mathcal{L}$-institution I

Categories of institutions / proof systems

$(co)\mathbb{I}ns$	the category of institution (co)morphisms
$(co)\mathbb{P}f\mathbb{I}ns$	the category of institution with proofs (co)morphisms
$(co)\mathbb{S}\mathbb{I}ns$	the category of stratified institution (co)morphisms
$(co)\mathbb{R}l\mathbb{S}ys$	the category of system-of-proof rules (co)morphisms
$(co)\mathbb{P}f\mathbb{S}ys$	the category of proof system (co)morphisms
$\mathbb{E}\mathbb{D}\mathbb{I}ns$	the category of institutions with diagrams

In concrete institutions

φ^{st}	the mapping on sort symbols of φ, a morphism of $\mathcal{FOL}$ signatures
φ^{op}	the mapping on operation symbols of φ, a morphism of $\mathcal{FOL}$ signatures
φ^{rl}	the mapping on relation symbols of φ, a morphism of $\mathcal{FOL}$ signatures
$\|M\|$	the set of the elements of the carriers of the model M
$(x: s)$	the constant/variable x has sort s
$\overrightarrow{S}$	the set of all (higher-order) types constructed from the sorts S
$=_f$	the kernel of homomorphism f
$=_\Gamma$	the least Γ-congruence
T_Σ	the set of Σ-terms
$t \stackrel{e}{=} t'$	existence equation in $\mathcal{PA}$
S_w	class of (plain) injective model homomorphisms in $\mathcal{FOL}$ and $\mathcal{PA}$
S_c	class of closed injective model homomorphisms in $\mathcal{FOL}$ and $\mathcal{PA}$
S_f	class of full subalgebras in $\mathcal{PA}$
H_r	class of surjective model homomorphisms in $\mathcal{FOL}$ and $\mathcal{PA}$
H_s	class of strong surjective homomorphisms in $\mathcal{FOL}$
H_c	class of closed surjective homomorphisms in $\mathcal{FOL}$

Internal logic

$\rho_1 \wedge \rho_2$	the conjunction of ρ_1 and ρ_2
$\rho_1 \vee \rho_2$	the disjunction of ρ_1 and ρ_2
$\rho_1 \Rightarrow \rho_2$	the implication of ρ_2 by ρ_1
$\rho_1 \Leftrightarrow \rho_2$	the equivalence between ρ_1 and ρ_2
$\bigwedge E$	the conjunction of the set of sentences E
$\neg\rho$	the negation of ρ
$\neg E$	$\{\neg\rho \mid \rho \in E\}$
$(\forall\chi)\rho$	universal quantification of Σ'-sentence ρ for $\chi : \Sigma \to \Sigma'$ signature morphism
$(\exists\chi)\rho$	existential quantification of Σ'-sentence ρ for $\chi : \Sigma \to \Sigma'$ sign. morphism
M_χ	the model representing the signature morphism χ
i_χ	the natural isomorphism defining a representable signature morphism χ
$Mod\psi$	the model reduct translation part of the substitution ψ
$Sen\psi$	the sentence translation part of the substitution ψ
M_E	the model defining a basic set of sentences E
$M \models^{\mathrm{inj}} h$	M is injective with respect to h
$Inj(H)$	the class of objects / models injective with respect to all $h \in H$

Other notations

$FX\ \mathcal{U}$	the set of the fixed points of the semantic operator $\mathcal{U}$
$Up\ \mathbb{M}$	the class of all ultraproducts of models of $\mathbb{M}$
Ur	the ultraradical relation on models
$\mathsf{Univ}\Sigma$	the set of universal Σ-sentences in $\mathcal{FOL}$
$\mathsf{Exist}\Sigma$	the set of existential Σ-sentences in $\mathcal{FOL}$
$M[Sen^0]N$	$M^* \cap Sen^0\Sigma \subseteq N^* \cap Sen^0\Sigma$
$M \xrightarrow{Sen^0} N$	there exists a Σ-model homomorphism $h : M \to N$ such that $M_M[Sen^0]N_h$
$\mathcal{J}^\sharp$	the Grothendieck institution of the indexed (co)institution $\mathcal{J}$
$sig[\mathrm{SP}]$	the signature of the specification SP
$Mod[\mathrm{SP}]$	the class of the models of the specification SP
$\mathrm{SP} \models \mathrm{SP}'$	$sig[SP] = sig[\mathrm{SP}']$ and $Mod[\mathrm{SP}] \subseteq Mod[\mathrm{SP}']$
$\mathrm{SP} \models\!\!\mid \mathrm{SP}'$	equivalence of specifications, i.e. $\mathrm{SP} \models \mathrm{SP}'$ and $\mathrm{SP}' \models \mathrm{SP}$
$\mathrm{SP} \cup \mathrm{SP}'$	union of specifications
$\mathrm{SP} \star \varphi$	renaming of specification SP
$\varphi \Box \mathrm{SP}$	hiding of specification SP
$Spec^I$	the category of structured specifications of institution I
$Spec_{\mathcal{T},\mathcal{D}}$	the category of $\langle\mathcal{T}, \mathcal{D}\rangle$-specifications
I^{spec}	the institution of structured specifications over I
$I^{\mathrm{spec}}_{\langle\mathcal{T},\mathcal{D}\rangle}$	the institution of $\langle\mathcal{T}, \mathcal{D}\rangle$-specifications over I

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